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Weak Sequential Convergence in $L_p(\mu, X)$

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We provide some new results on the weak convergence of sequences or nets lying in $L_p((T, \Sigma, \mu), X) \equiv L_p(\mu, X)$, $1 \leqslant p < \infty$, i.e., the space of equivalence classes of X-valued (X is a Banach space) Bochner integrable functions on the finite measure space (T, Σ, μ). Our theorems generalize in several directions recent resuls on weak sequential convergence in $L_1(\mu, X)$ obtained by M. A. Khan and M. Majumdar [*J. Math. Anal. Appl.* **114** (1986), 569–573] and Z. Artstein [*J. Math. Econ.* **6** (1979), 277–282], and they can be used to obtain dominated convergence results for the Aumann integral. Our results have useful applications in Economics and Game Theory.

1. INTRODUCTION

The purpose of this paper is to prove some new results on the weak convergence of sequences or nets lying in $L_p(\mu, X)$, $1 \leqslant p < \infty$, i.e., the space of equivalence classes of X-valued (X is a Banach space) Bochner integrable functions $x: T \to X$ on a finite measure space (T, Σ, μ). In particular, the main theorem of the paper asserts that:

If X is a separable Banach space, (T, Σ, μ) is a finite positive measure space, and $\{f_\lambda : \lambda \in \Lambda\}$ is a net in $L_p(\mu, X)$, $1 \leqslant p < \infty$, such that f_λ converges weakly to $f \in L_p(\mu, X)$, and for all $\lambda \in \Lambda$, $f_\lambda(t) \in F(t)$ μ-a.e., where $F: T \to 2^X$ is a weakly compact, integrably bounded, convex, nonempty valued correspondence. Then we can extract a sequence $\{f_{\lambda_n} : n = 1, 2, ...\}$ from the net $\{f_\lambda : \lambda \in \Lambda\}$ such that f_{λ_n} converges weakly to f and for almost all t in T, $f(t)$ is an element of the closed convex hull of the weak limit superior of the sequence $f_{\lambda_n}(t)$, i.e., $f(t) \in \overline{\text{con}}\ \text{w-Ls}\{f_{\lambda_n}(t)\}$ μ-a.e.

The above theorem generalizes in several directions a recent result of Khan–Majumdar [12], which in turn is an extension of a theorem of

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Artstein [2]. Moreover, versions of the above theorem can be used to prove Lebesgue–Aumann-type dominated convergence results either for the set of all integral selections of a correspondence or for the integral of a correspondence. The latter results extend the previous dominated convergence theorems for the integral of a correspondence obtained by Aumann [3], Pucci–Vitillaro [16], and Yannelis [22]. Our results have useful applications in Economics and Game Theory (see for instance Khan–Yannelis [13], Khan [14, 15], and Yannelis [21]).

The paper is organized as follows: Section 2 contains notation and definitions. In Section 3 the main results of the paper are stated, and finally the proofs of all the results are collected in Sections 4 and 5.

2. Notation and Definitions

2.1. *Notation*

2^A denotes the set of all nonempty subsets of the set A; $\varnothing$ denotes the empty set; dist denotes distance; R denotes the set of real numbers; R^l denotes the l-fold Cartesian product of R. If A is a subset of a Banach space, clA denotes the norm closure of A, and $\overline{\text{con}}\, A$ denotes the closed convex hull of A. If X is a linear topological space, its dual is the space X^* of all continuous linear functionals on X, and if $p \in X^*$ and $x \in X$ the value of p at x is denoted by $\langle p, x \rangle$. If $\{F_n := 1, 2, ...\}$ is a sequence of nonempty subsets of a Banach space X, we will denote by w-Ls F_n and s-Li F_n the set of its weak limit superior and strong limit inferior points respectively, i.e.,

$$\text{w-Ls } F_n = \{x \in X : x = \text{w-}\lim_{k \to \infty} x_{n_k},\ x_{n_k} \in F_{n_k},\ k = 1, 2, ...\}$$
$$\text{s-Li } F_n = \{x \in X : x = \text{s-}\lim_{n \to \infty} x_n,\ x_n \in F_n,\ n = 1, 2, ...\}.$$

2.2. *Definitions*

Let (T, Σ, μ) be a finite measure space and X be a separable Banach space. The correspondence $\varphi: T \to 2^X$ is said to have a *measurable graph* if the set $G_\varphi = \{(t, x) \in T \times X : x \in \varphi(t)\}$ belongs to $\Sigma \otimes \beta(X)$, where $\beta(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes product σ-algebra. The correspondence $\varphi: T \to 2^X$ is said to be *lower measurable* if for every open subset V of X, the set $\{t \in T : \varphi(t) \cap V \neq \varnothing\}$ belongs to Σ. It is a standard result (see Himmelberg [10, p. 47]) that if $\varphi(\cdot)$ has a measurable graph, then $\varphi(\cdot)$ is lower measurable, and if $\varphi(\cdot)$ is closed valued and lower measurable then $\varphi(\cdot)$ has a measurable graph. Moreover, if T is a complete finite measure space and $\varphi(\cdot)$ has a measurable graph and it is nonempty valued, then there exists a *measurable selection* for $\varphi(\cdot)$; i.e., there exists a measurable function $f: T \to X$ such that $f(t) \in \varphi(t)$ μ-a.e. (see [10, Theorem 5.2, p. 60]).

Following Diestel–Uhl [7] we define the notion of a Bochner integrable function. Let (T, Σ, μ) be a finite measure space and X be a Banach space. A function $f: T \to X$ is called *simple* if there exist $y_1, y_2, ..., y_n$ in X and $\alpha_1, \alpha_2, ..., \alpha_n$ in Σ such that $f = \sum_{i=1}^n y_i \chi_{\alpha_i}$, where $\chi_{\alpha_i}(t) = 1$ if $t \in \alpha_i$ and $\chi_{\alpha_i}(t) = 0$ if $t \notin \alpha_i$. A function $f: T \to X$ is said to be *μ-measurable* if there exists a sequence of simple functions $f_n: T \to X$ such that $\lim_{n \to \infty} \|f_n(t) - f(t)\| = 0$ for almost all $t \in T$. *A* μ-measurable function $f: T \to X$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n: n = 1, 2, ...\}$ such that

$$\lim_{n \to \infty} \int_T \|f_n(t) - f(t)\| \, d\mu(t) = 0.$$

In this case we define for each $E \in \Sigma$ the integral to be $\int_E f(t) \, d\mu(t) = \lim_{n \to \infty} \int_E f_n(t) \, d\mu(t)$. It can be shown (see Diestel–Uhl [7, Theorem 2, p. 45]) that if $f: T \to X$ is a μ-measurable function then f is Bochner integrable if and only if $\int_T \|f(t)\| \, d\mu(t) < \infty$. For $1 \leqslant p < \infty$, we denote by $L_p(\mu, X)$ the space of equivalence classes of X-valued Bochner integrable functions $x: T \to X$ normed by

$$\|x\|_p = \left[\int_T \|x(t)\|^p \, d\mu(t) \right]^{1/p}.$$

As was noted in Diestel–Uhl [7, p. 50], it can be easily shown that normed by the functional $\|\cdot\|_p$ above, $L_p(\mu, X)$ becomes a Banach space.

A Banach space X has the *Radon–Nikodym Property* (RNP) *with respect to the measure space* (T, Σ, μ) if for each μ-continuous vector measure $G: \Sigma \to X$ of bounded variation there exists $g \in L_1(\mu, X)$ such that $G(E) = \int_E g(t) \, d\mu(t)$ for all $E \in \Sigma$. A Banach space X has the *Radon–Nikodym Property* if X has the RNP with respect to every finite measure space. Recall now (see Diestel–Uhl [7, Theorem 1, p. 98]) that if (T, Σ, μ) is a finite measure space $1 \leqslant p < \infty$, and X is a Banach space, then X^* has the RNP if and only if $(L_p(\mu, X))^* = L_q(\mu, X^*)$ where $1/p + 1/q = 1$. For $1 \leqslant p < \infty$ denote by S_φ^p the *set of all selections of the correspondence* $\varphi: T \to 2^X$ that belong to the space $L_p(\mu, X)$, i.e.,

$$S_\varphi^p = \{x \in L_p(\mu, X): x(t) \in \varphi(t) \ \mu\text{-}a.e.\}.$$

We will also consider the set $S_\varphi^1 = \{x \in L_1(\mu, X): x(t) \in \varphi(t) \ \mu\text{-a.e.}\}$, i.e., S_φ^1 is the *set of all integrable selections* of $\varphi(\cdot)$. Using the above set and following Aumann [3] we can define the *integral of the correspondence* $\varphi: T \to 2^X$ as

$$\int_T \varphi(t)\, d\mu(t) = \left\{ \int_T x(t)\, d\mu(t) : x \in S_\varphi^1 \right\}.$$

In the sequel we will denote the above integral by $\int \varphi$. Recall that the correspondence $\varphi: T \to 2^X$ is said to be *integrably bounded* if there exists a map $g \in L_1(\mu, R)$ such that $\sup\{\|x\| : x \in \varphi(t)\} \leqslant g(t)$ μ-a.e. Furthermore, if T is a complete finite measure space, X is a separable Banach space and $\varphi: T \to 2^X$ is an integrably bounded nonempty valued correspondence having a measurable graph, by virtue of the measurable selection theorem we can conclude that S_φ^1 is nonempty and so $\int \varphi$ is nonempty as well. Now let $\{F_n : n = 1, 2, ...\}$ be a sequence of nonempty subsets of a Banach space X. We will say this F_n *converges* in F (written as $F_n \to F$) if and only if s-Li $F_n =$ w-Ls $F_n = F$.

With all these preliminaries now out of the way we are ready to state our main results.

3. The Main Theorems

We begin by stating the following result on weak sequential convergence in $L_p(\mu, X)$, $1 \leqslant p < \infty$.

Theorem 3.1. *Let (T, Σ, μ) be a finite positive measure space and X be a separable Banach space. Let $\{f_\lambda : \lambda \in \Lambda\}$ (Λ is a directed set) be a net in $L_p(\mu, X)$, $1 \leqslant p < \infty$, such that f_λ converges weakly to $f \in L_p(\mu, X)$. Suppose that for all $\lambda \in \Lambda$, $f_\lambda(t) \in F(t)$ μ-a.e., where $F: T \to 2^X$ is a weakly compact, integrably bounded, convex, nonempty valued correspondence. Then we can extract a sequence $\{f_{\lambda_n} : n = 1, 2, ...\}$ from the net $\{f_\lambda : \lambda \in \Lambda\}$ such that:*

(i) *f_{λ_n} converges weakly to f, and*

(ii) $f(t) \in \overline{\text{con}}\ \text{w-Ls}\{f_{\lambda_n}(t)\}$ μ-a.e.

As an immediate conclusion of Theorem 3.1 we can obtain the following generalization of Theorem 1 in Khan–Majumdar [12].

Corollary 3.1. *Let (T, Σ, μ) be a finite positive measure space and X be a separable Banach space. Let $\{f_n : n = 1, 2, ...\}$ be a sequence of functions in $L_p(\mu, X)$, $1 \leqslant p < \infty$, such that f_n converges weakly to $f \in L_p(\mu, X)$. Suppose that for all n ($n = 1, 2, ...$), $f_n(t) \in F(t)$ μ-a.e., where $F: T \to 2^X$ is a weakly compact, integrably bounded, nonempty valued correspondence. Then*

$$f(t) \in \overline{\text{con}}\ \text{w-Ls}\{f_n(t)\}\ \mu\text{-a.e.}$$

Corollary 3.1 generalizes Theorem 1 of Khan–Majumdar [12] in several directions. In particular, the measure space (T, Σ, μ) need not be atomless or complete, the sequence $\{f_n : n = 1, 2, ...\}$ need not be in a fixed weakly compact subset of X, and finally the sequence $\{f_n : n = 1, 2, ...\}$ need not lie only in $L_1(\mu, X)$.

Using Corollary 3.1 we can prove the following dominated convergence result for the set of integrable selections.

THEOREM 3.2. *Let (T, Σ, μ) be a complete finite positive measure space and X be a separable Banach space. Let $\varphi_n : T \to 2^X$ $(n = 1, 2, ...)$ be a sequence of closed valued and lower measurable correspondences such that*:

(i) *For each n $(n = 1, 2, ...)$, $\varphi_n(t) \subset F(t)$ μ-a.e., where $F: T \to 2^X$ is an integrably bounded, weakly compact, convex, nonempty valued correspondence,*

(ii) $\varphi_n(t) \to \varphi(t)$ μ-a.e., *and*

(iii) $\varphi(\cdot)$ *is convex valued.*

Then

$$S^1_{\varphi_n} \to S^1_{\varphi}.$$

As a corollary of Theorem 3.2 we can obtain a dominated convergence result for the integral of a correspondence.

COROLLARY 3.2. *Let $\varphi_n : T \to 2^X$ $(n = 1, 2, ...)$ be a sequence of closed valued and lower measurable correspondences satisfying all the assumptions of Theorem 3.2. Then*

$$\int \varphi_n \to \int \varphi.$$

The above corollary may be seen as an extension of a result of Aumann [3, Theorem 5, p. 3] to correspondences taking values in a separable Banach space. Recall that in [3], $X = R^l$.

A version of the above dominated convergence result for the integral of a correspondence has been obtained by Pucci and Vitillaro [16]. In their paper the upper and lower limits of a sequence of correspondences were defined in terms of support functions. Moreover, they assumed that X is a separable reflexive Banach space and that (T, Σ, μ) is atomless. Hence, their result does not subsume ours. Finally, Corollary 3.2 generalizes Theorem 5.2 in Yannelis [22], where it was assumed that (T, Σ, μ) is atomless.

It should be noted that Theorem 3.2 follows easily from Lemmata 5.1–5.3 (see Section 5) which are w-Ls and s-Li versions of the Fatou Lemma for the set of integrable selections. In particular, Lemma 5.1, i.e., the w-Ls version of the Fatou Lemma, is a direct consequence of Corollary 3.1 and it can be easily shown that it implies the w-Ls versions of the Fatou Lemma for the integral of a function or correspondence, obtained by Khan–Majumdar [12], Balder [4], and Yannelis [22].

We can now turn to the proofs of our main theorems.

4. Proof of Theorem 3.1

We begin by stating the following result of Artstein which will be used for the proof of Proposition 4.2 below:

PROPOSITION 4.1. *Let (T, Σ, μ) be a finite positive measure space and let $f_n: T \to R^l$ $(n = 1, 2, ...)$ be a uniformly integrable sequence of functions converging weakly to f. Then,*

$$f(t) \in \operatorname{con} \text{w-Ls}\{f_n(t)\} \ \mu\text{-a.e.}$$

Proof. See [2, Proposition C, p. 280].

PROPOSITION 4.2. *Let (T, Σ, μ) be a finite positive measure space and X be a separable Banach space whose dual X^* has the* RNP. *Let $\{f_n: n = 1, 2, ...\}$ be a sequence in $L_p(\mu, X)$, $1 \leq p < \infty$, such that f_n converges weakly to $f \in L_p(\mu, X)$. Suppose that for all n $(n = 1, 2, ...)$, $f_n(t) \in F(t)$ μ-a.e., where $F: T \to 2^X$ is a weakly compact nonempty valued correspondence. Then*

$$f(t) \in \overline{\operatorname{con}} \text{ w-Ls}\{f_n(t)\} \ \mu\text{-a.e.}$$

Proof. Since f_n converges weakly to f and X^* has the RNP, for any $\varphi \in (L_p(\mu, X))^* = L_q(\mu, X^*)$ (where $1/p + 1/q = 1$), we have that $\langle \varphi, f_n \rangle = \int_T \langle \varphi(t), f_n(t) \rangle \, d\mu(t)$ converges to $\langle \varphi, f \rangle = \int_T \langle \varphi(t), f(t) \rangle \, d\mu(t)$. Define the functions $h_n: T \to R$ and $h: T \to R$ by $h_n(t) = \langle \varphi(t), f_n(t) \rangle$ and $h(t) = \langle \varphi(t), f(t) \rangle$, respectively. Since for each n, $f_n(t) \in F(t)$ μ-a.e. and $F(\cdot)$ is weakly compact, h_n is bounded and uniformly integrable. Also, it is easy to check that h_n converges weakly to h. In fact, let $g \in L_\infty(\mu, R)$ and let $M = \|g\|_\infty$, then

$$\begin{aligned} &\left| \int_T g(t)(h_n(t) - h(t)) \, d\mu(t) \right| \\ &= \left| \int_T g(t)(\langle \varphi(t), f_n(t) \rangle - \langle \varphi(t), f(t) \rangle) \, d\mu(t) \right| \\ &\leq M |\langle \varphi, f_n \rangle - \langle \varphi, f \rangle| \end{aligned} \tag{4.1}$$

and (4.1) can become arbitrarily small since as it was noted above that $\langle \varphi, f_n \rangle$ converges to $\langle \varphi, f \rangle$.

By Proposition 4.1, we have that μ-a.e., $h(t) \in \text{con w-Ls}\{h_n(t)\} \subset \overline{\text{con}}\text{ w-Ls}\{h_n(t)\}$, i.e., μ-a.e., $\langle \varphi(t), f(t) \rangle \in \overline{\text{con}}\text{ w-Ls}\{\langle \varphi(t), f_n(t) \rangle\} = \langle \varphi(t), \overline{\text{con}}\text{ w-Ls}\{f_n(t)\} \rangle$ and consequently,

$$\int_T \langle \varphi(t), f(t) \rangle \, d\mu(t) \in \int_T \langle \varphi(t), x(t) \rangle \, d\mu(t), \tag{4.2}$$

where $x(\cdot)$ is a selection from $\overline{\text{con}}\text{ w-Ls}\{f_n(\cdot)\}$.

It follows from (4.2) that

$$f \in S^p_{\overline{\text{con}}\text{ w-Ls}\{f_n\}}. \tag{4.3}$$

To see this, suppose by way of contradiction that $f \notin S^p_{\overline{\text{con}}\text{ w-Ls}\{f_n\}}$; then by the separating hyperplane theorem[1] (see for instance [1, p. 136]), there exists $\psi \in (L_p(\mu, X))^* = L_q(\mu, X^*)$, $\psi \neq 0$, such that $\langle \psi, f \rangle > \sup\{\langle \psi, x \rangle : x \in S^p_{\overline{\text{con}}\text{ w-Ls}\{f_n\}}\}$, i.e., $\int_T \langle \psi(t), f(t) \rangle \, d\mu(t) > \int_T \langle \psi(t), x(t) \rangle \, d\mu(t)$, where $x(\cdot)$ is a selection from $\overline{\text{con}}\text{ w-Ls}\{f_n(\cdot)\}$, a contradiction to (4.2). Hence, (4.3) holds and we can conclude that $f(t) \in \overline{\text{con}}\text{ w-Ls}\{f_n(t)\}$ μ-a.e. This completes the proof of Proposition 4.2.

Remark 4.1. Proposition 4.2 remains true without the assumption that X^* has the RNP. The proof proceeds as follows: Since f_n converges weakly to f we have that $\langle \varphi, f_n \rangle$ converges to $\langle \varphi, f \rangle$ for all $\varphi \in (L_p(\mu, X))^*$. It follows from a standard result (see for instance Dinculeanu [8, p. 112]) that φ can be represented by a function $\psi: T \to X^*$ such that $\langle \psi, x \rangle$ is measurable for every $x \in X$ and $\|\psi\| \in L_q(\mu, R)$. Hence, $\langle \varphi, f_n \rangle = \int_T \langle \psi(t), f_n(t) \rangle \, d\mu(t)$ and $\langle \varphi, f \rangle = \int_T \langle \psi(t), f(t) \rangle \, d\mu(t)$. Define the functions $h_n: T \to R$ and $h: T \to R$ by $h_n(t) = \langle \psi(t), f_n(t) \rangle$ and $h(t) = \langle \psi(t), f(t) \rangle$, respectively. One can now proceed as in the proof of Proposition 4.2 to complete the argument.

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Denote the net $\{f_\lambda : \lambda \in \Lambda\}$ by B. Since by assumption for all $\lambda \in \Lambda$, $f_\lambda(t) \in F(t)$ μ-a.e., where $F: T \to 2^X$ is an integrably bounded, weakly compact, convex valued correspondence, we can conclude that for all $\lambda \in \Lambda$, f_λ lies in the weakly compact set S^p_F (recall Diestel's

[1] Note that the set $S^p_{\overline{\text{con}}\text{ w-Ls}\{f_n\}}$ is nonempty. In fact, since w-Ls$\{f_n\}$ is lower measurable and nonempty valued so is $\overline{\text{con}}$ w-Ls$\{f_n\}$. Hence, $\overline{\text{con}}$ w-Ls$\{f_n\}$ admits a measurable selection (recall the Kuratowski and Ryll–Nardzewski measurable selection theorem). Obviously the measurable selection is also integrable since $\overline{\text{con}}$ w-Ls$\{f_n\}$ lies in a weakly compact subset of X. Therefore, we can conclude that $S^p_{\overline{\text{con}}\text{ w-Ls}\{f_n\}}$ is nonempty.

theorem on weak compactness; see for example [20] for an exact reference). Hence, the weak closure of B, i.e., w-cl B, is weakly compact. By the Eberlein–Smulian Theorem (see [9, p. 430] or [1, p. 156]), w-cl B is weakly sequentially compact. Obviously the weak limit of f_λ, i.e., f, belongs to w-cl B. From Whitley's theorem[2] [1, Lemma 10.12, 155], we know that if $f \in$ w-cl B, then there exists a sequence $\{f_{\lambda_n}: n = 1, 2, \ldots\}$ in B such that f_{λ_n} converges weakly to f. Since the sequence $\{f_{\lambda_n}: n = 1, 2, \ldots\}$ satisfies all the assumptions of Proposition 4.2 and Remark 4.1 we can conclude that $f(t) \in \overline{\text{con}}$ w-Ls$\{f_{\lambda_n}(t)\}$ μ-a.e. This completes the proof of the theorem.

5. Proof of Theorem 3.2

For the proof of Theorem 3.2 we need to prove w-Ls and s-Li versions of Fatou's Lemma for the set of integrable selections.

LEMMA 5.1. *Let (T, Σ, μ) be a finite positive measure space, X be a separable Banach space and $\varphi_n: T \to 2^X$ $(n = 1, 2, \ldots)$ be a sequence of non-empty, closed valued correspondences such that:*

(i) *For all n $(n = 1, 2, \ldots)$, $\varphi_n(t) \subset F(t)$ μ-a.e., where $F: T \to 2^X$ is an integrably, bounded weakly compact, convex, nonempty-valued correspondence. Then,*

$$\text{w-Ls } S^1_{\varphi_n} \subset S^1_{\overline{\text{con}}\ \text{w-Ls}\ \varphi_n}.$$

Proof. Let $x \in$ w-Ls $S^1_{\varphi_n}$; i.e., there exists $x_k \in S^1_{\varphi_{n_k}}$ $(k = 1, 2, \ldots)$ such that x_k converges weakly to x. We wish to know that $x \in S^1_{\overline{\text{con}}\ \text{w-Ls}\ \varphi_n}$. Since x_k converges weakly to x and x_k lies in a weakly compact set, it follows from Proposition 4.2 that $x(t) \in \overline{\text{con}}$ w-Ls$\{x_k(t)\}$ μ-a.e. which implies that $x(t) \in \overline{\text{con}}$ w-Ls $\varphi_n(t)$ μ-a.e. Since by assumption for each n, $\varphi_n(\cdot)$ lies in the integrably bounded convex set $F(\cdot)$, we can conclude that $x \in S^1_{\overline{\text{con}}\ \text{w-Ls}\ \varphi_n}$. This completes the proof of the lemma.

With additional assumptions to those in Lemma 5.1, we are now able to obtain an exact w-Ls version of Fatou's Lemma for the set of integrable selections.

LEMMA 5.2. *Let $\varphi_n: T \to 2^X$ $(n = 1, 2, \ldots)$ be a sequence of correspondences satisfying all the assumptions of Lemma 5.1. Moreover, assume that* w-Ls $\varphi_n(\cdot)$ *is closed and convex valued. Then*

$$\text{w-Ls } S^1_{\varphi_n} \subset S^1_{\text{w-Ls}\ \varphi_n}.$$

[2] See also Kelly–Namioka [11, exercise L, p. 165].

Proof. It follows from Lemma 5.1 that

$$\text{w-Ls}\, S^1_{\varphi_n} \subset S^1_{\overline{\text{con}}\,\text{w-Ls}\,\varphi_n}. \tag{5.1}$$

Since w-Ls $\varphi_n(\cdot)$ is closed and convex (hence weakly closed) we have that w-Ls $\varphi_n(\cdot) = \overline{\text{con}}$ w-Ls $\varphi_n(\cdot)$ and therefore,

$$S^1_{\text{w-Ls}\,\varphi_n} = S^1_{\overline{\text{con}}\,\text{w-Ls}\,\varphi_n}. \tag{5.2}$$

Combining now (5.1) and (5.2) we can conclude that w-Ls $S^1_{\varphi_n} \subset S^1_{\text{w-Ls}\,\varphi_n}$. This completes the proof of the lemma.

The result below is a s-Li version of Fatou's Lemma for the set of integrable selections. It generalizes Proposition 4.2 in [3] to separable Banach spaces.

LEMMA 5.3. *Let (T, Σ, μ) be a complete finite measure space and let X be a separable Banach space. If $\varphi_n: T \to 2^X$ $(n = 1, 2, ...)$ is a sequence of integrably bounded correspondences having a measurable graph, i.e., $G_{\varphi_n} \in \Sigma \otimes \beta(X)$, then*

$$S^1_{\text{s-Li}\,\varphi_n} \subset \text{s-Li}\, S^1_{\varphi_n}.$$

Proof. Let $x \in S^1_{\text{s-Li}\,\varphi_n}$, i.e., $x(t) \in \text{s-Li}\,\varphi_n(t)$ μ-a.e.; we must show that $x \in \text{s-Li}\, S^1_{\varphi_n}$. First note that $x(t) \in \text{s-Li}\,\varphi_n \mu$-a.e. implies that there exists a sequence $\{x_n: n = 1, 2, ...\}$ such that $\text{s-lim}_{n \to \infty} x_n(t) = x(t)$ μ-a.e. and $x_n(t) \in \varphi_n(t)$ μ-a.e., which is equivalent to the fact that $\lim_{n \to \infty} \text{dist}(x(t), \varphi_n(t)) = 0$ μ-a.e. As in [17, p. 528 or 15a] for each n $(n = 1, 2, ...)$ define the correspondence $A_n: T \to 2^X$ by $A_n(t) = \{y \in \varphi_n(t): \|y - x(t)\| \leqslant \text{dist}(x(t), \varphi_n(t)) + 1/n\}$. Clearly for all n $(n = 1, 2, ...)$ and for all $t \in T$, $A_n(t) \neq \emptyset$. Moreover, $A_n(\cdot)$ has a measurable graph. Indeed, the function $g: T \times X \to [-\infty, \infty]$ defined by $g(t, y) = \|y - x(t)\| - \text{dist}(x(t), \varphi_n(t))$ is measurable in t and continuous in y and therefore by a standard result (see Himmelberg [10, Theorem 2, p. 378]) $g(\cdot, \cdot)$ is jointly measurable with respect to the product σ-algebra $\Sigma \otimes \beta(X)$. It is easy to see that

$$G_{A_n} = \left\{(t, y) \in T \times X: g(t, y) \leqslant \frac{1}{n}\right\} \cap G_{\varphi_n} = g^{-1}\left(\left[-\infty, \frac{1}{n}\right]\right) \cap G_{\varphi_n}.$$

Since $\varphi_n(\cdot)$ has a measurable graph and $g(\cdot, \cdot)$ is jointly measurable, we can conlclude that G_{A_n} belongs to $\Sigma \otimes \beta(X)$; i.e., $A_n(\cdot)$ has a measurable graph. By the Aumann measurable selection theorem (see for instance Himmelberg [10]) there exists a measurable function $f_n: T \to X$ such that $f_n(t) \in A_n(t)$ μ-a.e. Since $x(t) \in \text{s-Li}\,\varphi_n(t)$ μ-a.e., $\lim_{n \to \infty} \text{dist}(x(t), \varphi_n(t)) = 0$ μ-a.e. which implies that $\lim_{n \to \infty} \|f_n(t) - x(t)\| = 0$ μ-a.e. Since $f_n(t) \in$

$\varphi_n(t)$ μ-a.e. and $\varphi_n(\cdot)$ is integrably bounded, by the Lebesgue dominated convergence theorem (see Diestel–Uhl [7, p. 45]), $f_n(\cdot)$ is Bochner integrable, i.e., $f_n \in L_1(\mu, X)$. Hence, $x \in$ s-Li $S^1_{\varphi_n}$ and this completes the proof of the lemma.

We are now ready to complete the proof of Theorem 3.2.

Proof of Theorem 3.2. First note that since for each n $(n = 1, 2, ...)$, $\varphi_n(\cdot)$ is closed valued and lower measurable, $G_{\varphi_n} \in \Sigma \otimes \beta(X)$ (see [10, Theorem 3.5]); i.e., $\varphi_n(\cdot)$ has a measurable graph and so does s-Li $\varphi_n(\cdot)$. Now if $\varphi(t) = \text{s-Li}\, \varphi_n(t) = \text{w-Ls}\, \varphi_n(t)$ μ-a.e., it follows from Lemmata 5.2 and 5.3 that

$$S^1_\varphi = S^1_{\text{s-Li}\, \varphi_n} \subset \text{s-Li}\, S^1_{\varphi_n} \subset \text{w-Ls}\, S^1_{\varphi_n} \subset S^1_{\text{w-Ls}\, \varphi_n} = S^1_\varphi.$$

Therefore

$$S^1_\varphi = \text{s-Li}\, S^1_{\varphi_n} = \text{w-Ls}\, S^1_{\varphi_n},$$

and we can conclude that $S^1_{\varphi_n} \to S^1_\varphi$. This completes the proof of the theorem.

Proof of Corollary 3.2. Define the mapping $\psi: L_1(\mu, X) \to X$ by $\psi(x) = \int x(t)\, d\mu(t)$. From Theorem 3.2 we have that

$$S^1_\varphi = \text{s-Li}\, S^1_{\varphi_n} = \text{w-Ls}\, S^1_{\varphi_n}. \tag{5.3}$$

Taking into account (5.3), it follows directly from the definition of the integral of a correspondence that

$$\psi(S^1_\varphi) = \{\psi(x): x \in S^1_\varphi\} = \int \varphi(t)\, d\mu(t) = \psi(\text{s-Li}\, S^1_\varphi)$$

$$= \text{s-Li} \int \varphi_n(t)\, d\mu(t) = \psi(\text{w-Ls}\, S^1_{\varphi_n}) = \text{w-Ls} \int \varphi_n(t)\, d\mu(t),$$

i.e.,

$$\int \varphi_n \to \int \varphi$$

as was to be shown.

6. Concluding Remarks

Remark 6.1. If (T, Σ, μ) in Lemma 5.1 is assumed to be atomless, then by virtue of Result 2 in [16] one can obtain a generalized version of

Fatou's Lemma proved in Khan–Majumdar [12]. The proof is similar with that in [12].

Remark 6.2. In finite dimensional spaces Balder [5] has shown that the Chacon biting lemma (see [5] for a reference) can be used to generalize Schmeidler's [19] version of Fatou's Lemma in several dimensions. Recently, Balder [6] has extended the biting lemma to $L_1(\mu, X)$ where X is a reflexive Banach space. It is of interest to know whether Balder's extension of the biting lemma can be used to prove Lemma 5.1, or even versions of Theorem 3.1.

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