Using the Hexagonal Grid for Three-Dimensional Images:
Direct Fourier Method Reconstruction and Weighted Distance Transform

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Abstract

An image reconstruction technique for computed tomography (CT) images, the direct Fourier method, is shown to apply to non-standard grids. In CT, the 3D image is obtained by reconstructing 2D slices separately. We propose to use the Hexagonal grid for the 2D slices, resulting in 3D images on non-standard grids. Low-level image processing is also considered for these grids – optimal weights to be used for computing the weighted distance transform are calculated.

1 Introduction

When generating images using computed tomography (CT), objects are reconstructed from a number of projections. There are several ways to do this, e.g., algebraic methods, often called iterative methods, and transform methods including, e.g., the filtered backprojection method (FBM) and direct Fourier methods (DFM).

In [5], it is shown that the body-centered cubic (bcc) grid is optimal for an iterative method, the algebraic reconstruction technique. The reason is that the bcc grid in spatial domain corresponds to a closest packing lattice in the frequency domain, the face-centered cubic (fcc) grid.

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When constructing images using the FBM, the interpolation to the grid that should be used to represent the image is done in the very last step, so generalizing these methods to non-standard grids is straight-forward. For the FBM, projections corresponding to the Radon transform are considered. By the projection slice theorem, each of these projections corresponds to a slice in the frequency domain. Therefore, each of these 1D slices can be filtered separately in frequency domain. When this is done, the 1D inverse Fourier transform is applied to the slices and when backprojecting the corresponding slices in spatial domain, the grid points get the values.

In the DFM, after filtering the 1D slices, the interpolation is done in the frequency domain. To get the image in the spatial domain, a 2D inverse Fourier transform is performed. The CT reconstruction technique that is most frequently used is the FBM. The reason is that it is easier to achieve accurate images with a straight forward implementation using FBM compared to DFM, [3]. However, with high-precision (not necessarily computationally expensive) data sampling and interpolation, DFM is faster and less memory demanding than other reconstruction techniques, [3]. Observe that the filtering is not considered in this paper, since it is done in the 1D transform and can thus be done as for standard grids.

Since DFM is used for reconstructing each slice separately, the optimal 2D grid, which is the hexagonal grid, should be used for each slice. By using the hexagonal grid instead of the standard square grid, 13.4% less samples can be used without information loss, [4]. Also, by interpolating the 1D transforms to a hexagonal grid in the frequency domain, the grid on which the reconstructed 3D image is represented is not a cubic grid, but either a generalization of the fcc grid or of the 3D Hexagonal grid.

The weighted distance transform (WDT) is a fast low-level image processing tool that is used in many applications, [2, 7] which is applied to the non-standard grids obtained from the reconstruction. The weights used in this algorithm decide how “round” the balls in the distance transform are. The WDT was applied to the fcc and bcc grids in [7]. In this paper, optimal weights are calculated for the grids resulting from the image reconstruction.

This paper is organized as follows. The theory needed for adopting the CT reconstruction algorithm based on the DFM to the hexagonal grid is presented in Section 1.2. With this theory, the efficient implementation of the hexagonal discrete Fourier transform HDFT, presented in Section 1.3, can be employed for reconstructing each slice. A 3D grid can be obtained from these slices in different ways, which is explained in Section 2. Optimal weights for the weighted distance transform are calculated in Section 3.
1.1 Basic Notions

Any set of $N$ linearly independent vectors in $\mathbb{R}^N$ defines a lattice. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ be $N$ such linearly independent vectors defining the lattice $\mathcal{V}$. The matrix

$$\mathbf{V} = \left( \begin{array}{ccc} | & | & | \\ v_1 & v_2 & \cdots & v_N \end{array} \right)$$

is the matrix generating the lattice $\mathcal{V}$. Let $\mathbf{U}$ and $\mathbf{V}$ be matrices generating the lattices $\mathcal{U}$ and $\mathcal{V}$. The lattices $\mathcal{U}$ and $\mathcal{V}$ are reciprocal [4] if

$$\mathbf{U}^{-1} = \mathbf{V}^T.$$  \hspace{1cm} (1)

1.2 The Projection Slice Theorem for a Hexagonal Basis

The Fourier transform of a function $f(x, y)$ of two variables is defined as

$$F(u, v) = \int \int f(x, y) \exp \{-i2\pi (ux + vy)\} \, dx \, dy.$$  \hspace{1cm} (2)

Using the coordinates $\omega_1$ and $\omega_2$ defined as

$$\left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right) = \mathbf{U} \left( \begin{array}{c} x \\ y \end{array} \right), \quad \mathbf{U} = \sqrt{\frac{2}{\sqrt{3}}} \left( \begin{array}{ccc} 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{array} \right),$$

the coordinates $\epsilon_1, \epsilon_2$ defined as

$$\left( \begin{array}{c} \epsilon_1 \\ \epsilon_2 \end{array} \right) = \mathbf{V} \left( \begin{array}{c} u \\ v \end{array} \right), \quad \mathbf{V} = \sqrt{\frac{3}{\sqrt{3}}} \left( \begin{array}{ccc} -\frac{\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} & 0 \\ \frac{2\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \end{array} \right),$$

(3)

$$g(x, y) = f \left( \frac{\sqrt{3}}{\sqrt{3}} y, \frac{\sqrt{3}}{\sqrt{3}} x + \frac{1}{\sqrt{3}\sqrt{2}} \right), \quad (4)$$

and

$$G(u, v) = F \left( -\frac{1}{\sqrt{3}\sqrt{2}} u + \frac{\sqrt{3}}{\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{2}} v, \frac{\sqrt{3}}{\sqrt{2}} u \right)$$

(5)

gives

$$G(\epsilon_1, \epsilon_2) = \int \int g(\omega_1, \omega_2) \exp \{-i2\pi (\epsilon_1\omega_1 + \epsilon_2\omega_2)\} \, d\omega_1 \, d\omega_2.$$  \hspace{1cm} (6)

There is an obvious duality between the coordinates defined in (2) and (3) – the change of coordinates (2) in the spatial domain corresponds to the change of coordinates (3) in the frequency domain, see Figure 1. The lattices generated by the matrices in (2) and (3) satisfy the relation in (1) and are thus reciprocal. Observe that the change of coordinates implies that the restriction $x, y \in \mathbb{Z}$ and $u, v \in \mathbb{Z}$ both give points $(\omega_1, \omega_2)$ and $(\epsilon_1, \epsilon_2)$ in hexagonal grids, thus both $\mathbf{U}$ and $\mathbf{V}$ generate hexagonal grids denoted $\mathbb{H}_\mathbf{U}$ and $\mathbb{H}_\mathbf{V}$, respectively.

![Figure 1. Coordinate systems in spatial domain (a) and frequency domain (b).](image)

For fixed $(r, \theta)$, the Radon transform $p(r, \theta)$ of $f$ is the line integral of $f(x, y)$ along the line $(r \cos \theta - s \sin \theta, r \sin \theta + s \cos \theta)$, $s \in \mathbb{R}$, i.e.

$$p(r, \theta) = \int f(r \cos \theta - s \sin \theta, r \sin \theta + s \cos \theta) \, ds.$$  \hspace{1cm} (7)

For a fixed $\theta$, the Radon transform is the projection through $f$ at angle $\theta$.

**Theorem 1 (The Hexagonal Projection Slice Theorem)**

The Fourier transform $P(R, \theta)$ of a parallel projection $p(r, \theta)$ of an image $f(x, y)$ taken at angle $\theta$ is found on a line subtending the angle $\theta$ with the $u$-axis in the transformed image $F(u, v)$ when the Fourier Transform with hexagonal basis in (6) is applied.

A proof can be found in, e.g., [3]. By a change of coordinates, it is easy to verify that the theorem is valid with the bases defined in (2) and (3).

1.3 The Hexagonal Discrete Fourier Transform

In this section, $f_D, F_D, g_D$, and $G_D$ are the restrictions of $f, F, g$, and $G$ in Section 1.2 to a finite set of grid points $D$. When the restriction is clear from the context, the subscript will be omitted. Discrete versions of the Fourier transforms on hexagonal basis derived in Section 1.2, are presented in [6, 1]. Since the image is assumed to be periodic, a periodicity matrix $N$ is defined. A function $f$ has period $N$ if the relation $f(x, y) = f((x, y) + N(x', y'))$ holds for any two grid points $(x, y)$ and $(x', y')$.

Let $N$ be a periodicity matrix and $D$ a suitable subset of $\mathbb{Z}^2$. For any $(k, l) \in D$, the discrete Fourier transform (DFT) is

$$G(k, l) = \sum_{(m, n) \in D} g(m, n) \exp \left\{ -i2\pi (k, l)N^{-1} \begin{pmatrix} m \\ n \end{pmatrix} \right\},$$

where $(k, l)$ and $(m, n)$ are grid points in the frequency and spatial domain, respectively. The inverse transform, is for any $(m, n) \in D,$
\[ g(m, n) = \frac{1}{|\det N|} \sum_{(k, l) \in D} G(k, l) \exp \left\{ i2\pi(m, n)N^{-1} \left( \begin{array}{c} k \\ l \end{array} \right) \right\}. \]

For an image on the square grid, \((k, l), (m, n) \in Z^2\), \(N\) is usually a diagonal matrix such that the diagonal elements of \(N\) are the width and the height of the image. This results in a separable Fourier kernel, which is used for the fast Fourier transform (FFT). For a hexagonal image, \((k, l), (m, n) \in H\), the periodicity matrix

\[ N = \left( \begin{array}{cc} 2N & N \\ N & 2N \end{array} \right), \]

where \(N\) is a positive integer, was proposed in [6]. Using this approach, the kernel is not separable, so the standard FFT algorithm can not be applied. A FFT specially designed for the hexagonal grid was proposed in [6]. In [1], a diagonal periodicity matrix was used for the the hexagonal grid allowing the use of the standard FFT.

Consider the set of grid points in the hexagonal grid denoted \(\mathcal{R}\) in Figure 2. The function values at the grid points are periodic, so the set \(\mathcal{R}'\) can just as well be considered for the Fourier transform. Let \(D = \{(n_1, n_2) : 0 \leq n_1 < N, 0 \leq n_2 < N\}\). One period in the grids defined by \(\mathcal{U}\) and \(\mathcal{V}\) is the set of grid points

\[ U\left( \begin{array}{c} i \\ j \end{array} \right), (i, j) \in D \quad \text{and} \quad V\left( \begin{array}{c} i \\ j \end{array} \right), (i, j) \in D, \]

respectively. Thus, the HDFT with diagonal \(N\) is

\[ G(k, l) = \sum_{(m, n) \in D} g(m, n) \exp \left\{ -i \frac{2\pi}{N} (km + ln) \right\}. \]

Similarly, for any \((m, n) \in D\), the inverse transform is

\[ g(m, n) = \frac{1}{|\det N|} \sum_{(k, l) \in D} G(k, l) \exp \left\{ i \frac{2\pi}{N} (km + ln) \right\}. \]

2 3D Grids with Embedded Hexagonal Grid

Let \(\vec{w}_1 = (1, 0, 0), \vec{w}_2 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right)\) be the vectors generating a 2D Hexagonal grid. We call the lattice generated by \(\vec{w}_1, \vec{w}_2, \text{and} \ (1, 0, h)\) the generalized 3D hexagonal grid (GH). The lattice generated by \(\vec{w}_1, \vec{w}_2, \text{and} \ (0, \frac{\sqrt{3}}{2}, h)\) is an fcc grid for \(h = \sqrt{\frac{2}{3}}\), so we call it the generalized fcc grid (GFCC).

3 Weighted Distance Transform

In [7], optimal weights for the weighted distance transform is calculated for, e.g., the fcc and bcc grids. The same method will be applied here to the grids obtained by reconstructing each slice of a CT volume on a hexagonal grid. The efficient two-scan algorithm examined for the fcc and bcc grids in [7] can be used also for the grids considered here.

Given a grid and a neighbourhood, the set of prime vectors is the set of vectors from a grid point to its neighbours. In \(\mathbb{R}^N\), \(N\) linearly independent prime vectors are adjacent if the region spanned by these vectors using only positive coefficients does not contain any other prime vectors. Given a set of \(N\) adjacent prime vectors \(\{\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N\}\), the cone with apex in grid point \(v\) spanned by these prime vectors is the set of grid points that can be written on the form \(v + \sum \alpha_i \vec{p}_i\), where \(\alpha_i\) are positive integers. For the grids considered here, all grid points that can be written \(v + \sum \alpha_i \vec{p}_i\), where the \(\beta_i\)s are positive real numbers are in the cone with apex in grid point \(v\) spanned by \(\{\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N\}\). This property is important for the Chamfer algorithm to produce correct distance maps, [7].

Definition 1 (Weighted distance) Given a set of prime vectors \(\mathcal{P} = (\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_m)\), the weighted distance is defined as

\[ d_w(v_1, v_2) = d_w(0, v_2 - v_1) = \min_{n_i \in \mathbb{R}_+} \left( \sum_{i=1}^{m} n_i |s\vec{p}_i| : \sum_{i=1}^{m} n_i \vec{p}_i = v_2 - v_1 \right), \]

where \(v_1, v_2 \in \mathbb{R}^3\), \(n_i \in \mathbb{R}_+\) is the scale factor that corresponds to the number of “steps” by prime vector \(\vec{p}_i\), and \(|s\vec{p}_i|\) is the weight associated with \(\vec{p}_i\). When computing the weighted distance \(d_w\) in a grid the \(n_i\)s are restricted to positive integers.

Figure 2. One period in spatial domain (a) and frequency domain (b), using \(N = 4\).
3.1 Optimization

Now, the value of $s$ that minimizes the absolute difference is calculated using the approach in [7]. The following error function is considered:

$$E = \min_s \left\{ \max_{r \in S} \left\{ |d_E(0,r) - d_W(0,r)| \right\} \right\},$$

where $d_E$ denotes the Euclidean distance and $S$ is a Euclidean sphere of fixed radius. See [7] for details. A plot on the maximum difference and the shape of balls with these metrics are shown in Figure 3.

The optimal weights will of course depend on the value of $h$, i.e., the distance between the hexagonal grids in the layers.

3.1.1 Generalized FCC Grid

We consider GFCC with 18 neighbours (for $h = \sqrt{2/3}$, these are the neighbours at distance 1 and $\sqrt{2}$).

The optimal $h_{\text{opt}} = \arg \min_h (E)$ is $\sqrt{2}/3$.

For $h < h_{\text{opt}}$ and $h \geq h_{\text{opt}}$, the scale factor $s$ is given by

$$6 \frac{h}{\sqrt{3} + 9 h^2 + 3 h} + 2 \left( \frac{2/3}{h \sqrt{21 + 63 h^2 - 12 \sqrt{3 + 9 h^2} \sqrt{1 + 3}} + 1} \right),$$

respectively. For $h_{\text{opt}}$, $s \approx 0.8990$.

3.1.2 Generalized 3D Hexagonal Grid

For the GH, the 20 closest neighbours are considered. For $h = 1$, this corresponds to the neighbours at distance 1 and $\sqrt{2}$. On this grid, $h_{\text{opt}} \approx 0.618496$.

For $h < h_{\text{opt}}$ and $h \geq h_{\text{opt}}$, the scale factor $s$ giving the optimal weight is

$$h_{\text{opt}} = \min \{ \frac{6}{\sqrt{3} + 9 h^2 + 3 h} + 2 \left( \frac{2/3}{h \sqrt{21 + 63 h^2 - 12 \sqrt{3 + 9 h^2} \sqrt{1 + 3}} + 1} \right) \},$$

respectively. For $h_{\text{opt}}$, $s \approx 0.9136$.

4 Conclusions

The theoretic tools needed for using 3D grids with embedded hexagonal grids for CT reconstruction with DFM and optimal weights for WDTs are presented. The definition of HDFT used in this paper is not more complicated than the DFT for the square grid. The weights obtained in Section 3 might look complicated, but since the formulas are valid for any $h$, the complexity is justified.

We conclude that the complexity for generating and processing images on these non-standard grids is approximately the same as for the standard grid, but the storage efficiency is significantly better.

References


