On The Euclidean and the Spectral Norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices

Hasan ÖĞÜNMEZ
Department of Mathematics, Faculty of Science and Arts, Kocatepe University, Afyon-TURKEY
e-posta: hogunmez@aku.edu.tr

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Abstract
In this study, we have established upper and lower bounds for the Euclidean and the Spectral norms of Quaternion Cauchy-Toeplitz ($T_n$) and Quaternion Cauchy-Hankel ($H_n$) matrices respectively. Besides, by assuming complex matrix $T_n = A_1 + A_2 j$, we have defined the matrix $[T_n] = \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix}$ and similarly, by assuming complex matrix $H_n = B_1 + B_2 j$, we have also defined another matrix $[H_n] = \begin{bmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{bmatrix}$. Then we have obtained the bounds of the spectral norms for these matrices.

Quatereion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrislerinin Euclideans ve Spektral Normları Üzerine

Anahtar Kelimeler
Quaternion Cauchy-Toeplitz Matrisler, Quaternion Cauchy-Hankel Matrisler, Euclidean Norm, Spectral Norm. 15A45, 15A60

Özet
Bu çalışmada, sırasıyla Quaternion Cauchy-Toeplitz ($T_n$), Quaternion Cauchy-Hankel ($H_n$) Matrislerinin Spektal ve Euclidean normlar için alt ve üst sınırlar elde ettik. Ayrıca, $T_n = A_1 + A_2 j$, kompleks matrisi yardımıyla $[T_n] = \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix}$ matrisini ve benzer şekilde $H_n = B_1 + B_2 j$, kompleks matrisi yardımıyla $[H_n] = \begin{bmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{bmatrix}$ matrisini tanımlayıp bu matrislerin spektral normlar için sınırlar elde ettik.

1. Introduction and Preliminaries

In quantum physic, the family of quaternions plays an important role. But in mathematics they generally play a role in algebraic systems, skew fields or noncommutative division algebras, matrices in commutative rings take attention but, matrices with quaternion entries has not been investigated very much yet. But in recent times quaternions are in order of day. The main obstacles in the study of quaternion matrices, as expected come from the noncommutative multiplication of quaternions. One will find that working on a quaternion matrix problem is often equivalent to dealing with a pair of complex matrices [Zhang(1997), Lee(1949)].

Recently, the studies concern with matrix norms, has been given by several authors, see for instance [Moenck(1977), Mathias(1990), Visick(2000), Zielke(1988), Horn and Johnson(1991), Bozkurt(1996), Solak and Bozkurt(2003), Türkmen and Bozkurt (2002)] and references cited therein. In this paper, we have obtained some a lower and an upper bounds for the Euclidean and spectral of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices. Now, we need the following definitions and preliminaries.

Definition 1. Let $\mathbb{C}$ and $\mathbb{R}$ denote the fields of the complex and real numbers respectively. Let $\mathbb{Q}$ be a
four-dimensional vector space over \( \mathbb{R} \) with an ordered basis, denoted by \( e, i, j \) and \( k \). A real quaternion, simply called quaternion, is a vector
\[
x = x_0 e + x_1 i + x_2 j + x_3 k \in \mathbb{Q}
\]
with real coefficients \( x_0, x_1, x_2, \) and \( x_3 \).

Besides the addition and the scalar multiplication of the vector space \( \mathbb{Q} \) over \( \mathbb{R} \), the product of any two quaternions \( e, i, j \) and \( k \) are defined by the requirement that \( e \) act as a identity and by the table
\[
i^2 = j^2 = k^2 = -1
\]
\[
ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]

Let \( M_{mn} (\mathbb{Q}) \), simply \( M_n (\mathbb{Q}) \) when \( m = n \), denote the collection of all \( m \times n \) matrices with quaternion entries.

**Definition 2.** Let \( A = A_1 + A_2 j \in M_n (\mathbb{Q}) \), where \( A_1, A_2 \) are \( n \times n \) complex matrices. We shall call the \( 2n \times 2n \) complex matrix
\[
\begin{bmatrix}
  A_1 & A_2 \\
  -A_2 & A_1
\end{bmatrix}
\]
uniquely determined by \( A \), the complex adjoint matrix or adjoint of the quaternion matrix \( A \) [Lee(1949)].

Now we give some preliminaries related to our study. Let \( A \) be any \( n \times n \) matrix. The \( \ell_p \) norms of the matrix \( A \) are defined as
\[
\| A \|_p = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p} \quad (1 \leq p < \infty).
\]
If \( p = \infty \), then
\[
\| A \|_{\infty} = \lim_{n \to \infty} \| A \|_p = \max_{i,j} |a_{ij}|.
\]
The well-known Euclidean norm of matrix \( A \) is
\[
\| A \|_2 = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}
\]
and also the spectral norm of matrix \( A \) is
\[
\| A \|_F = \sqrt{\max_{1 \leq i \leq n} \lambda_i (A^T A)}
\]
where \( A \) is \( m \times n \) and \( A^T \) is the conjugate transpose of the matrix \( A \). The following inequality holds:
\[
\frac{1}{\sqrt{n}} \| A \|_E \leq \| A \|_2 \leq \| A \|_E
\]
[Zielke (1988)]. A function \( \Psi \) is called a psi (or digamma) function if
\[
\Psi(x) = \frac{d}{dx} \{ \ln \Gamma(x) \}
\]
where
\[
\Gamma(x) = \int_0^x e^{-t} t^{x-1} dt.
\]
The \( n \)th derivatives of a \( \Psi \) function is called a polygamma function
\[
\Psi(n, x) = \frac{d}{dx^n} \Psi(x) = \frac{d}{dx} \left( \frac{d}{dx} \ln \Gamma(x) \right).
\]
If \( n = 0 \) then \( \Psi(0, x) = \Psi(x) = \left\{ \frac{d}{dx} \ln \Gamma(x) \right\} \). On the other hand, if \( a > 0 \), \( b \) is any number and \( n \) is positive integer, then
\[
\lim_{n \to \infty} \Psi(a, n + b) = 0
\]
[Moenck(1977)]. Throughout the paper \( Z^+ \) and \( \mathbb{R}^+ \) will represent the sets of positive integers and positive real numbers, respectively.

Let \( A \) and \( B \) be \( n \times n \) matrices. The Hadamard product of \( A \) and \( B \) is defined by
\[
A \circ B = [a_{ij} b_{ij}].
\]
If \( \| \cdot \| \) is any norm on \( n \times n \) matrices, then
\[
\| A \circ B \| \leq \| A \| \| B \| \quad \text{(1.6)}
\]
[Visick(2000)]. Define the maximum column length norm \( c_j (\cdot) \) and the maximum row length norm \( r_i (\cdot) \) of any matrix \( A \) by
\[
c_j (A) = \max_j \left( \sum_{i=1}^{m} |a_{ij}|^2 \right)^{1/2} \quad i = 1, 2, \cdots, m
\]
and
\[
r_i (A) = \max_i \left( \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad j = 1, 2, \cdots, n
\]
respectively [Horn and Johnson(1991)]. Let \( A \) and \( B \) be \( m \times n \) matrices. If \( A \circ B = C \) then
\[
\| A \|_2 \leq r_j (B) c_j (C) \quad \text{(1.9)}
\]
[Mathias(1990)].

## 2. Matrices of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel

**Definition 3.** The matrices in \( x \) quaternion from Definition 1, for \( 2 \leq t, l, m \in \mathbb{Z}^+ \) and \( p = 1, 2, \cdots, n \), \( r = 1, 2, \cdots, n \) and
\( x_0 = 0, \ x_1 = \frac{1}{i+p-r}, \ x_2 = \frac{1}{j+p-r}, \ x_3 = \frac{1}{k+p-r} \)

(2.1) \[
T = \begin{bmatrix}
i & j & k \\
\frac{1}{i+p-r} & \frac{1}{j+p-r} & \frac{1}{k+p-r} \\
\end{bmatrix}
\]
is called Quaternion Cauchy-Toeplitz matrix.

(2.2) \[
H = \begin{bmatrix}
i & j & k \\
\frac{1}{i+p+r} & \frac{1}{j+p+r} & \frac{1}{k+p+r} \\
\end{bmatrix}
\]
is called Quaternion Cauchy-Hankel matrix.

In this section we are going to find an upper and lower bounds for the Euclidean norm and the spectral norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel matrices in (2.1) and (2.2).

2.1. Euclidean and Spectral Norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices

**Theorem 1.** For Euclidean norm of \( T \) Quaternion Cauchy-Toeplitz matrix which is defined in (2.1),

\[
\|T\|_E \leq \pi \sqrt{n(csc^2 \frac{\pi}{t} + csc^2 \frac{\pi}{j} + csc^2 \frac{\pi}{k})}
\]
is valid upper bound, where \( 2 \leq t, l, m \in \mathbb{Z}^+ \).

**Proof.** From Euclidean norm the following

\[
\|T\|_E^2 = n(t^2 + l^2 + m^2) + \sum_{s=1}^{n-1} (n-s) \left[ \frac{t^2}{(1-st)^2} + \frac{l^2}{(1-sl)^2} + \frac{m^2}{(1-sm)^2} \right]
\]
is obtained. If we divide both of sides by \( n \) and if we take upper bound of right hand side to infinity, we obtain

\[
\frac{1}{n} \|T\|_E^2 \leq t^2 + l^2 + m^2 + \lim_{n \rightarrow \infty} \sum_{s=1}^{n-1} (1 - \frac{s}{n}) \left[ \frac{t^2}{(1-st)^2} + \frac{l^2}{(1-sl)^2} + \frac{m^2}{(1-sm)^2} \right]
\]

\[
= \pi^2 (csc^2 \frac{\pi}{t} + csc^2 \frac{\pi}{j} + csc^2 \frac{\pi}{k})
\]
or

\[
\frac{1}{n} \|T\|_E^2 \leq \pi^2 \left[ csc^2 \frac{\pi}{t} + csc^2 \frac{\pi}{j} + csc^2 \frac{\pi}{k} \right]
\]

if we take

\[
\frac{1}{\sqrt{n}} \|T\|_E \leq \pi \sqrt{csc^2 \frac{\pi}{t} + csc^2 \frac{\pi}{j} + csc^2 \frac{\pi}{k}}
\]
then we obtain

\[
\|T\|_E \leq \pi \sqrt{n(csc^2 \frac{\pi}{t} + csc^2 \frac{\pi}{j} + csc^2 \frac{\pi}{k})}
\]
which is an upper bound for Euclidean norm of Quaternion Cauchy-Toeplitz matrix.

**Corollary 1.** For spectral norm of \( T \) Quaternion Cauchy-Toeplitz matrices defined in (2.1),

\[
\pi \sqrt{csc^2 \frac{\pi}{t} + csc^2 \frac{\pi}{j} + csc^2 \frac{\pi}{k}} \leq \|T\|_2
\]
is valid lower bound, where \( 2 \leq t, l, m \in \mathbb{Z}^+ \).

**Proof.** Following relation, for spectral norm of \( T \) matrix will be

\[
\frac{1}{\sqrt{n}} \|T\|_E \leq \|T\|_2
\]

Then we obtain

\[
\pi \sqrt{csc^2 \frac{\pi}{t} + csc^2 \frac{\pi}{j} + csc^2 \frac{\pi}{k}} \leq \|T\|_2
\]
Now for in definition (2.1) of spectral norm of $T$ Quaternion Cauchy-Toeplitz matrix, we have obtained upper bound to give as a theorem before lets give some definition end concepts.

**Definition 4.** $J_i=\begin{bmatrix}1\end{bmatrix}_{n\times n}$ $(i=1,2,\cdots,n)$ let it be a square $J$ matrix with all entries 1.

Now, $A_1=\left[\frac{i}{2^{\frac{p-r}{p-r}}}\right]_{p,r=1}^{i}$ and $A_2=\left[\frac{i}{2^{\frac{p-r}{p-r}}}+\frac{i}{2^{\frac{n-n}{p-r}}}\right]_{p,r=1}^{i}$ write $T_n=A_1+A_2j$. From Definition 2,

$$[T_n]=\begin{bmatrix}A_1 & A_2 \\ -A_2 & A_1\end{bmatrix}$$

is being occurred as complex matrix in $2n\times 2n$. From definition (2.5) lets give the following theorem.

**Theorem 2.** Let $t=l=m=2$ in (2.5). Then the upper bound for the spectral norm of Quaternion Cauchy-Toeplitz matrix is

$$\|T_n\|_2 \leq \left[\left(2+2\sqrt{2}\right) \left(1 + \ln 2 + \frac{\gamma}{2} + \frac{1}{2}\Psi\left(n-\frac{1}{2}\right)\right)\right.\left.\times \left(\pi^2 + 2\ln 2 + \gamma - 2\Psi\left(1,n+\frac{1}{2}\right) + \Psi\left(n+\frac{1}{2}\right)\right)\right]^{\frac{1}{2}}.$$  

**Proof.** Consider $2n\times 2n$ complex matrix in (2.5), with $A_1=\left[\frac{i}{2^{\frac{p-r}{p-r}}}\right]_{p,r=1}^{n}$,

$$A_1=\begin{bmatrix}2i & -2i & \cdots & -\frac{2}{2n-5}i & -\frac{2}{2n-5}i \\ \frac{2}{2n-3}i & 2i & \cdots & -\frac{2}{2n-7}i & -\frac{2}{2n-7}i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{2}{2n-3}i & \frac{2}{2n-5}i & \cdots & 2i & -2i \\ \frac{2}{2n-1}i & \frac{2}{2n-3}i & \cdots & \frac{2}{3}i & 2i \end{bmatrix}_{n\times n}$$

and with $A_2=\left[\frac{1}{2^{\frac{p-r}{p-r}}}+\frac{i}{2^{\frac{n-n}{p-r}}}\right]_{p,r=1}^{n}$,

$$A_2=\begin{bmatrix}2(1+i) & -2(1+i) & \cdots & -\frac{2}{2n-7}(1+i) & -\frac{2}{2n-7}(1+i) \\ \frac{2}{3}(1+i) & 2(1+i) & \cdots & -\frac{2}{2n-7}(1+i) & -\frac{2}{2n-7}(1+i) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{2}{2n-3}(1+i) & \frac{2}{2n-5}(1+i) & \cdots & \frac{2}{3}(1+i) & 2(1+i) \\ \frac{2}{2n-1}(1+i) & \frac{2}{2n-3}(1+i) & \cdots & \frac{2}{3}i & 2(1+i) \end{bmatrix}_{n\times n}$$

in this case, we write $2n\times 2n$ matrix as.

$$[T_n]=\begin{bmatrix}2i & -2i & \cdots & -\frac{2}{2n-3}i & 2(1+i) & -2(1+i) & \cdots & -\frac{2}{2n-3}(1+i) \\ \frac{2}{3}i & 2i & \cdots & -\frac{2}{2n-5}i & \frac{2}{3}(1+i) & 2(1+i) & \cdots & -\frac{2}{2n-5}(1+i) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{2}{2n-1}i & \frac{2}{2n-3}i & \cdots & 2i & \frac{2}{2n-1}(1+i) & \frac{2}{2n-3}(1+i) & \cdots & 2(1+i) \\ -2(1-i) & 2(1-i) & \cdots & -\frac{2}{2n-3}(1-i) & -2i & 2i & \cdots & \frac{2}{2n-3}i \\ \frac{2}{3}(1-i) & -2(1-i) & \cdots & -\frac{2}{2n-5}(1-i) & -\frac{2}{3}i & -2i & \cdots & -\frac{2}{2n-5}i \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{2n-1}(1-i) & -\frac{2}{2n-3}(1-i) & \cdots & -2(1-i) & -\frac{2}{2n-1}i & -\frac{2}{2n-3}i & \cdots & -2i \end{bmatrix}$$

Then, the Hadamard Product (1.5) of
For

\[ T_n = [B_n \circ D_n] \]

\[ B_n = \begin{bmatrix} \sqrt{A_1} & \sqrt{A_2} \\ J_i & J_i \end{bmatrix} \]

and

\[ D_n = \begin{bmatrix} \sqrt{A_1} & \sqrt{A_2} \\ -A_2 & A_1 \end{bmatrix} \]

\[ T_n = \begin{bmatrix} \sqrt{A_1} & \sqrt{A_2} \\ J_i & J_i \end{bmatrix} \circ \begin{bmatrix} \sqrt{A_1} & \sqrt{A_2} \\ -A_2 & A_1 \end{bmatrix} \]

From (1.7) and (1.8), we obtain

\[ r_i^2(B_n) = 2 + 2(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}) + 2\sqrt{2} + 2\sqrt{2}(1 + \frac{1}{3} + \cdots + \frac{1}{2^{n-1}}) \]

\[ = (2 + 2\sqrt{2}) \left( 1 + \sum_{i=1}^{n-1} \frac{1}{2^{i-1}} \right) \]

\[ = (2 + 2\sqrt{2}) \left( 1 + \ln 2 + \frac{\pi}{2} + \frac{1}{2} \Psi(n - \frac{1}{2}) \right) \]

in this case we write

\[ (2.7) \]

\[ r_i(B_n) = \left[ (2 + 2\sqrt{2}) \left( 1 + \ln 2 + \frac{\pi}{2} + \frac{1}{2} \Psi(n - \frac{1}{2}) \right) \right]^2 \]

similarly

\[ c_i^2(D_n) = 2(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}) + 2^3 \left( 1 + \frac{1}{3} + \cdots + \frac{1}{(2^{n-1})^2} \right) \]

\[ = 2 \sum_{i=1}^{n} \frac{1}{2^{i-1}} + 2^3 \sum_{i=1}^{n} \frac{1}{(2^{i-1})^2} \]

\[ = \pi^2 + 2\ln 2 + \gamma - 2\Psi(1, n + \frac{1}{2}) + \Psi(n + \frac{1}{2}) \]

Then

\[ (2.8) \]

\[ c_i(D_n) = \left[ \pi^2 + 2\ln 2 + \gamma - 2\Psi(1, n + \frac{1}{2}) + \Psi(n + \frac{1}{2}) \right]^2 \]

we can see the following relation and we utilize from (1.9) that from (2.7) and (2.8).

\[ \|T_n\| \leq \left[ (2 + 2\sqrt{2}) \left( 1 + \ln 2 + \frac{\gamma}{2} + \frac{1}{2} \Psi(n - \frac{1}{2}) \right) \right]^2 \]

\[ = \left( \pi^2 + 2\ln 2 + \gamma - 2\Psi(1, n + \frac{1}{2}) + \Psi(n + \frac{1}{2}) \right)^2 \]

is being obtained as upper bound.

**Theorem 3.** Let \( 2 \leq t, l, m \in \mathbb{Z}^+ \) hold in (2.2). Then for the Euclidean norm of Quaternion Cauchy-Hankel matrix \( H \), we have

\[ (2.9) \]

\[ \|H\|_E \rightarrow \infty. \]

**Proof.** From the definition of Euclidean

\[ \|H\|_E^2 = \sum_{s=1}^{n} \frac{t^2}{(1 + (s + 1)t)^2} + \frac{l^2}{(1 + (s + 1)l)^2} + \frac{m^2}{(1 + (s + 1)m)^2} \]

\[ + \sum_{s=1}^{n-1} (n - s) \left( \frac{t^2}{(1 + (n + s + 1)t)^2} + \frac{l^2}{(1 + (n + s + 1)l)^2} + \frac{m^2}{(1 + (n + s + 1)m)^2} \right) \]

is clear. If we calculate right hand of this equality, then
\[
\|H\|_E^2 = \left(1 + t\right) \psi\left(1, n + 1 + \frac{1 + t}{t}\right) + \psi\left(n + 1 + \frac{1 + t}{t}\right) + \left(1 + l\right) \psi\left(1, n + 1 + \frac{1 + l}{l}\right) + \psi\left(n + 1 + \frac{1 + l}{l}\right) \\
+ \psi\left(n + 1 + \frac{1 + l}{l}\right) + \left(1 + m\right) \psi\left(1, n + 1 + \frac{1 + m}{m}\right) + \psi\left(n + 1 + \frac{1 + m}{m}\right) \\
- \left(1 + t\right) \psi\left(1, 1 + \frac{1 + t}{t}\right) - \psi\left(1 + \frac{1 + t}{t}\right) - \left(1 + l\right) \psi\left(1, 1 + \frac{1 + l}{l}\right) - \psi\left(1 + \frac{1 + l}{l}\right) \\
- \left(1 + m\right) \psi\left(1, 1 + \frac{1 + m}{m}\right) - \psi\left(1 + \frac{1 + m}{m}\right) - \left(1 + t + 2m\right) \psi\left(1, n + 1 + \frac{1 + t + m}{t}\right) \\
- \psi\left(1, n + 1 + \frac{1 + t + m}{t}\right) - \left(1 + l + 2nl\right) \psi\left(1, n + 1 + \frac{1 + l + nl}{l}\right) - \psi\left(1, n + 1 + \frac{1 + l + nl}{l}\right) \\
- \left(1 + m + 2nm\right) \psi\left(1, n + 1 + \frac{1 + m + nm}{m}\right) - \psi\left(1, n + 1 + \frac{1 + m + nm}{m}\right) \\
+ \left(1 + t + 2m\right) \psi\left(1, 1 + \frac{1 + t + m}{t}\right) + \psi\left(1 + \frac{1 + t + m}{t}\right) + \left(1 + l + 2nl\right) \psi\left(1, 1 + \frac{1 + l + nl}{l}\right) \\
+ \psi\left(1 + \frac{1 + l + nl}{l}\right) + \left(1 + m + 2nm\right) \psi\left(1, 1 + \frac{1 + m + nm}{m}\right) + \psi\left(1 + \frac{1 + m + nm}{m}\right)
\]

is being found above. Therefore, we arrive

\[
\lim_{n \to \infty} \psi\left(n + 1 + \frac{1 + t}{t}\right) = \lim_{n \to \infty} \psi\left(n + 1 + \frac{1 + l}{l}\right) = \lim_{n \to \infty} \psi\left(n + 1 + \frac{1 + m}{m}\right) = \infty
\]

and

\[
\lim_{n \to \infty} \psi\left(1, n + 1 + \frac{1 + t}{t}\right) = \lim_{n \to \infty} \psi\left(1, n + 1 + \frac{1 + l}{l}\right) = \lim_{n \to \infty} \psi\left(1, n + 1 + \frac{1 + m}{m}\right) = 0
\]

This implies,

\[
\|H\|_E \to \infty
\]

which is desired. Now, let's show

\[
B_1 = \left[\frac{i}{\frac{1}{r} + p + r}\right]_{p, r = 1}^n
\]

and

\[
B_2 = \left[\frac{1}{\frac{1}{r} + p + r} + \frac{i}{\frac{1}{m} + p + r}\right]_{p, r = 1}^n.
\]

Then, obtain

\[
H_n = B_1 + B_2 j.
\]

From Definition 2,

\[
(2.10) \quad [H_n] = \begin{bmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{bmatrix}
\]

is being occurred as \(2n \times 2n\) complex matrix. Now, we ready to give the upper bound below.

**Theorem 4.** Let \(l = m = m = 2\) for spectral norm of \([H_n]\), Quaternion Cauchy-Hankel matrix in (2.10) than,

\[
\|H_n\|_2 \leq \left(2 + 2\sqrt{2}\right) \left(\ln 2 - \frac{4}{9} + \frac{\pi}{2} + \psi\left(n + \frac{\pi}{2}\right)\right)
\]

\[
\times \left(\pi^2 - \frac{10\pi}{3} + 2\ln 2 + \gamma - 2\psi\left(1, n + \frac{\pi}{2}\right) + \psi\left(n + \frac{\pi}{2}\right)\right)^{\frac{1}{2}}
\]
is valid.

**Proof.** To use (2.10), define

\[ B_1 = \left[ \frac{i}{2^{+}+r} \right]^{n}, \]

\[ B_i = \begin{bmatrix} \frac{2}{5} i & \frac{2}{5} i & \cdots & \frac{2}{5} i \\ \frac{2}{7} i & \frac{2}{9} i & \cdots & \frac{2}{5} i \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{5} i & \frac{2}{7} i & \cdots & \frac{2}{5} i \end{bmatrix}_{2n \times 2n}, \]

and

\[ B_2 = \left[ \frac{1}{2^{+}+r} + \frac{i}{2^{+}+r} \right]^{n}, \]

\[ B_i = \begin{bmatrix} \frac{2}{5} (1+i) & \frac{2}{5} (1+i) & \cdots & \frac{2}{5} (1+i) \\ \frac{2}{7} (1+i) & \frac{2}{9} (1+i) & \cdots & \frac{2}{5} (1+i) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{5} (1+i) & \frac{2}{7} (1+i) & \cdots & \frac{2}{5} (1+i) \end{bmatrix}_{2n \times 2n} \]

in this case, we write \( 2n \times 2n \) matrix as.

\[ [H_n] = \begin{bmatrix} \frac{2}{5} i & \frac{2}{7} i & \cdots & \frac{2}{5} i \\ \frac{2}{7} i & \frac{2}{9} i & \cdots & \frac{2}{5} i \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{5} i & \frac{2}{7} i & \cdots & \frac{2}{5} i \end{bmatrix} \]

finding Hadamard Product

\[ [M_n] = \begin{bmatrix} \sqrt{B_1} & \sqrt{B_2} \\ J_i & J_i \end{bmatrix} \]

and

\[ [K_n] = \begin{bmatrix} \sqrt{B_1} & \sqrt{B_2} \\ -B_2 & B_1 \end{bmatrix} \]

Since, these two matrices are \( 2n \times 2n \), we write

\[ [H_n] = [M_n] \circ [K_n] \]

or

\[ [H_n] = \begin{bmatrix} \sqrt{B_1} & \sqrt{B_2} \\ J_i & J_i \end{bmatrix} \circ \begin{bmatrix} \sqrt{B_1} & \sqrt{B_2} \\ -B_2 & B_1 \end{bmatrix} \]

From (1.7) and (1.8), we obtain
On The Euclidean and the Spectral Norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices, Öğünmez

\[ r_1^2(M_n) = 2\left(\frac{1}{3} + \frac{1}{7} + \cdots + \frac{1}{2n+3}\right) + 2\sqrt{2}\left(\frac{1}{2} + \frac{1}{7} + \cdots + \frac{1}{2n+3}\right) \]
\[ = \left(2 + 2\sqrt{2}\right)\left(\sum_{s=1}^{n} \frac{1}{2s+3}\right) \]
\[ = \left(2 + 2\sqrt{2}\right)\left(\ln 2 - \frac{\pi}{3} + \frac{\gamma}{2} + \frac{1}{2}\Psi(n + \frac{3}{2})\right) \]

in this case

(2.12)
\[ r_1^2(M_n) = \left(2 + 2\sqrt{2}\right)\left(\ln 2 - \frac{4}{3} + \frac{\gamma}{2} + \frac{1}{2}\Psi\left(n + \frac{5}{2}\right)\right) \]

we found

\[ c_1^2(K_n) = 2\left(\frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{(2n+3)\gamma}\right) + 2\left(\frac{1}{25} + \frac{1}{29} + \cdots + \frac{1}{(2n+3)^2}\right) \]
\[ = 2\sum_{s=1}^{n} \frac{1}{2s+3} + 2\sum_{s=1}^{n} \frac{1}{(2s+3)^2} \]
\[ = \left(\pi^2 - \frac{104}{9} + 2\ln 2 + \gamma - 2\Psi\left(1, n + \frac{3}{2}\right) + \Psi\left(n + \frac{5}{2}\right)\right) \]

Finally,

(2.13)
\[ c_1(K_n) = \left(\pi^2 - \frac{104}{9} + 2\ln 2 + \gamma - 2\Psi\left(1, n + \frac{5}{2}\right) + \Psi\left(n + \frac{5}{2}\right)\right)^{\frac{1}{2}} \]

is written. We utilize between these norms following relation and from (1.9). By the way from (2.12) and (2.13)

\[ \|H_n\|_2 \leq \left(2 + 2\sqrt{2}\right)\left(\ln 2 - \frac{4}{3} + \frac{\gamma}{2} + \frac{1}{2}\Psi\left(n + \frac{5}{2}\right)\right) \]
\[ \times \left(\pi^2 - \frac{104}{9} + 2\ln 2 + \gamma - 2\Psi\left(1, n + \frac{5}{2}\right) + \Psi\left(n + \frac{5}{2}\right)\right)^{\frac{1}{2}} \]

is obtained as upper bound which is desired.

3. Numerical Results

In this section, have compared our findings with the known bounds of the norms of matrices in the illustrative examples below. We have found between in theorems we have given real norms of matrices in second section.

Example 1. For \( t = l = m = 2, \alpha = E \) comparative values in Theorem 1.

\[ \Delta = \pi\sqrt{n\left(\csc^2\frac{\pi}{7} + \csc^2\frac{\pi}{11} + \csc^2\frac{\pi}{m}\right)} \]

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<th>( |F|_E )</th>
<th>( \Delta )</th>
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<tr>
<td>200</td>
<td>76.63022697</td>
<td>76.95298983</td>
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</tbody>
</table>
For \( t = 2, \ l = 3, \ m = 4, \ \alpha = E \) comparative values in Theorem 1.

\[
\Delta = \pi \sqrt{n(\csc^2 \frac{\pi}{t} + \csc^2 \frac{\pi}{l} + \csc^2 \frac{\pi}{m})}
\]

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</table>

Example 2. For \( t = l = m = 2, \ \alpha = 2 \) comparative values in Theorem 2.

\[
\mathbf{a} = \left[\left(2 + 2\sqrt{2}\right)\left(1 + \ln 2 + \frac{\gamma}{2} + \frac{1}{2} \Psi\left(n - \frac{1}{2}\right)\right)\right]^\frac{3}{2}
\]

\[
\left(\pi^2 + 2\ln 2 + \gamma - 2\Psi\left(1, n + \frac{1}{2}\right) + \Psi\left(n + \frac{1}{2}\right)\right]^\frac{3}{2}
\]

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Example 3. For \( t = l = m = 2, \ \alpha = 2 \) comparative values in Theorem 4.

\[
\mathbf{b} = \left[(2 + 2\sqrt{2})\left(\ln 2 - \frac{4}{3} + \frac{\gamma}{2} + \frac{1}{2} \Psi\left(n + \frac{5}{2}\right)\right)\right]^\frac{3}{2}
\]

\[
\times \left(\pi^2 - \frac{104}{9} + 2\ln 2 + \gamma - 2\Psi\left(1, n + \frac{5}{2}\right) + \Psi\left(n + \frac{5}{2}\right)\right)
\]

<table>
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<th>( \mathbf{b} )</th>
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References


