Abstract

In this study, the consensus problem of a class of discrete-time multi-agent systems with switching topology is considered. A consensus algorithm is constructed for the multi-agents rendezvous. The algorithm based on nearest neighbor rules is composed of odd time steps and even time steps. In odd time steps, each agent maintains its state toward the states of its neighbors, and in even time steps its statemaybe be updated by the condition of connection of last odd time step with its neighbors. If an agent is disconnected with one of its neighbors in an even time step which was connected in the last odd time step, its state should be restored as the last odd time step, otherwise its state should not be changed. We provide a convergence analysis for the disagreement network dynamics of a undirected network with switching topology by applying the matrix theory and the graph theory. Some sufficient criteria are obtained for the discrete-time multi-agent systems with switching topology. Simulation results show the correctness of results.

Keywords: Consensus, Multi-agent Systems, Switching System

1. Introduction

As one of the most important problems of distributed coordination of multi-agent systems [1-2], consensus problem, which has received significant research attention in a wide range include biological sciences, physical sciences, systems and control sciences, computer sciences. Consensus means the output of a group of agents reaches a common value by the information exchange between an agent and all of its neighbors on the network in a finite time. A consensus protocol is a distributed interaction rule to make agents to reach consensus. In the study of the consensus problem, some typical methods were used to describe and analyze the consensus of the multi-agent systems including graph theory, matrix analysis, control theory and so on. During the past decade, the theory of consensus has developed in a variety of directions under fixed and switching topologies.

Olfati-Saber and Murray [3] discuss consensus problems for networks of dynamic agents with fixed and switching topologies, and introduce two consensus protocols with and without time delays, provide a convergence analysis to solve the average-consensus problem by defining a disagreement function in all cases. Moreau [4] proposed a model of network of agents interacting via time-dependent communication links, and presented necessary and sufficient conditions for consensus of states under fixed and switching topologies which be guaranteed as long as the network remains connected at all times. Ando [5] present a distributed algorithm for converging autonomous mobile robots with limited visibility toward a single point. Zhiyun Lin [6] proposed three formation strategies to achieve specified formation among a group of mobile autonomous agents by distributed control. J. Fang [7] proposed a new model to solve the discrete multi-agent rendezvous problem and it is concerned with a specified set of points called “dwell-points” which a set of mobile autonomous agents with limited sensing range.

Accordingly, these works [8-12] use point models for agents. It is based on the connectivity of network at every moment, if one agent lost communication with another agent, the communication between them would never restore. The system would converge to a several limiting values and not a common value. In this paper multi-agents may be positioned at a point. The agents can restore the communication via a new consensus algorithm when they lost communication with another agent at a time step.
2. Preliminaries

Let $G(V,E,A)$ be a undirected graph of order $n$ with a set of vertices $V = \{1,2,...,n\}$ and a weighted adjacency matrix $A = [a_{ij}]$. A matrix is defined as $a_{ii} = 0$ and $a_{ij} > 0$, if $(j,i) \in E$, where $i \neq j$. A matrix is non-negative symmetric. The Laplacian matrix of undirected graph $G$ is defined as $L = \left[ l_{ij} \right]$ where $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ where $i \neq j$.

The set of neighbors of node $i$ is denoted by $N_i = \{ j \in V : a_{ij} \neq 0 \}$. The number of neighbors of agent $i$ is denoted by $\left| N_i \right| = \left| \{ j \in V : (i,j) \in E \} \right|$. The set of neighbors at time $k$ are denoted by $N_i(k) = \{ j \in V : \| x_i(k) - x_j(k) \| \leq r \}$. The number of neighbors of agent $i$ is denoted by $\left| N_i(k) \right| = \left| \{ j \in V : (i,j) \in E(k) \} \right|$. We can use a function $\psi_i(k)$, where $k \geq 2$ to describe whether there is a active link or lost link between agent $i$ and agent $j$ at time $k$ which is defined as follows

\[ \psi_i(k) = \begin{cases} 0, & \text{if } \exists j, j \in N_i(k) \text{ and } j \notin N_i(k+1) \\ 1, & \text{else.} \end{cases} \]

For multi-agents rendezvous with discrete-time models with switching topologies, each agent dynamics is given as follows

\[
\begin{align*}
  x_i(k+1) &= x_i(k) + \hat{\epsilon} \sum_{j \in N_i(k)} a_{ij} (x_j(k) - x_i(k)) \\
  x_i(k+2) &= (1 - \psi_i(k+2)) x_i(k) + \psi_i(k+2) x_i(k+1), \quad k = 0,2,4,...
\end{align*}
\]
where $\partial$ is the feedback gains, $\partial > 0$.

The discrete-time network dynamics which can be written in matrix form is given as follows

$$
\begin{align*}
X(k+1) &= X(k) + \partial L(k) X(k) = (I + \partial L(k)) X(k) \\
X(k+2) &= (I + \psi(k+2)) X(k) + \psi(k+2) X(k+1), \quad k = 0, 2, 4, \ldots.
\end{align*}
$$

(2)

where $X(k) = [x_i(k)]$, $\psi(k) = [\psi_j(k)]$.

Definition 1: the multi-agent system (1) is said to achieve consensus for all of the agents if we have

$$
\|x_i(k) - x_j(k)\| \to 0, \text{ for } i, j \in \text{Agent} \text{ and } j \neq i \text{ and } k \to \infty
$$

3. Convergence analysis

Theorem 1: Let $x_i(k)$ be the position of mobile autonomous agent $i$ evolving according to (1), if

$$
0 < \partial \leq \frac{1}{n}
$$

and $x_i(0) \leq x_j(0)$ then $x_i(k) \leq x_j(k)$ for all $k$

Proof: We use induction. Without loss generality, we assume that

(i) $x_i(0) \leq x_j(0)$, where $j \geq i$ and $i, j \in \text{Agent}$;
(ii) $\|x_i(0) - x_j(0)\| \leq L(0)$;
(iii) $\|x_i(k) - x_j(k)\| \leq r$, where $i \in \{1, 2, \ldots, n-1\}$

(1) Suppose $x_i(k) \leq x_j(k), k = 0, 2, 4, \ldots$

$$
x_i(k+1) = x_i(k) + \partial \sum_{p \in \mathcal{N}_i(k)} (x_p(k) - x_i(k))
$$

$$
x_j(k+1) = x_j(k) + \partial \sum_{p \in \mathcal{N}_j(k)} (x_p(k) - x_j(k))
$$

Let $\mathcal{N}_i(k)$ be the set of agents connected to $i$ and not to $j$, $\mathcal{N}_j(k)$ the set of agents connected to $j$ and not to $i$, $\mathcal{N}_{ij}(k)$ the set of agents connected both $i$ and $j$.

$$
x_i(k+1) = x_i(k) + \partial \sum_{p \in \mathcal{N}_i(k)} x_p(k) + \partial \sum_{q \in \mathcal{N}_{ij}(k)} x_p(k) - \partial \left( n_i(k) + n_j(k) \right) x_i(k)
$$

$$
x_j(k+1) = x_j(k) + \partial \sum_{p \in \mathcal{N}_j(k)} x_p(k) + \partial \sum_{q \in \mathcal{N}_{ij}(k)} x_p(k) - \partial \left( n_i(k) + n_j(k) \right) x_j(k)
$$

$$
x_i(k+1) - x_j(k+1) = x_i(k) - x_j(k) + \partial \sum_{p \in \mathcal{N}_i(k)} x_p(k) + \partial \left( n_i(k) + n_j(k) \right) x_j(k)
$$

$$
- \partial \sum_{m \in \mathcal{N}_{ij}(k)} x_m(k) - \partial \left( n_i(k) + n_j(k) \right) x_i(k)
$$

Since $x_i(k) \leq x_j(k)$, $\sum_{p \in \mathcal{N}_i(k)} x_p(k) \leq n_i(k) x_i(k)$ and $\sum_{m \in \mathcal{N}_{ij}(k)} x_m(k) \geq n_j(k) x_i(k)$,

$$
x_i(k+1) - x_j(k+1) \leq \left( x_i(k) - x_j(k) \right) \left( 1 - \partial \left( n_i(k) + n_j(k) \right) \right)
$$

Since $x_i(k) \leq x_j(k)$, $2 \leq n_i(k) + n_j(k)$ and $n_j(k) \leq n$, if $\partial \leq \frac{1}{n}$, $x_i(k+1) \leq x_j(k+1)$.

(2) $x_i(k+2) = (1 - \psi_j(k+2)) x_i(k) + \psi_j(k+2) x_j(k+1)$

$$
x_j(k+2) = (1 - \psi_j(k+2)) x_j(k) + \psi_j(k+2) x_j(k+1), i \neq j
$$

(i) If $\psi_j(k+2) = 1$ and $\psi_j(k+2) = 1$ then $x_i(k+2) = x_i(k+1)$ and $x_j(k+2) = x_j(k+1)$

$$
x_i(k+2) - x_j(k+2) = x_i(k+1) - x_j(k+1)
$$
from (1) we can draw a conclusion

If \( x_i(k) \leq x_j(k) \) then \( x_i(k + 1) \leq x_j(k + 1), k = 0, 2, 4, \ldots \)

(ii) If \( \psi_i(k + 2) = 1 \) and \( \psi_j(k + 2) = 0 \) then \( x_i(k + 2) = x_j(k + 1) \) and \( x_j(k + 2) = x_j(k) \)

\[
x_i(k + 2) - x_j(k + 2) = x_i(k + 1) - x_j(k) = x_i(k) + \partial \sum_{p \in N_i(k)} (x_p(k) - x_i(k)) - x_j(k)
\]

\[
x = x_i(k) - x_j(k) + \partial \sum_{p \in N_i(k)} x_p(k) - \partial (n_i(k) + n_j(k)) x_i(k)
\]

Since \( x_i(k) \leq x_j(k) \),

\[
\sum_{p \in N_i(k)} x_p(k) \leq (n_i(k) + n_j(k)) x_i(k)
\]

\[
x_i(k + 2) - x_j(k + 2) \leq (x_i(k) - x_j(k)) [1 - \partial (n_i(k) + n_j(k))]
\]

Since \( x_i(k) - x_j(k) \leq 0, n_i(k) + n_j(k) \leq n \)

If \( \partial \leq \frac{1}{n} \) then \( x_i(k + 2) \leq x_j(k + 2) \)

(iii) If \( \psi_i(k + 2) = 0 \) and \( \psi_j(k + 2) = 1 \) then \( x_i(k + 2) = x_i(k + 1) \) and \( x_j(k + 2) = x_j(k + 1) \)

\[
x_i(k + 2) - x_j(k + 2) = x_i(k) - x_j(k) - \partial \sum_{p \in N_i(k)} x_p(k) + \partial (n_i(k) + n_j(k)) x_j(k)
\]

Since \( x_i(k) \leq x_j(k) \),

\[
\sum_{p \in N_i(k)} x_p(k) \geq (n_i(k) + n_j(k)) x_i(k)
\]

\[
x_i(k + 2) - x_j(k + 2) \leq (x_i(k) - x_j(k)) [1 - \partial (n_i(k) + n_j(k))]
\]

Since \( x_i(k) - x_j(k) \leq 0, n_i(k) + n_j(k) \leq n \)

If \( \partial \leq \frac{1}{n} \) then \( x_i(k + 2) \leq x_j(k + 2) \)

(iv) If \( \psi_i(k + 2) = 0 \) and \( \psi_j(k + 2) = 0 \) then \( x_i(k + 2) = x_i(k) \) and \( x_j(k + 2) = x_j(k) \)

\[
x_i(k + 2) - x_j(k + 2) = x_i(k) - x_j(k)
\]

Since \( x_i(k) - x_j(k) \leq 0, x_i(k + 2) - x_j(k + 2) \leq 0 \)

It follows from (i)-(iv) that

If \( 0 < \partial \leq \frac{1}{n} \) and \( x_i(0) \leq x_j(0) \) then \( x_i(k) \leq x_j(k) \) for all \( k \)

For all of above, Theorem 1 is proved.

**Theorem 2**: Suppose \( C(0) = \sum_{i=1}^{\infty} x_i(0) / n \), \( 0 < \partial \leq \frac{1}{n} \) and the communication network \( G(V, E(k), A(k)) \) is connected at time \( k = 0, 2, 4, \ldots, 2m \), where \( m \geq 0 \). Then system (1) can solve the consensus problem. The convergence of mobile autonomous agents of system (1) equals to the global geometric center \( C(0) \).

**Proof**: Without loss generality, we assume that \( x(0) \) is sorted, from Theorem 1

if \( \partial \in \left(0, \frac{1}{n}\right) \) and \( x_i(0) \leq x_j(0) \) then \( x_i(k) \leq x_j(k) \) for all \( k \)

Suppose the smallest position is \( x_i(0) \), the largest position is \( x_j(0) \). Let \( x_i(k) \) be the position of mobile autonomous agent i evolving according to (1). Then we have

\[
x_i(k + 1) = x_i(k) + \partial \sum_{p \in N_i(k)} (x_p(k) - x_i(k))
\]

\[
x_i(k + 2) = (1 - \psi_i(k + 2)) x_i(k) + \psi_i(k + 2) x_i(k + 1), k = 0, 2, 4, \ldots
\]
(i) Suppose graph \( G(V,E(k),A(k)) \) of system (1) is a complete graph

Then we have \( N_i(k) = N_j(k) = \text{Agent, } i, j \in \text{Agent and } i \neq j \).

\[
x_i(k+1) = x_i(k) + \partial \sum_{p \in N_i(k)} x_p(k) - \partial \times n \times x_i(k)
\]

Suppose \( C(k) = \sum_{i=1}^{n} x_i(k)/n \)

We have \( x_i(k+1) = x_i(k) + \partial n (C(k) - x_i(k)) \) and \( x_j(k+1) = x_j(k) + \partial n (C(k) - x_j(k)) \)

If \( x_i(k) = x_j(k) = C(k) \) then \( x_i(k+1) = x_j(k+1) = x_i(k) = C(k), i \neq j \)

Since \( C(0) = \sum_{i=1}^{n} x_i(0)/n \), we suppose \( C(k) = \sum_{i=1}^{n} x_i(k)/n = C(0) \)

Then \( C(k+1) = \sum_{i=1}^{n} x_i(k+1)/n = \sum_{i=1}^{n} x_i(k)/n = C(k) = C(0) \)

\[
C(k+2) = \sum_{i=1}^{n} x_i(k+2)/n = \sum_{i=1}^{n} ((1 - \psi_i(k+2)) x_i(k) + \psi_i(k+2) x_i(k+1))/n
\]

Since \( x_i(k) = x_j(k) = C(k) \) \( C(k+2) = \sum_{i=1}^{n} x_i(k)/n = C(k) = C(0) \)

It follows from above that \( \| x_i(k) - x_j(k) \| \to 0 \), for \( i, j \in \text{Agent and } j \neq i \) and \( k \to \infty \)

(ii) Suppose the communication network \( G(V,E(k),A(k)) \) of system (1) is connected.

Since \( x_i(0) \) is the smallest position, \( x_i(0) \leq x_i(0), i \in [2,n] \)

Suppose \( x_i(k) \leq x_i(k), i \in [2,n] \)

\[
x_i(k+1) = x_i(k) + \partial \sum_{p \in N_i(k)} (x_p(k) - x_i(k)) \geq x_i(k)
\]

\[
x_i(k+2) = (1 - \psi_i(k+2)) x_i(k) + \psi_i(k+2) x_i(k+1)
\]

If \( \psi_i(k+2) = 1 \) then \( x_i(k+2) = x_i(k+1) \geq x_i(k) \)

If \( \psi_i(k+2) = 0 \) then \( x_i(k+2) = x_i(k) \)

Since \( x_i(0) \) is the largest position is. \( x_i(0) \leq x_i(0), i \in [1,n-1] \)

Suppose \( x_i(k) \leq x_i(k), i \in [2,n] \)

\[
x_i(k+1) = x_i(k) + \partial \sum_{p \in N_i(k)} (x_p(k) - x_i(k)) \leq x_i(k)
\]

\[
x_i(k+2) = (1 - \psi_i(k+2)) x_i(k) + \psi_i(k+2) x_i(k+1)
\]

If \( \psi_i(k+2) = 1 \) then \( x_i(k+2) = x_i(k+1) \geq x_i(k) \)

If \( \psi_i(k+2) = 0 \) then \( x_i(k+2) = x_i(k) \)

Hence \( x_i(k) \) is non-decreasing with time and \( x_i(k) \) is non-increasing with time. For any time \( k = 0,2,4,...,2m \), where \( m \geq 0 \), if the communication network \( G(V,E(k),A(k)) \) is connected and it is still not a complete graph, the smallest position \( x_i(k) \) is increasing and \( x_i(k) \) is decreasing.

The network will be a complete graph which evolves according to system (1) at last and is connected at time \( k = 0,2,4,...,2m \). It is a relevant point worth mentioning that if the interconnection between two agents at odd time steps is disconnected, the interconnection between two agents would be restored at even time steps. Thus when \( \psi_i(k+2) = 0 \), it means to the agent \( i \) disconnects one neighbour agent \( j \), Agent \( i \) should restore its value of last step. When \( \psi_i(k+2) = 1 \), agent \( i \) holds its position until next
step. That is to said, the multi-agent system does not require a real-time connectivity, the network can disconnect at odd time steps.

It follows from above that \( x_i(k) \) converges a limit \( C(0) \) in finite time.

4. Simulation Results

1. \( \delta = \frac{1}{6} \), Agent positions are either uniformly spaced, on an interval of length 6, range from \([1,6]\),

\[
\text{Agent} = [1; 2; 3; 4; 5; 6]
\]

![Figure 1. Time evolution of 6 agent positions](image)

According to the model (1).Initial agent positions are either uniformly spaced, on an interval of length 6. Agents converge to limiting values 3.5 that is the global geometric center of agent positions. Agent 3 is disconnected with one of its neighbors Agent 4 at time step 27 and the states of Agent 3 and 4 would be restored at time step 28.

2. \( \delta = 0.05 \), agent positions are either uniformly spaced, on an interval of length 12, range from \([1,12]\)

\[
\text{Agent} = [1; 1.2; 1.5; 1.8; 2; 3; 4; 5; 6; 6.2; 6.8; 7; 8; 9; 10; 11; 11.2; 11.5; 11.8; 12]
\]

According to the model (1).Initial agent positions are either uniformly spaced, on an interval of length 12. Agents converge to limiting values 6.5 that is the global geometric center of agent positions after almost 1000 time steps. Many Agents are disconnected with one of its neighbors at odd time step and restored at even time step.
5. Conclusion

In this paper, we have analyzed the model of system dynamics (1). Our motivation was to provide an analysis of a class of discrete-time multi-agent systems with switching topology. A consensus algorithm is constructed for the multi-agents rendezvous. We gave conditions for when the agents will rendezvous under the consensus algorithm. Numerical experiments again show the convergence of mobile autonomous agents of system (1) that equals to the global geometric center.

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7. References


