Verified Transformations on Functional Programs
Using the Higher-Order Abstract Syntax Approach

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Abstract
We describe an approach to the verified implementation of transformations on functional programs that exploits the higher-order representation of syntax. In this approach, transformations are specified using the logic of hereditary Harrop formulas (HOHH). On the one hand, these specifications serve directly as implementations, being programs in the language λProlog. On the other hand, they can be used as input to the Abella system, which allows us to prove properties about them and thereby about the implementations. We argue that this approach is especially effective in realizing transformations that analyze binding structure. We consider in detail the case of typed closure conversion, a transformation that changes such structure in programs. We present a concise encoding of the transformation in λProlog and show how Abella can be used to prove the correctness of the encoding.

1. Introduction
Program transformations play an important role in the compilation process. Being able to implement them easily and to prove the correctness of their implementation is therefore of considerable interest. A complicating factor in the functional programming setting is that transformations must often work on higher-order programs, manipulating their binding structure in complicated ways. The commonly used programming environments do not provide any direct support for such manipulations. The programmer typically has to use first-order representations of program syntax and must build in auxiliary support for treating binding within these representations.

We propose an alternative in this paper. We advocate the use of a framework that supports a higher-order representation of syntax in both implementation and reasoning. This framework comprises two parts: the λProlog language \[\text{\texttt{\lambda}Prolog}\] that is implemented, for example, in the Teyjus system \[\text{\texttt{Teyjus}}\], and the Abella proof assistant \[\text{\texttt{Abella}}\]. The λProlog language is a realization of the logic of hereditary Harrop formulas or HOHH. This logic provides support for a higher-order treatment of abstract syntax and is a suitable vehicle for specifying transformations on functional programs. Moreover, HOHH specifications have a computational interpretation, and are therefore \textit{implementations} of compiler transformations. The Abella system is also based on a logic that supports higher-order abstract syntax. A defining characteristic of this logic, called \(G\), is that it incorporates a treatment of fixed-point definitions which can also be interpreted inductively or co-inductively. The Abella system uses these definitions to embed HOHH within \(G\) and thereby to reason directly about the specifications written in HOHH.

In the context of this paper, this yields a means for verifying our implementations of compiler transformations.

The rest of this paper is organized to make a more detailed argument for the proposal that we have presented above. In Section 2, we elaborate on the framework, focusing specifically on those aspects of it that are relevant to the goals of this paper. We then show how this framework can be used to implement and to verify the closure conversion transformation on typed functional programs. Section 3 introduces the transformation and presents an informal proof of its correctness. Section 4 shows how closure conversion can be implemented in λProlog. Section 5 presents the formal verification of semantics preservation for closure conversion in Abella. Although we consider only closure conversion in detail, our framework can also be used to realize other transformations. Section 6 discusses a few other transformations to highlight this aspect. We conclude the paper in Section 7 with a comparison with other approaches to verified implementation of transformations on functional programs and an indication of future work.

2. The Framework
In the subsections below, we describe, in turn, the specification logic and λProlog, the reasoning logic and the manner in which the Abella system embeds the specification logic.

2.1 The specification logic and \texttt{\lambda}Prolog
The specification logic, called HOHH, is an intuitionistic and predicative fragment of Church’s Simple Theory of Types \[\text{\texttt{\lambda\textit{types}}}]\]. The types in its setting are formed using the function type constructor \(\to\) over user defined primitive types and the distinguished type \(\pi\) for formulas. Expressions are formed from a user-defined \texttt{signature} of typed constants whose argument types do not contain \(\pi\) and the \texttt{logical constants} \(\Rightarrow\) and \& of type \(\pi\to\pi\to\pi\) and, for each type \(\tau\) not containing \(\pi\), \(\Pi\) of type \((\tau\to\pi)\to\pi\). We write \(\Rightarrow\) and \&, which denote implication and conjunction respectively, in infix form. Further, we write \(\Pi\lambda(x:\tau)M\), which represents the universal quantification of \(x\) over \(M\), as \(\Pi_x M\). HOHH is characterized by two sets of formulas called \texttt{goal} formulas and \texttt{program} clauses that are given by the following rules:

\[
\begin{align*}
G &::= A | G\land G | D \Rightarrow G | \Pi_x G \\
D &::= A | A \Rightarrow A | \Pi_x x D
\end{align*}
\]

Here, \(A\) represents \texttt{atomic} formulas that have the form \((p t_1 \ldots t_n)\) where \(p\) is a (user defined) \texttt{predicate constant}, \textit{i.e.} a constant with target type \(\pi\). Note that our program clauses extend Horn clauses by replacing first-order terms with typed \(\lambda\)-terms as predicate arguments and permitting hypothetical and universal goals in clause bodies.

A collection of \(D\) formulas constitute a \texttt{program}. A program and a signature represent a specification of all the goal formulas that can be derived from them. The derivability of a goal formula \(G\) is expressed formally by the judgment \(\Sigma; \Theta; \Gamma \vdash G\) in which \(\Sigma\) is a signature, \(\Theta\) is a collection of program clauses defined by the user.
and $\Gamma$ is a collection of dynamically added program clauses. We assume the reader to be familiar with rules such as backchaining that characterize derivability in the Horn clause setting. HOHH has two additional rules for deriving universal and hypothetical goals:  

$$
\begin{align*}
\frac{\Sigma; \Theta; \Gamma \vdash D \implies G}{\Sigma; \Theta; \Gamma \vdash D} & \quad (\text{IR}) \\
\frac{\Sigma; \Theta; \Gamma \vdash \Pi \cdot x \cdot G}{\Sigma; \Theta; \Gamma \vdash \Pi \cdot x} & \quad (\text{II})
\end{align*}
$$

In the IR rule, $c$ is a new constant not already contained in $\Sigma$. Note that these rules can cause the program and the signature to grow in the course of looking for a derivation.

In presenting HOHH specifications in this paper we will show programs as a sequence of clauses each terminated by a period. We will leave the outermost universal quantification in these clauses implicit, indicating the variables they bind by using tokens that begin with uppercase letters. We will write program clauses of the form $G \Rightarrow A$ as $A : - G$. We will show goals of the form $G_1 \land G_2$ and $\Pi \cdot y \cdot G$ as $G_1 \cdot G_2$ and $\text{pl } y : \tau \land G$, respectively, dropping the type annotation in the latter if it can be filled in uniquely based on the context. Finally, we will write abstractions as $\lambda y.M$ instead of $\lambda y.x$.

Horn clauses already provide a means for encoding rule based specifications. By moving to HOHH we obtain the additional ability to accommodate binding structure cleanly in such encodings. This is, in fact, the main content of the $\lambda$-tree syntax approach: we use meta-language $\lambda$-abstraction to explicitly represent binding in object-language syntax, we realize recursion over such structure with attendant freshness conditions by introducing new constants using universal goals and we record auxiliary properties for such constants via hypothetical goals. This kind of encoding is concise and has logical properties that we can use in reasoning.

As an illustration, let us consider the specification of the typing relation for the simply typed $\lambda$-calculus (STLC). Let $N$ be the only primitive type. We designate the type $ty$ for representations of object language types that we build using the constants $n : ty$ and $\text{arr } : ty \rightarrow ty \rightarrow ty$. Similarly, we use the type $tm$ for encodings of object language terms that we build using the constants $\text{app } : tm \rightarrow tm \rightarrow tm$ and $\text{abs } : ty \rightarrow (tm \rightarrow tm) \rightarrow tm$. The type of the latter constructor follows our chosen approach to encoding binding: for example, we represent $(\lambda x : N : N \cdot x : N (y \cdot x))$ by $(\text{abs } (\text{arr } n \ n) \ (y \ \\ (\text{app } y \ (x \ (\text{app } \text{arr } n \ n)))))$. Typing for the simply typed $\lambda$-terms is a judgment: $\Gamma \vdash \lambda x : A : \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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\right
The treatment of fixed-point definitions in \( \mathcal{G} \) results in universal quantifiers being given an \textit{extensional} interpretation. However, many arguments concerning binding require the capability of reasoning over \textit{open} structures in which free variables are treated \textit{intensionally}, i.e., each name is distinct and not further analyzable. To support this possibility, \( \mathcal{G} \) includes the \textit{generic} quantifier \( \forall \), for each type \( \tau \) \cite{13}. In writing this quantifier, we will, once again, elide the type \( \tau \). The rules for treating this quantifier in an assumed formula and a formula in the conclusion are similar: a “goal” with \( (\forall x, M) \) in it reduces to one in which this formula has been replaced by \( M[c/x] \) where \( c \) is a fresh, unanalyzable constant called a \textit{nominal constant}. As an example of the difference between \( \forall \) and \( \forall \), \( (\forall x y, x = y \rightarrow \text{false}) \) is provable in \( \mathcal{G} \) but \( (\forall x y, x = y \rightarrow \text{false}) \) is not.

\( \mathcal{G} \) actually allows the \( \forall \) quantifier to be used in the heads of definitions. The full form for a definitional clause is in fact \( \forall \exists \forall z \exists A \equiv B \), where the \( \forall \) quantifiers scope only over \( A \). In generating an instance of such a clause, the variables in \( \exists \) must be replaced with nominal constants. The quantification order then means that the instantiations of the variables in \( \exists \) cannot contain the constants used for \( \exists \). This extension makes it possible to encode structural properties of terms in definitions. For example, the clause \( (\forall x, \text{name } x) \) defines \text{name} to be a recognizer of nominal constants. As another example, the clause \( (\forall x, \text{fresh } x B) \) defines \text{fresh} such that \( (\text{fresh } X B) \) holds just in the case that \( X \) is a nominal constant and \( B \) is a term that does not contain \( X \).

As a final example, consider the following definition in which \( \forall \) is the typing predicate from the previous subsection.

\[
\text{ctx nil}; \\
\forall x, \text{ctx } (\text{of } x T :: L) \equiv \text{ctx } L.
\]

These clauses define \text{ctx} such that \( (\text{ctx } L) \) holds exactly when \( L \) is a list of type assignments to distinct variables.

### 2.3 The two-level logic approach

Our framework allows us to write specifications in HOHH and reason about them using \( \mathcal{G} \). This \textit{two level logic approach} fits well with the goal of verified implementation: specifications can be proved correct using Abella and then executed as programs using Tejus.

Abella supports the two level logic approach by encoding HOHH derivability in a definition and providing a convenient interface to it. The user program and signature for these derivations are obtained from a Prolog program file. The state in a derivation is represented by a judgment of the form \( \Gamma \vdash \mathcal{G} \) where \( \Gamma \) is the list of dynamically added clauses; additions to the signature are treated implicitly via nominal constants. If \( \Gamma \) is empty, the judgment is abbreviated to \( \{ \mathcal{G} \} \). The theorems that are to be proved mix such judgments with other ones defined directly in Abella. For example, a generalized form of uniqueness of typing for the STLC based on its encoding in HOHH can be stated as follows.

\[
\forall L \ M T T, \text{ctx } L \rightarrow (\text{ctx } L \rightarrow \text{ctx } L T T) \rightarrow T = T'.
\]

This formula talks about the typing of \textit{open} terms relative to a dynamic collection of clauses that assign unique types to (potentially) free variables.

The ability to mix specifications in HOHH and definitions in Abella provides considerable expressivity to the reasoning process. This expressivity is further enhanced by the fact that both HOHH and \( G \) support the \texttt{A-tree} syntax approach. We illustrate this observations by considering the explicit treatment of substitutions: we will use these ideas in later sections where we will consider transforming programs with free variables and will use their instantiations by closed substitutions in characterizing correctness. We use the type \text{map} and the constant \text{map} : \text{tm} \rightarrow \text{tm} \rightarrow \text{map} to represent mappings for individual variables (encoded as nominal constants) and a list of such mappings to represent a substitution; for simplicity, we overload the constructors \text{nil} and \( :: \) at this type. The predicate \text{subst} such that \( \text{subst } ML M' \) holds exactly when \( M' \) is the result of applying the substitution \( ML \) to \( M \) can then be defined by the following clauses.

\[
\text{subst } \text{nil } M; \\
\forall x, \text{subst } (\text{map } x V) :: ML (R x) M \equiv \text{subst } ML (R x) M.
\]

Observe how quantifier ordering is used in this definition to create a “hole” where a free variable appears in a term and application is then used to plug the hole with the substitution. This definition makes it extremely easy to prove structural properties of substitutions. For example, the fact that substitution distributes over applications and abstractions can be stated as follows.

\[
\forall M L M' L', \text{subst } ML (\text{app } M L M') \rightarrow E ML M' L', \text{subst } ML (\text{app } M L M') \land \text{subst } ML L M' L'. \\
\forall R T T', \text{subst } ML (\text{abs } T R T') \rightarrow E ML R T T', \text{subst } ML (\text{abs } T R T') \land \forall x, \text{subst } ML (R x) (R' x). \\
\]

Moreover, these properties are proved by an easy induction over the definition of substitution.

As another example, we may want to characterize relationships between closed terms and substitutions. For this, we can first define well-formed terms through the following HOHH clauses.

\[
tm (\text{fun } M N) \equiv \text{tm } M, \text{tm } N. \\
tm (\text{abs } T R) \equiv \text{pl } x \ \text{tm } x \Rightarrow \text{tm } (R x).
\]

Then we can define the context used in \text{tm} derivations.

\[
\text{tm-context } \text{nil}; \\
\forall x, \text{tm-context } (\text{tm } x :: L) \equiv \text{tm-context } L.
\]

Intuitively, if \( \text{tm-context } L \) and \( \{ L \vdash \text{tm } M \} \), then \( M \) is a well-formed term whose free variables are given by \( L \). Clearly, if \( \{ \text{tm } M \} \) holds, then \( M \) is closed. The fact that a closed term is unaffected by a substitution can be stated as below and proved again by an easy induction on the definition of substitutions.

\[
\forall M M', (\{ M \} \rightarrow \text{subst } ML M' M').
\]

Another useful property is that cascaded substitutions can be effected simultaneously if the range of the first substitution is closed. This property can be stated as follows and, again, proved easily.

\[
\forall M M' V, \exists n, \{ M \vdash \text{tm } V \} \rightarrow \text{subst } (\text{map } n V) :: ML (M n) (M' n) \rightarrow \text{subst } (\text{map } n V) :: ML (M n) (M' n).
\]

### 3. Closure Conversion

The closure conversion transformation replaces each function in the overall expression by a \textit{closure} that consists of a function and an environment. The function part of the closure represents a transformation of the original function into a form where the free variables are replaced by projections into a new environment parameter. The environment component, on the other hand, encodes the construction of a value for this parameter in the enclosing context. As a concrete example, when this transformation is applied to the following pseudo OCaml code segment

\[
\text{let } x = 2 \ \text{in } \text{let } y = 3 \ \text{in } (\text{fun } z. \ z + x + y)
\]

it will yield

\[
\text{let } x = 2 \ \text{in } \text{let } y = 3 \ \text{in } (\text{fun } z. \ z + e.1 + e.2, (x,y))
\]

The notation \( \langle F, E \rangle \) used here represents a closure whose function part is \( F \) and environment part is \( E \). Further, the expression \( e.i \) represents the \( i \)-th projection applied to \( e \).
The syntax for our source and target languages is shown in Figure 1. The source language syntax

\[ T ::= N \mid T_1 \to T_2 \mid \text{unit} \mid T_1 \times T_2 \]

\[ M ::= n \mid x \mid \text{pred} M \mid M_1 + M_2 \mid \text{if} M_1 \text{ then } M_2 \text{ else } M_3 \]

\[ \{()\} \mid \{M_1, M_2\} \mid \text{fst} M \mid \text{snd} M \]

\[ \text{let } x = M_1 \text{ in } M_2 \mid \text{fix } f \cdot x.\ M \mid \{M_1, M_2\} \]

\[ V ::= n \mid \text{fix } f \cdot x.\ M \mid \{()\} \mid \{V_1, V_2\} \]

For the target language, they are the following:

\[ M_1 \to M_1' \]

\[ M_2 \to M_2' \]

\[ V_1 \to V_1' \]

\[ M_2 \to M_2' \]

\[ \text{fix } f \cdot x.\ M \to M' \text{fix } f \cdot x.\ M / f / V / x \]

The only ones that merit explicit mention are those for typing the introduction and elimination of closures in the target language:

\[ \Gamma \vdash M : \{T_1 \to T_2\} \times T_1 \times T_2 \Rightarrow T_3 \]

\[ \Gamma \vdash \text{fix } f \cdot x.\ M : T_2 \]

\[ \Gamma \vdash \{M_1, M_2\} : T_1 \to T_2 \]

\[ \Gamma, x_1 : (\{T_1 \to T_2\} 	imes T_1 \times T_2) \Rightarrow T_2, x_2 : l \vdash M_2 : T_1 \]

\[ \Gamma \vdash \text{open} (x_1, x_2) = M_1 \text{ in } M_2 : T \]

In cof-clos, the function part of a closure must be typable in a closed context. In cof-open, \( x_1, x_2 \) must be names that are new to \( \Gamma \). This rule also uses a “type” \( l \) whose meaning must be explained. This symbol represents a new type constant, different from \( N \) and any other type constant used in the typing derivation. This constraint in effect captures the requirement that the environment of a closure should be opaque to its user. Finally, as already observed, abstractions in the target language have types of the form \( T_1 \Rightarrow T_2 \).

The operational semantics for both the source and the target language is based on a left to right, call-by-value evaluation strategy. We assume that this is given in small-step form and, overloading notation again, we write \( M \rightsquigarrow M' \) to denote that \( M \) evaluates to \( M' \) in one step in whichever language is under consideration. The only evaluation rules that may be non-obvious are the ones for applications. For the source language, they are the following:

\[ M_1 \rightsquigarrow M_1' \]

\[ M_2 \rightsquigarrow M_2' \]

\[ V_1 \rightsquigarrow V_1' \]

\[ M_2 \rightsquigarrow M_2' \]

Using the one-step evaluation definitions we can define the \( n \)-step evaluation that we denote by \( M \rightsquigarrow_n M' \). Finally, we denote the evaluation of an expression \( M \) to a value by \( M \rightsquigarrow V \). This relation means that either \( M \) is already (the value) \( V \) or that \( M \rightsquigarrow_n V \) for some \( n > 0 \).

3.2 The closure conversion transformation

In the general case, we must transform terms under mappings for their free variables. For example, for a function (fixed point) expression, this mapping represents the replacement of the free variables by projections from the environment variable for which a new abstraction will be introduced into the term. Accordingly, we specify the transformation as a 3-place relation written as \( \rho \triangleright M \rightsquigarrow M' \), where \( M \) and \( M' \) are source and target language terms and \( \rho \) is a mapping from source language variables to target language terms. We write \( (\rho, x \mapsto M) \) to denote the extension of \( \rho \) with a mapping for \( x \) and \( (x \mapsto M) \in \rho \) to mean that \( \rho \) contains a mapping of \( x \) to \( M \). Figure 3 defines the \( \rho \triangleright M \rightsquigarrow M' \) relation in SOS style; these rules use the auxiliary relation \( \rho \triangleright (x_1, \ldots, x_n) \rightsquigarrow M \), that determines an environment corresponding to a tuple of variables. The cc-let and cc-fix rules have a proviso: the bound variables, \( x \) and \( f, x \) respectively, should have been renamed to avoid clashes with the domain of \( \rho \). Most of the rules have an obvious structure. We comment only on the ones for transforming fixed point expressions and applications. The former translates into a closure. The function part of the closure is obtained by transforming the body of the abstraction, but under a new mapping for its free variables; the expression \( (x_1, \ldots, x_n) \triangleright_{\text{vars}} \text{fix } f \cdot x.\ M \) means that all the free variables of \( \text{fix } f \cdot x.\ M \) appear in the tuple. The environment part of the closure correspondingly contain mappings for the variables in the tuple that are determined by the enclosing context. Note also that the parameter for the function part of the
The transformation we have described relates function expressions because the index decreases. The cumulative notion of equivalence, target terms and equivalence relation \(\approx\) idea of step indexing [2, 3]. Specifically, we define the following semantics preservation. A further complication is that our source use a logical relation style definition of equivalence in proving in the source and target languages. Following [19], we therefore expression. The transformation of a source language application to the function being defined recursively in the source language.

\[
\begin{align*}
\rho \triangleright n & \rightsquigarrow n \text{ cc-nat} \quad \frac{(x \mapsto M) \in \rho}{\rho \triangleright x} \rightsquigarrow \rho(n) \quad \rho \triangleright x \rightsquigarrow M_n \quad \frac{\rho \triangleright x \rightsquigarrow M_n \quad \rho \triangleright x_1 \rightsquigarrow M_1 \quad \ldots \quad \rho \triangleright x_n \rightsquigarrow M_n}{\rho \triangleright \langle x_1, \ldots, x_n \rangle \rightsquigarrow \langle M_1, \ldots, M_n \rangle} \quad \frac{\rho \triangleright M \rightsquigarrow M'}{\rho \triangleright \text{pred } M \rightsquigarrow \text{pred } M'} \quad \text{cc-pred}
\end{align*}
\]

\[
\begin{align*}
\rho \triangleright M_1 \rightsquigarrow M'_1 \quad \rho \triangleright M_2 \rightsquigarrow M'_2 \quad \rho \triangleright M_1 + M_2 \rightsquigarrow M'_1 + M'_2 \quad \text{cc-plus}
\end{align*}
\]

\[
\begin{align*}
\rho \triangleright M \rightsquigarrow M' \quad \frac{\text{if } \text{ then } M_1 \text{ else } M_2 \rightsquigarrow M'_1 \text{ else } M'_2}{\rho \triangleright \text{if } M \text{ then } M_1 \text{ else } M_2 \rightsquigarrow M'_1 \text{ else } M'_2} \quad \text{cc-ifz}
\end{align*}
\]

\[
\begin{align*}
\rho \triangleright f x. M \rightsquigarrow \langle \lambda p. \text{let } g = \pi_1(p) \text{ in let } y = \pi_2(p) \text{ in let } x_1 = \pi_3(p) \text{ in } M' \rangle \quad \text{cc-fix}
\end{align*}
\]

\[
\begin{align*}
\rho \triangleright \text{fix } f x. M \rightsquigarrow M' \quad \text{cc-fix}
\end{align*}
\]

where \(\rho' = (x \mapsto y, f \mapsto g, x_1 \mapsto \pi_3(x_1), \ldots, x_n \mapsto \pi_n(x_n))\) and \(p, g, y, x\) are fresh variables

\[
\begin{align*}
\rho \triangleright \text{let } x = M_1 \text{ in } M_2 \rightsquigarrow \text{let } y = M'_1 \text{ in } M'_2 \quad \text{cc-let}, y \text{ is fresh}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
(x_1, \ldots, x_n) \ni \text{vars}(f x. M) \quad \rho \triangleright (x_1, \ldots, x_n) \rightsquigarrow \rho'_n \quad \rho' \triangleright M \rightsquigarrow M'
\end{array}
\end{align*}
\]

Figure 3. Closure Conversion Rules

3.3 Correctness of the transformation

The transformation we have described relates function expressions in the source and target languages. Following [19], we therefore use a logical relation style definition of equivalence in proving semantics preservation. A further complication is that our source language includes recursion. To handle this situation, we use the idea of step indexing [2,3]. Specifically, we define the following mutually recursive simulation relation \(\sim\) between closed source and target terms and equivalence relation \(\sim\) between closed source and target values, each indexed by a type and a step measure.

\[
M \sim_{T, k} M' \iff \forall j \leq k. \forall V. M \rightsquigarrow V \vdash V' \sim_{T, k-j} V';
\]

\[
n \ni (\text{fix } f x. M) \rightsquigarrow (\text{let } g = \pi_1(p) \text{ in let } y = \pi_2(p) \text{ in let } x_1 = \pi_3(p) \text{ in } M' \rangle \quad \text{cc-fix}
\]

where \(\rho' = (x \mapsto y, f \mapsto g, x_1 \mapsto \pi_3(x_1), \ldots, x_n \mapsto \pi_n(x_n))\) and \(p, g, y, x\) are fresh variables.

By analyzing the simulation relation and using the evaluation rules, we can prove the following "composition" lemma for various constructs in the source language.

**Lemma 1.** 1. If \(M \sim_{n, k} M'\) then \(\text{pred } M \sim_{n, k} \text{ pred } M'\).
2. If \(M \sim_{T, k} M'\) then \(\text{fst } M \sim_{T, k} \text{ fst } M'\) and \(\text{snd } M \sim_{T, k} \text{ snd } M'\).
3. If \(M \sim_{T, k} M'\) and \(N \sim_{T, k} N'\) then \((M, N) \sim_{T, T, k} (M', N')\).
4. If \(M \sim_{N, k} M'\), \(M_1 \sim_{T, k} M'_1\) and \(M_2 \sim_{T, k} M'_2\) then \(M_1 + M_2 \sim_{T, k} M'_1 + M'_2\).
5. If \(M \sim_{T, k} M'\) and \(M_1 \sim_{T, k} M'_1\) then \(M_2 \sim_{T, k} M'_2\) then \(M_1 \times M_2 \sim_{T, k} \text{open } (x, x) \equiv M'_1 \times M'_2\).

The proof of the last of these properties requires us to consider the evaluation of a fixed point expression which involves "feeding" the expression to its own body. In working out the details, we make use of the easily observed property that the simulation and equivalence relations are closed under decreasing indices.

Our notion of equivalence only relates closed terms. However, our transformation typically operates on open terms, albeit under mappings for the free variables. To handle this situation, we consider semantics preservation for possibly open terms under closed substitutions. We will take substitutions in both the source and target settings to be simultaneous mappings of closed values for a finite collection of variables, written as \((V_1/x_1, \ldots, V_n/x_n)\). In defining a correspondence between source and target language substitutions, we need to consider the possibility that a collection of free variables in the first may be reified into an environment variable in the second. This motivates the following definition in which \(\gamma\) represents a source language substitution:

\[
\gamma \approx_{e_m, e_m, \ldots, e_m, e_m} (V_1, \ldots, V_n) \iff \forall 1 \leq i \leq n. \gamma(x_i) \approx_{T, k} V'_i.
\]

Writing \(\gamma_1 \circ \gamma_2\) for the concatenation of two substitutions viewed as lists, equivalence between substitutions is then defined as follows:

\[
(V_1/x_1, \ldots, V_n/x_n) \equiv \gamma \approx_{e_m, e_m, \ldots, e_m, e_m} (V'_1/y_1, \ldots, V'_n/y_n, V'_n/x_n) \iff (\forall 1 \leq i \leq n. V'_i) \equiv (\gamma \approx_{T, k} V_i) \land \gamma \approx_{T, k} V_e.
\]

Note that both relations are indexed by a source language typing context and a step measure.

We write the application of a substitution \(\gamma\) to a term \(M\) as \(M[\gamma]\).

The first part of the following lemma, proved by an easy use of the definitions of \(\approx\) and evaluation, provides the basis for justifying the treatment of free variables via their transformation into projections over environment variables introduced at function boundaries in the closure conversion transformation. The second part of the lemma is a simple corollary of the first part, which relates a source substitution and an environment computed during the closure conversion of fixed points.

**Lemma 2.** Let \(\delta = (V_1/x_1, \ldots, V_n/x_n); \gamma = (V'_1/y_1, \ldots, V'_n/y_n, V'_n/x_n)\) be source and target language substitutions and let \(\Gamma = \langle x_1 : T_m, \ldots, x_1 : T_1 \rangle : T_1, x_0 : T_n, \ldots, x_1 : T_1 \rangle\) be a source language typing context such that \(\delta \equiv_{T, k} \delta\). Further, let \(\rho = (x_1 \mapsto y_1, \ldots, x_n \mapsto y_n, x_1 \mapsto \pi_1(x_1), \ldots, x_n \mapsto \pi_n(x_n))\).

1. If \(x : T \in \Gamma\) then there exists a value \(V'\) such that \(\rho(x))[\delta] \mapsto V'\) and \(\delta(x) \equiv_{T, k} \delta'\).
2. If \(\Gamma' = (x_1 : T_1, \ldots, x_1 : T_1)\) for \(\Gamma' \subseteq \Gamma\) and \(\rho \triangleright (x_1 : T_1, \ldots, x_1 : T_1)\) then there exists \(V'\) such that \(M \mapsto V'\) and \(\delta \equiv_{T, k} \delta'\).
We first consider the encoding of types. We will use the source and target language substitutions and let $T = (x_1 : T_1, \ldots, x_n : T_n \mid T_0 : T_1 \vdash M : T)$ be a source language typing context such that $\sigma \vdash_M \delta$. Further, let $\rho = \{(x_1 \mapsto y_1, \ldots, x_n \mapsto y_n) : \pi_1(x_1), \ldots, \pi_n(x_n)\}$. The correctness theorem can now be stated as follows.

**Theorem 1.** Let $\delta = (V_1 / x_1, \ldots, V_n / x_n)$; $\gamma$ and $\delta' = (V'_1 / y_1, \ldots, V'_n / y_n, V' / x)$ be source and target language substitutions and let $\Gamma = (x'_1 : T'_1, \ldots, x'_n : T'_n, x'_0 : T_0, \ldots, x_1 : T_1) \vdash M : T$. We outline the main steps in the argument for this theorem: these will encode, respectively, the natural number, unit and pair types. The case for a number is obvious and for a variable we use Lemma 2.1. In the remaining cases, other than when $M$ is of the form $let x = M_1 in M_2$ or $fix M_1$, the argument follows a set pattern: we observe that substitutions distribute to the sub-components of expressions, we invoke the induction hypothesis over the sub-components and then we use Lemma 1 to conclude. If $M$ is of the form $let x = M_1 in M_2$, then $M_2$ must be of the form $let y = M'_1 in M'_2$. Here again the substitutions distribute to $M_1$ and $M_2$ and to $M'_1$ and $M'_2$, respectively. Then apply the induction hypothesis first to $M_1$ and $M'_1$ and then to $M_2$ and $M'_2$; in the latter case, we need to consider extended substitutions but these obviously remain equivalent. Finally, if $M$ is of the form $fix f \cdot M_1$, then $M'$ must have the form $(M'_1, M'_2)$. We can prove that the abstraction $M_1$ is a closed and therefore that $M'[\sigma'] = (M'_1, M'_2[\sigma'])$. We then apply the induction hypothesis. In order to do so, we generate the appropriate typing judgment using Lemma 3 and a new pair of equivalent substitutions (under a suitable step index) using Lemma 2.

### 4. Implementing Closure Conversion

We now show how closure conversion can be implemented concisely and perspicuously in $\lambda$Prolog. The general observation is that the logic underlying the language provides a means for encoding the $\texttt{SOS}$ style description of this transformation. A complicating factor is that the transformation involves the analysis of binding structure. More explicitly, the rule for transforming fixed points requires calculating the free variables of expressions and this rule and the $\texttt{cc-let}$ rule require newly introduced names to be fresh. However, these aspects can be treated elegantly by exploiting the support that $\lambda$Prolog provides for the $\lambda$-tree syntax approach. Moreover, our encoding will have a logical character that will be exploited later when we want to reason about it.

#### 4.1 Encoding the source and target languages

We first consider the encoding of types. We will use the $\lambda$Prolog type $\texttt{ty}$ for both languages. The constructors $\texttt{tnat}$, $\texttt{tunit}$ and $\texttt{tprod}$ will encode, respectively, the natural number, unit and pair types. There are two arrow types to be treated. We will represent $\rightarrow$ by $\texttt{arr}$ and $\Rightarrow$ by $\texttt{arr}'$. These decisions are summarized in the following $\lambda$Prolog signature.

| tnat, tunit | : ty |
| arr, tprod, arr' | : ty $\rightarrow$ ty $\rightarrow$ ty |

We will use the $\lambda$Prolog type $\texttt{tm}$ for encodings of source language terms. The particular constructors of this type that we need are the following.

| nat | : nat $\rightarrow$ tm |
| unit | : tm |

The only constructors that need further explanation here are $\texttt{let}$ and $\texttt{fix}$. These encode binding constructs in the source language and, as expected, we use $\lambda$Prolog abstraction to capture their binding structure. Thus, let $x = f \cdot x$ is encoded as $(\texttt{let} \ (\texttt{nat} \ n) \ (x\ x))$. Similarly, we encode $(\texttt{fix} \ f \ x \ x)$ as $(\texttt{fix} \ (f\ x\ \texttt{\&}) \ x)$.

To encode target language terms, we will use the $\lambda$Prolog type $\texttt{tm}'$. To represent the constructs the target language shares with the source language, we will use “primed” versions of the $\lambda$Prolog constants seen earlier, e.g. $\texttt{unit}'$ of type $\texttt{tm}'$ will represent the null tuple. Of course, there will be no constructor corresponding to $\texttt{fix}$. We will also need the following additional constructors.

| abs' | : $(\texttt{tm}' \rightarrow \texttt{tm}') \rightarrow \texttt{tm}' |
| clos' | : $(\texttt{tm}' \rightarrow \texttt{tm}' \rightarrow \texttt{tm}') \rightarrow \texttt{tm}' |
| open' | : $(\texttt{tm}' \rightarrow \texttt{tm}' \rightarrow \texttt{tm}' \rightarrow \texttt{tm}') \rightarrow \texttt{tm}' |

The last two encode closures and their application and $\texttt{abs}'$ encodes $\lambda$-abstraction. Note again the use of a $\lambda$-tree syntax representation for binding constructs.

Definitions of typing and evaluation are needed in the correctness argument. We can encode them in either the specification or the reasoning logic. We choose to do the former so that we can use meta-theorems about specification logic derivability to simplify the reasoning process.

Following Section 2 we represent typing judgments as relations between terms and types, treating contexts implicitly via dynamically added clauses that assign types to free variables. We use the predicates of and of' to encode typing in the source and target language respectively. The clauses defining these predicates are routine and we show only a few pertaining to the binding constructs. The rule for typing fixed points in the source language translates into the following.

$$\texttt{of} \ (\texttt{fix} \ R) \ (\texttt{arr} \ T \ T_2) :: \texttt{pl f} \ \texttt{\& pl x} \ \texttt{pl e} \ \texttt{pl l} \ \texttt{of f} \ (\texttt{arr} \ T \ T_2) \Rightarrow \texttt{of f} \ (\texttt{arr} \ T \ T_2) \Rightarrow \texttt{of f} \ (R \ f \ x) \ T_2.$$  

Observe how the freshness constraint that goes with the rule is realized in this clause; the universal quantifiers over $f$ and $x$ introduce new names and the application $(R \ f \ x)$ replaces the bound variables with these names to generate the new typing judgment that must be derived. For the target language, the main interesting rule is for typing the application of closures. The following clause encodes this rule.

$$\texttt{of} \ (\texttt{open'} \ M \ R) \ T :: \texttt{of} \ (\texttt{arr} \ T \ T_2) \Rightarrow \texttt{of f} \ (\texttt{prod} \ (\texttt{arr} \ T \ T_2) \ (\texttt{prod} \ T_1 \ T_2)) \ T_2 \Rightarrow \texttt{of f} \ (\texttt{prod} \ (\texttt{prod} \ T_1 \ T_2)) \ T_2 \Rightarrow \texttt{of f} \ (R \ f \ e) \ T.$$  

Here again we use universal quantifiers in goals to encode the freshness constraint. Note also how the universal quantifier over the variable $1$ captures the opaqueness quality of the type of the environment of the closure involved in the construct.

We encode the one step evaluation rules for the source and target languages using the predicates $\texttt{step}$ and $\texttt{step}'$. We again consider only a few interesting cases in their definition. Assuming that $\texttt{val}$ and $\texttt{val}'$ recognize values in the source and target languages, the clauses for evaluating the application of a fixed point and a closure are the following.

$$\texttt{step} \ (\texttt{app} \ (\texttt{fix} \ R) \ V) \ (R \ (\texttt{fix} \ R) \ V) :: \texttt{val V} \ \texttt{step'} \ (\texttt{open'} \ (\texttt{clos'} \ F \ E) \ R) \ (R \ E) :: \texttt{val'} \ (\texttt{clos'} \ F \ E).$$  

Note here how application in the meta-language realizes substitution.
We will use the type maps and will be obviated when the reasoning logic is extended to permit maps. The cc-fix combine nstep.' The predicates nstep' and eval' play a similar role for the target language and their definitions are similar.

4.2 Closure conversion

Closure conversion is defined relative to a mapping for source variables. We use the type map for mappings for individual variables. The map here is from source terms to target terms. Thus it is different from the map in Section 2. We overload this notation to simplify the representation. Later in Section 5.2 we will overload map to represent maps from target terms to target terms. The constant map : tm -> tm' -> map encodes the actual mapping.

We will use the type map_list for lists of such mappings, the constructors nil and for constructing such lists and the predicate member for checking membership in them. We also need to represent lists of source language variables. We will use the type tm_list for these and for simplicity of discussion, we will overload the list constructors and predicates at this type; such overloading will be obviated when the reasoning logic is extended to permit polymorphic typing.

The crux in formalizing the definition of closure conversion is capturing the content of the cc-fix rule. A key part of this rule is identifying a list of variables that contain all the free variables in a given source term. We actually define a predicate fvars identifying a list of variables that contain all the free variables of a given source term. We define fvars in Section 5.2 and in Section 5.1.

4.2.1 Auxiliary predicates needed in the formalization

The cc-fix rule requires us to construct an environment representing the mappings for the variables found by fvars. The predicate mapenv:tm_list -> map_list -> tm -> o specified by the following clauses provides this functionality.

mapenv nil -> nil.
mapenv (X : L) Map (pair' M ML) -> member (map X M) Map, mapenv L Map ML.

The cc-fix rule also requires us to create a new mapping from the variable list to projections from an environment variable. Representing the list of projection mappings as a function from the environment variable, this relation is given by the following clauses.

mapvar (X :: L') (e\ (map X (f\ e))) :: (Map (snd' e)) -> mapvar L Map.

We can now specify the closure conversion transformation. In fact, we provide clauses below that define the predicate cc such that (cc Map Vs M M') holds if M is a transformed version of M under the mapping Map for the variables in Vs; we assume that Vs contains all the free variables of M.

cc _ _ (nat N) (nat' N).
cc Map Vs X M :-member (map X M) Map.
cc Map Vs (pred M) (pred' M') :- cc Map Vs M M'.
cc Map Vs (plus M1 M2) (plus' M1' M2') :- cc Map Vs M1 M1', cc Map Vs M2 M2'.
cc Map Vs (ifz M1 M2) (ifz' M1' M2') :- cc Map Vs M1 M1', cc Map Vs M2 M2'.
cc Map Vs unit unit'.
cc Map Vs (pair M1 M2) (pair' M1' M2') :- cc Map Vs M1 M1', cc Map Vs M2 M2'.
cc Map Vs (nat M) (nat' M') :- cc Map Vs M M'.
cc Map Vs (and M) (and' M') :- cc Map Vs M M'.
cc Map Vs (let M R) (let' M' R') :- cc Map Vs M M', pi x, pi y.
cc ((map x y) :: Map) (x :: Vs) (R x) (R' y).
cc Map Vs (fix R) (clos' (abs' (p\ (let' (fst' p) (g\ (let' (fst' (snd' p)) (y\ (let' (snd' (snd' p)) (e\ (R' g y))))))))) E) :-
  fvars (fix R) Vs Fs, mapenv Fs Map E, mapvar Fs NMap, pi x, pi y, pi e,
  cc ((map x y) :: (map f g)) :: (NMap e) (x :: f :: Fs) (R x) (R' g y e).
cc Map Vs (app' M1 M2)
  (open' M1' (f\e\ (app' f (pair' M1' (pair' M2' e))))):-
  cc Map Vs M1 M1', cc Map Vs M2 M2'.

These clauses evidently correspond very closely to the rules in Figure 3. Note especially the clause for transforming an expression of the form (fix R) that encodes the content of the cc-fix rule. In the body of this clause, fvars is used to identify the free variables of the expression, and mapenv and mapvar are used to create the reified environment and the new mapping. In both this clause and in the one for transforming a let expression, the λ-tree representation, universal goals and (meta-language) applications are used to encode freshness and renaming requirements related to bound variables in a concise and logically precise way. This specification can be executed using a system such as Teyjus and thus it also serves directly as an implementation of the transformation.

5. Verification of Closure Conversion

We now outline a correctness proof for our implementation of closure conversion. This proof closely follows the informal one discussed in Section 3.
and \( \text{tm}_\text{ctx}' \) formalize the contexts used in these derivations. A source (target) term \( M \) is closed if \( \text{tm} M \) \((\text{tm}' M')\) is derivable. The predicates \text{val} and \text{val}' identify source and target values and \( \text{is}_\text{sty} \) recognizes source types. Finally, \( \text{vars}_\text{of}_\text{ctx} \) is a predicate such that \( (\text{vars}_\text{of}_\text{ctx} \ L \ V s) \) holds if \( L \) is a source language typing context and \( V s \) is the list of variables it pertains to and \( \text{var}_\text{of}_\text{tm}_\text{ctx} \) encodes a similar relation between a well-formedness context and a list of variables.

Step indexing uses ordering on natural numbers. We represent natural numbers using \( z \) for 0 and \( s \) for the successor constructor. The predicate \( \text{lt} \) and \( \text{le} \) represent the “less than” and the “less than or equal to” relations; their definitions are routine and we do not show them here.

5.2 The simulation and equivalence relations

The following clauses define the simulation and equivalence relations.

\[
\begin{align*}
\text{sim} \ T & \ K \ M \ M' \triangleq \forall V, \text{eval} V \to (\text{val} V) \to \exists V', \text{equiv} T \ N \ V \ V'; \\
\text{equiv} & \text{prod} T1 \ T2 \ K (\text{pair} V1 \ V2) \triangleq \text{equiv} T1 \ K \ V1' \ W \equiv \text{equiv} T2 \ K \ V2 \ W' \equiv (\text{tm \ V1} \ W \equiv (\text{tm \ V2} \ W'); \\
\text{equiv} & \text{arr} T1 \ T2 \ x \ (\text{fix} R) \ (\text{clo} \ (\text{abs \ R}) \ VE) \triangleq \{\text{val \ V} \} \land (\text{tm \ (fix \ R)}) \land \{\text{tm'} \ (\text{clo} \ (\text{abs'} \ R) \ VE)); \\
\text{equiv} & \text{arr} \text{prod} T1 \ T2 \ (\text{K} \ (\text{fix} R) \ (\text{clo} \ (\text{abs} \ R') \ VE) \land \forall V1 \ V2 \ V2', \text{equiv} T1 \ K \ V1 \ V1' \to \text{equiv} (\text{arr} T1 \ T2) \ K \ V2 \ V2' \to \text{sim} \ T2 \ K \ (\text{R} \ \text{V2} \ \text{V1}) \ (\text{R'} \ (\text{pair} \ V2' \ (\text{pair} \ V1' \ VE))));
\end{align*}
\]

The proposition \( (\text{sim} \ T \ K \ M \ M') \) is intended to mean that \( M \) simulates \( M' \) at type \( T \) in \( K \) steps; \( (\text{equiv} \ T \ K \ V \ V') \) has a similar interpretation. Note the exploitation of \( \lambda \)-tree syntax, specifically the use of application, to realize substitution in definition of \text{equiv}. It is easy to show that \text{sim} holds only between closed source and target terms and similarly \text{equiv} holds between closed source and target values.

The definition of the \text{equiv} relation uses itself negatively in the last clause. If we were to view this as a fixed-point definition, this violates the stratification condition. However, we use it only as a recursive definition, i.e., as a definition based on which we can do unfolding or rewriting but not case analysis. Weaker conditions suffice for such definitions and our clauses satisfy them: this is, in fact, the reason why we “build” the relation up over the natural numbers rather than mirroring directly the structure of the informal definition.

Composition lemmas in the style of Lemma\ref{lem:composition} are easily stated for \text{sim}. For example, the one for pairs is the following.

\[
\forall T1 \ T2 \ K \ M1 \ M2 \ M1' \ M2', \{\text{is}_\text{nat} \ K\} \to \{\text{is}_\text{sty} \ T1\} \to \{\text{is}_\text{sty} \ T2\} \to \text{sim} \ T1 \ K \ M1 \ M1' \to \text{sim} \ T2 \ K \ M2 \ M2' \to \text{sim} \ (\text{prod} T1 \ T2) \ K (\text{pair} M1 \ M2) (\text{pair} \ M1' \ M2').
\]

These lemmas have straightforward proofs.

5.3 Representing substitutions

We treat substitutions as discussed in Section\ref{sec:substitutions}. For example, source substitutions satisfy the following definition.

\[
\text{subst} \ nil; \\
\text{subst} \ (\text{map} \ X \ V) :: \text{ML} \triangleq \text{subst} \ (\text{ML} \land \text{name} \ X \land \{\text{val} \ V\} \land \{\text{tm} \ V\} \land \forall V', \text{member} \ (\text{map} \ X \ V') \to \forall V, \text{ML} \land V \to V' \equiv V.
\]

By definition, these substitutions map variables to closed values. For technical reasons we allow a mapping for a variable to be indicated more than once, but we require all of them to be to the same value.

The application of a source substitution is also defined as discussed in Section\ref{sec:application}.

\[
\begin{align*}
\text{app}_\text{subst} & \text{nil} \ M; \\
\forall x, \text{app}_\text{subst} \ (\text{map} \ x \ V) :: (\text{ML} \ x) \to (R x) \ M \equiv \forall x, \text{app}_\text{subst} \ (\text{ML} \ x) \to (R x) \ M.
\end{align*}
\]

As before, we can easily prove useful properties about substitution application based on this definition: that such application distributes over term structure, that closed terms are not affected by substitution and that the cascaded application of closed substitutions can be replaced by their simultaneous application.

The predicates \text{subst}' and \text{app}_\text{subst}' encode target substitutions and their application. The formalization of these notions is similar to that of source substitutions.

5.4 The equivalence relation on substitutions

We first define the relation \text{subst}_\text{env}_\text{equiv} between source substitutions and target environments:

\[
\begin{align*}
\text{subst}_\text{env}_\text{equiv} & \text{nil} \ K \ ML \triangleq \text{unit}; \\
\text{subst}_\text{env}_\text{equiv} \ (\text{of} \ X \ T :: L) \ K \ ML \ (\text{pair} V' \ VE) \triangleq \exists V, \text{subst}_\text{env}_\text{equiv} \ K \ ML \ VE \land \text{member} \ (\text{map} X \ V) \ ML \land \text{equiv} \ T \ K \ V \ V'.
\end{align*}
\]

Using \text{subst}_\text{env}_\text{equiv}, the needed relation between source and target substitutions is defined as follows.

\[
\begin{align*}
\forall \ve, \text{subst}_\text{env}_\text{equiv} \ K \ ML \ (\text{map} \ ve :: \text{ML}) & \triangleq \forall \ve, \text{subst}_\text{env}_\text{equiv} \ K \ ML \ VE; \\
\forall x, \text{subst}_\text{equiv} \ (\text{of} \ x :: L) \ K & \triangleq \exists V', \forall x, \text{subst}_\text{equiv} \ (\text{of} \ x :: L) \ K \ (\text{map} \ x \ V') :: \text{ML} \land \text{equiv} \ T \ K \ V' \land \text{subst}_\text{env}_\text{equiv} \ K \ ML \ ML'.
\end{align*}
\]

5.5 Lemmas about \text{fvars}, \text{mapvar} and \text{mapenv}

Lemma\ref{lem:fvars} translates into a lemma about \text{fvars} in the implementation. To state it, we define a strengthening relation between source typing contexts:

\[
\begin{align*}
\text{prune}_\text{ctx} & \text{nil} \ L \equiv \text{nil}; \\
\text{prune}_\text{ctx} \ (X :: \text{Vs}) \ L & \equiv \text{member} \ (\text{of} \ X \ T) \ L \land \text{prune}_\text{ctx} \ Vs \ L'.
\end{align*}
\]

\(\text{prune}_\text{ctx} \ Vs \ L'\) holds if \( L' \) is a typing context that “strengthens” \( L \) to contain assignments only for the variables in \( Vs \). The lemma about \text{fvars} is then the following.

\[
\forall \ve \text{Vs} \text{ M T FVs}, \text{ctx} \ L \to \text{vars}_\text{of}_\text{ctx} \ Vs \ L' \to \{L \land \text{of} \ M \ T\} \to \{\text{fvars} \ M \ Vs \ FVs\} \to \exists L', \text{prune}_\text{ctx} \ FVs \ L' \land \{L' \land \text{of} \ M \ T\}.
\]

To prove this theorem, we generalize it so that the HOHOF derivation of \( \langle \text{fvars} \ M \ Vs \ FVs \rangle \) relativized to a context that marks some variables as not free. The resulting generalization is proved by induction on the \text{fvars} derivation.

A formalization of Lemma\ref{lem:fvars} is also needed for the main theorem.

We start with a lemma about \text{mapvar}.

\[
\forall \ve \text{Vs} \text{Map M L K VE} \to \text{ctx} \ L \to \text{subst}_\text{ML} \to \text{subst}_\text{env}_\text{equiv} \ L \ K \ ML \to \text{vars}_\text{of}_\text{ctx} \ Vs \ L' \to \{\text{mapvar} \ Vs \ Map \} \to \text{member} \ (\text{of} \ X \ T) \ L \to \text{app}_\text{subst} \ L \ X \ V \to \{\text{member} \ (\text{map} \ X \ M') \} \ (\text{Map} \ e) \to \exists V', \forall x, \text{eval} \ (\text{M'} \ VE) \ V' \land \text{equiv} \ T \ K \ V' V'.
\]

In words, this lemma states the following. If \( L \) is a source typing context for the variables \( x_1, \ldots, x_n \), \( ML \) is a source substitution and
VE is an environment equivalent to ML at L, then mapvar determines a mapping for \((x_1, \ldots, x_n)\) that are projections over an environment with the following character: if the environment is to be VE, then, for \(1 \leq i \leq n\), \(x_i\) is mapped to a projection that must evaluate to a value equivalent to the substitution for \(x_i\) in ML. The lemma is proved by induction on the derivation of \(\{\text{mapvar} Vs Map\}\).

Lemma 2 is now formalized as follows.

\[\forall L ML KL ML' K Vs Vs' Map, \{\text{is_nat K}\} \rightarrow \text{ctx L} \rightarrow \text{subt} ML \rightarrow \text{subt} ML' \rightarrow \text{subt equiv} L K ML ML' \rightarrow \text{vars_of_ctx} L Vs \rightarrow \text{mapvar} Vs Map \rightarrow \text{to_mapping} Vs Map \rightarrow (Y X T V M' M'''), \]

\[\begin{align*}
&\text{member (of X T) L} \rightarrow \{\text{member (map X M') Map}\} \\
&\text{app_subt} ML X V \rightarrow \text{app_subt'} ML' M' M'' \rightarrow \exists V', \{\text{eval' V' V}\} \land \text{equiv T K V' V'} \land \\
&\{L', \text{NFVs E E'}\}, \text{prune_ctx} \text{NFVs L L'} \rightarrow \{\text{mapenv NFVs Map E}\} \rightarrow \text{app_subt'} ML' E E' \rightarrow \exists \text{VE'}, \{\text{eval' E' VE'}\} \land \text{subt equiv L' K ML VE'},
\end{align*}\]

Two new predicates are used in this statement. The judgment \(\text{vars_of_subst'} ML Vs'\) “collects” the variables in the target substitution ML' into Vs' and, given source variables Vs = \((x_1, \ldots, x_m)\) and target variables Vs' = \((y_1, \ldots, y_n)\), the predicate \(\pi\) maps \(x_i\) to \(y_i\), \(y_{i+1}\) to \(\pi_2(x_i)\), \(\ldots\), \(y_n\) to \(\pi_n(x_n)\). The conclusion of the lemma is a conjunction representing the two parts of Lemma 2. The first part can be proved by induction on \(\{\text{member (map X M') Map}\}\), using the lemma for mapvar when X is some \(x_i(1 \leq i \leq m)\). The second part is proved by induction on \(\{\text{mapenv NFVs Map E}\}\) using the first part.

### 5.6 The main theorem

The semantics preservation theorem is formalized as follows:

\[\forall L ML KL ML' K Vs Vs' Map T P P' M', \{\text{is_nat K}\} \rightarrow \text{ctx L} \rightarrow \text{subt} ML \rightarrow \text{subt} ML' \rightarrow \text{subt equiv} L K ML ML' \rightarrow \text{vars_of_ctx} L Vs \rightarrow \text{vars_of_subst'} ML' Vs' \rightarrow \text{to_mapping} Vs Map \rightarrow \{L \rightarrow \text{cc Map Vs M'}\} \rightarrow \{\text{app_subt} ML M P \rightarrow \text{app_subt'} ML' M' P' \rightarrow \text{sim T K P P'}\}.
\]

We use an induction on \(\{\text{cc Map Vs M'}\}\), the closure conversion derivation, to prove this theorem. As should be evident from the preceding development, the proof in fact closely follows the structure we outlined in Section 3.

### 6. Other Transformations

We have undertaken the verified implementation of other functional program transformations using our framework. One of these is the continuation-passing-style (CPS) transformation, and that is usually applied before closure conversion. A one-pass CPS transformation that identifies and eliminates on-the-fly the so-called “administrative redexes” is described in [11]. This version can be encoded concisely in Prolog by using meta-level redexes for administrative redexes. In fact, the following clauses implement it for our source language in the style of [13].

\[\begin{align*}
cps (\text{nat} M) K (K (\text{nat} N)) \\
cps (\text{pred} M) K M' \rightarrow \text{cps} M (\lambda x. \text{let} (\text{pred} x) (\lambda k v) M') \\
cps (\text{plus} M N) M' \rightarrow \{\text{pi} x\} \text{cps} M (\lambda x. \text{let} (\text{plus} x N) (\lambda v k) M') \\
cps (\text{ifz} M N K) K' \rightarrow \{\text{pi} x\} \text{cps} M (\lambda x. \text{ifz} x M N) K' \\
cps (\text{unit} K) (K (\text{unit})) \\
cps (\text{pair} M N) K M' \rightarrow \{\text{pi} x\} \text{cps} M (\lambda x. \text{let} (\text{pair} x N) (\lambda v k) M') \\
cps (\text{fix} K) (K (\text{fix} K))
\end{align*}\]

Given these clauses, \(\text{cps} M (\lambda x. M')\) is derivable if \(M'\) is a CPS form of \(M\). This transformation can be verified in a manner similar to closure conversion.

Another transformation whose verified implementation we have undertaken is code-hoisting. The CPS transformation, closure conversion and code hoisting are the first three phases in compiling functional programs. They translate higher-order functional programs into a form to which conventional compilation techniques are applicable.

### 7. Related Work and Conclusion

Work on compiler verification has a long history. Impressive strides have been taken recently in automating such verification as exemplified by the CompCert project [16]. Much of this work has been restricted to compiling imperative languages. Our focus in this paper has been on functional programs and, in particular, on the translation phases that actually transform binding structure so as to make the program amenable to more conventional compilation techniques.

Compiler verification for functional languages specifically has been considered by other researchers. [6] describes the verified implementation of a compiler for the STLC. Similarly, [12] describes a verified translator from a subset of ML into the intermediate language used by CompCert. Both these efforts use the Coq theorem prover [24] that is based in the Calculus of Inductive Constructions (CIC) [10]. The λ-tree syntax approach is difficult to support in this setting. The function spaces underlying the λ-terms in CIC are rich and the analysis of abstraction in these terms can therefore not be limited to examining just their syntactic structure, as would be needed if we want to use it to capture binding in object language syntax. Not surprisingly, therefore, both mentioned works use a first-order approach based on De Bruijn’s nameless scheme for representing bound variables [13]. [7] introduces parametric HOAS (PHOAS) to provide a partial support of higher-order abstract syntax. However, some of the transformations, such as closure conversion, seem eventually to use a first-order approach even under PHOAS. In the Coq formalization, the logical relation used in the proof of closure conversion must be encoded as a function because of its negative occurrence in the definition. This raises another difficulty: functions in CIC must be terminating but evaluation in the source language may not terminate. To overcome this problem, [8] uses a hybrid of an SOS-style semantics and an abstract machine in the formalization and reasoning process.

There has been other work devoted to implementing closure conversion for the STLC in a way that guarantees type preservation, a weaker property than the one we have considered in this paper. [13] implements closure conversion in Haskell using a De Bruijn representation in such a way that type preservation follows directly from type checking in Haskell. Another interesting implementation has been carried out in Beluga [22], a functional programming language that is based on contextual modal type theory and that supports higher-order abstract syntax [21]. Rich properties of programs can be embedded in types in Beluga and [5] shows how this feature can
be exploited to ensure type preservation. It remains to be seen if types can also be used to encode semantics preservation.

Our work to-date has focused on transformations in the STLC that manipulate binding structure. We plan to extend this work to yield a complete compiler. A more ambitious goal is to utilize these techniques in the verified compilation of richer functional languages that involve polymorphism.

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