The completion conjecture in equivariant cohomology

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Consider an RO(G)-graded cohomology theory k_G^*. We shall not insist on a detailed definition; suffice it to say that there is a suspension isomorphism for each real representation of G. The first examples were real and complex equivariant K-theory KO_G^* and K_G^*. The next example was equivariant stable cohomotopy theory π_G^*. There are RO(G)-graded ordinary cohomology theories with coefficients in Mackey functors.

The study of these theories is still in its infancy. They can all be defined for arbitrary compact Lie groups, but we shall restrict our attention to finite groups. When we localize away from the order of G, there are very powerful algebraic devices for the reduction of the calculation of k_G^* to nonequivariant calculations. If we localize at a prime dividing the order of G, there are techniques for reducing calculations to consideration of p-groups contained in G. There are no known general procedures for the calculation of k_G^* at p for p-groups G. Largely for this reason, the reservoir of known calculations is almost empty.

Let A(G) be the Burnside ring of finite G-sets. Then k_G^* takes values in the category of A(G)-modules. For some purposes, this is the main reason for interest in the RO(G)-grading. The assertion is false for Z-graded equivariant cohomology theories which fail to extend to RO(G)-graded theories. Let EG be a free contractible G-CW complex and let $\varepsilon\colon EG \to * = \{pt\}$ be the trivial map. We have an induced homomorphism of A(G)-modules

$$\varepsilon^*\colon k_G^*(*) \longrightarrow k_G^*(EG).$$

(We use unreduced theories until otherwise specified.) The completion conjecture for k_G^* asserts that, on integer gradings, ε^* becomes an isomorphism upon completion with respect to the topology given by the powers of the augmentation ideal of A(G). As we shall shortly make precise, when G is a p-group and the completion conjecture holds for k_G^*, the p-adic completion of k_G^* is computable in nonequivariant terms.

The first theorem in this direction was due to Atiyah [4]; see also Atiyah and Segal [5]. Their results are stated in terms of representation rings but, since G is finite, the Burnside ring gives the same topology.

<u>Theorem 1.</u> The completion conjecture holds for real and complex equivariant K-theory.

However, the completion conjecture certainly fails to hold in general.

Counterexample 2. The completion conjecture fails for ordinary cohomology $H_G^*(?;\underline{Z})$ with coefficients in the constant coefficient system $\underline{Z}$. On integer gradings, $H_G^*(*;\underline{Z}) = H_G^0(*;\underline{Z}) = Z$, whereas $H_G^*(EG;\underline{Z}) = H^*(BG;Z)$, the ordinary integral cohomology of the classifying space BG.

The Segal conjecture is the completion conjecture for equivariant cohomotopy theory. The central step in its proof has been supplied in a beautiful piece of work by Gunnar Carlsson [6].

Theorem 3. The completion conjecture holds for equivariant stable cohomotopy theory.

There is an equivalent nonequivariant reformulation and an interesting implied generalization that I will discuss at the end.

When I first heard about the Segal conjecture, my instinct was that it was unlikely to be true in general. I also felt that it was a much less important problem than the general one of explaining for which theories k_G^* the completion conjecture would or would not hold. However, Carlsson's work not only completed the proof of Theorem 3, it also led to very substantial progress on the general problem. This development is joint work of Jeff Caruso and myself [9] and follows up our simplification of Carlsson's work [8], which was undertaken in hopes of just such a generalization. My understanding of these matters also owes a great deal to joint work and conversations with Jeremy Gunawardena, Gaunce Lewis, Stewart Priddy, and Stefan Waner.

My purpose here is to explain Carlsson's work, our generalization of it, and related matters in conceptual terms, without getting bogged down in details of proofs. None of the steps presents any great difficulty any more, all of the work lying very close to the foundations, but there are quite a few steps. We shall stick to the main line of development, and this means that a great deal of earlier work on the Segal conjecture will go unmentioned. Adams [1] summarized what was known late in 1980.

We shall first explain the force of the completion conjecture and its reduction to a question about p-groups, following May and McClure [19].

For each subgroup H of G, there is an RO(H)-graded theory k_H^* associated to k_G^* (depending not just on H as an abstract group but on the inclusion $H \subset G$). In particular, with $H = e$, there is an associated nonequivariant cohomology theory k^*. Modulo interpretation in terms of restriction

$RO(G) \longrightarrow RO(H)$,

$$k_H^*(Y) = k_G^*(G \times_H Y).$$

The projection $G \times Y \to Y$ induces a map $\pi\colon k_G^*(Y) \to k^*(Y)$ of Z-graded cohomology theories on spaces Y. We say that k_G^* is split if there is a map $\zeta\colon k^*(Y) \to k_G^*(Y)$ of cohomology theories such that $\pi\zeta$ is the identity. We have a notion of an RO(G)-graded ring-valued cohomology theory. If k_G^* is a ring theory, so is each k_H^*. We say that k_G^* is a split ring theory if ζ is a map of ring-valued cohomology theories, and each k_H^* is then also a split ring theory.

We say that k_G^* is of finite type if each $k_H^q(*)$, $q \in Z$ and $H \subset G$, is a finitely generated Abelian group. This ensures that $k_G^\alpha(X)$ is finitely generated for all $\alpha \in RO(G)$ and all finite G-CW complexes X. The most interesting examples are split ring theories of finite type.

The connection between equivariant and nonequivariant cohomology is established by the following observation [19, lemma 12].

Lemma 4. If k_G^* is split and Y is a free G-CW complex, then, on integer gradings, $k_G^*(Y)$ is naturally isomorphic to $k^*(Y/G)$.

Thus the completion conjecture relates $k_G^*(*)$ to $k^*(BG)$. The relevance of the completion conjecture to the general calculation of $k_G^*(X)$ is given by the following result [19, Prop. 15].

Proposition 5. Assume that k_G^* is a split ring theory of finite type and that $\lim^1 k^*(BH^n) = 0$ for each $H \subset G$ (where BH^n denotes the n-skeleton of BH). If the completion conjecture holds for k_H^* for each H, then the projection $EG \times X \to X$ induces an isomorphism

$$\hat{k}_G^\alpha(X) \longrightarrow k_G^\alpha(EG \times X)$$

for all $\alpha \in RO(G)$ and all finite G-CW complexes X, where the left side is the completion of $k_G^\alpha(X)$ with respect to the Burnside ring topology.

Of course, by the lemma, $k_G^q(EG \times X) \cong k^q(EG \times_G X)$ for $q \in Z$.

For clarity of exposition, we shall henceforward restrict attention to integer gradings.

The completed Burnside ring Green functor satisfies induction with respect to the subgroups of G of prime power order. Don't worry if you don't understand the previous sentence. As a matter of pure algebra, it leads to a proof that the Burnside ring rapidly disappears from the picture. See [19, Thm 13 and Prop 14] and also Laitinen [14,15].

Theorem 6. The completion conjecture holds for k_G^* if it holds for k_H^* for all subgroups H of prime power order.

Proposition 7. If G is a p-group and k_G^* is split, then the completion conjecture holds for k_G^* if and only if $\varepsilon^*: k_G^*(*) \to k_G^*(EG)$ induces an isomorphism upon passage to p-adic completion.

Henceforward (until otherwise specified near the end), G is to be a p-group; $\hat{k}$ and cognate symbols will indicate completion at p. To avoid constant repetition of hypotheses, we assume once and for all that all theories k_G^*, given or constructed, are split and of finite type.

We are at the starting point of Carlsson's work, and some preliminary philosophical comments are in order. The stable part of algebraic topology has three main branches: homology and cohomology theory on spaces, homotopy theory on spectra, and infinite loop space theory. Carlsson's preprint [6] was written from the second point of view. Specifically, Carlsson worked in the stable category of G-spectra constructed by Lewis and myself [17]. This was done with my encouragement, and I must apologise to Carlsson for giving him very bad advice. As Caruso and I discovered, the mathematics simplifies considerably when the first point of view is taken, and the changed point of view is crucial to our generalization of Carlsson's work that is the theme of this paper.

Carlsson's theorem asserts that the Segal conjecture holds for all p-groups if it holds for all elementary Abelian p-groups, that is, for all p-groups of the form $(Z_p)^n$. The Segal conjecture was proven by Lin [18] for Z_2, by Gunawardena [13] for Z_p, and by Adams, Gunawardena, and Miller [3] for $(Z_p)^n$ with $n \geq 2$. These authors actually prove the equivalent nonequivariant reformulation of the conjecture to be discussed later. My work with Caruso led to work with Priddy that gives a geodesic proof of the Segal conjecture for $(Z_p)^n$ within Carlsson's context ([20] plus later corrections).

We shall see that a version of the reduction to elementary Abelian groups goes through for all theories k_G^* such that k_* is bounded below, in the sense that $k_q(*) = 0$ for all sufficiently small q. (This hypothesis serves only to ensure the convergence of certain Adams spectral sequences.) While we shall be able to shed some light on the elementary Abelian case, the general picture is still obscure. We shall describe a satisfactory necessary and sufficient condition for the completion conjecture to hold when k_* is cohomologically bounded above, in the sense that $H^q(k) = 0$ for all sufficiently large q, but without this unpleasant hypothesis it seems that "calculation is the way to the truth". Here k denotes the spectrum which represents the nonequivariant theory k^*.

We need some definitions to give content to the discussion. Let U be a countably infinite dimensional real G-inner product space which contains infinitely many copies of each irreducible representation of G. We take $U = R^\infty \oplus U'$, where U' contains no copies of the trivial representation and so fix $R^s \subset U$, $s \geq 0$. By an indexing G-space, we understand a finite dimensional G-inner product subspace of U. We assume given a G-prespectrum k_G, namely a collection of based G-spaces $k_G V$ for indexing G-spaces V and based G-maps $\sigma\colon \Sigma^{W-V} k_G V \to k_G W$ for $V \subset W$; here $W-V$ denotes the orthogonal complement of V in W. As usual, $\Sigma^V X = X \wedge S^V$, where S^V is the 1-point compactification of V; similarly, $\Omega^V X$ is the G-space of based maps $S^V \to X$. We require technical conditions on the spaces $k_G V$ and maps σ, but these result in no loss of generality and need not concern us here. For based G-CW complexes X and Y and for an integer q, write $q = r-s$, where $r \geq 0$, $s \geq 0$, and $r = 0$ or $s = 0$ (to avoid separate cases) and define

$$k_q^G(X;Y) = [\Sigma^r X, \operatorname*{colim}_{V \supset R^s} \Omega^{V-\mathbb{R}^s}(Y \wedge k_G V)]_G.$$

This is the Z-graded bitheory associated to k_G. It specializes to

$$k_q^G(Y) = k_q^G(S^0;Y) \quad \text{and} \quad k_G^q(X) = k_{-q}^G(X;S^0).$$

These are reduced theories, to which we switch henceforward. To define stable homotopy and cohomotopy, we take k_G to be the sphere G-prespectrum; its $V^{\underline{th}}$ space is S^V, and $\sigma\colon \Sigma^{W-V} S^V \to S^W$ is the evident identification.

Carlsson's first step was joint work with Cusick [7] and involved reduction from a problem in cohomotopy to a more tractable problem in homotopy. Independently and concurrently, Caruso and Waner arrived at an extremely elegant way of carrying out essentially the same reduction. They had observed earlier [10] that a model for EG could be obtained by taking the union over V of certain smooth compact G-manifolds with boundary $M(V)$ embedded with codimension zero in V. The essential property of $M(V)$ is that $M(V)/\partial M(V)$ is equivalent to S^V/T^V, where T^V denotes the singular set of S^V (namely the set of points with non-trivial isotropy subgroup). Using this model and an easy Spanier-Whitehead duality argument, one finds that $\varepsilon\colon \hat{k}_G^*(S^0) \to \hat{k}_G^*(EG^+)$ can be identified with a certain natural map of inverse limits

$$\lim \hat{k}_*^G(S^V;S^V) \longrightarrow \lim \hat{k}_*^G(S^V;S^V/T^V).$$

A simple cofibration argument then gives the following conclusion.

Proposition 8. The completion conjecture holds for k_G^* if and only if

$$\lim \hat{k}_*^G(S^V;T^V) = 0.$$

For Carlsson's second step, it is convenient to introduce the notation

$$\hat{k}_q^G(Y\wedge \underline{W}) = \lim_j \hat{k}_q^G(S^{jW};Y).$$

Here Y is a G-CW complex, W is a representation of G, and the inverse limit is taken over the homomorphisms

$$\hat{k}_q^G(S^{(j+1)W};Y) \xrightarrow{(1\wedge e)_*} \hat{k}_q^G(S^{jW}\wedge S^W;Y\wedge S^W) \cong \hat{k}_*^G(S^{jW};Y),$$

where $e: S^0 \to S^W$ is the evident inclusion. The case $Y = S^0$ is particularly important. By clever use of interchange of limits applied to bi-indexed limits involving smash products, Carlsson uses the criterion of the previous proposition to obtain the following conclusion.

Proposition 9. If the completion conjecture holds for k_H^* for all subgroups H of G, then $\hat{k}_*^G(\underline{W}) = 0$ for all $W \neq 0$. Conversely, if the completion conjecture holds for k_H^* for all proper subgroups H of G and $\hat{k}_*^G(\underline{W}) = 0$ for any one $W \neq 0$ such that $W^G = \{0\}$, then the completion conjecture holds for k_G^*.

Carlsson fixes a good choice of W with $W^G = 0$, which we shall call Z. Let $\bar{G}$ be the elementary Abelianization $G/[G,G] \otimes Z_p$ of G and take Z to be the pullback to G of the reduced regular representation of $\bar{G}$. The restriction of Z to any proper subgroup H contains a copy of the trivial representation, and a theorem of Serre [29] (or its consequence due to Quillen and Venkov [26]) implies that the Euler class $\alpha(Z)$ is nilpotent if G is not elementary Abelian. Proposition 9 has the following consequence.

Corollary 10. Assume that the completion conjecture holds for k_H^* for all proper subgroups H of G. Then the completion conjecture holds for k_G^* if and only if $\hat{k}_*^G(\underline{Z}) = 0$.

At this point, Carlsson introduces the key simplification. Let $\widetilde{EG}$ be the unreduced suspension of EG with one of its cone points as basepoint. Equivalently, $\widetilde{EG}$ is the cofibre of the evident map $EG^+ \to S^0$, and one obtains the fundamental long exact sequence

$$(*)\quad \cdots \longrightarrow \hat{k}^G_q(EG^+ \wedge \underline{Z}) \longrightarrow \hat{k}^G_q(\underline{Z}) \longrightarrow \hat{k}^G_q(\tilde{E}G \wedge \underline{Z}) \xrightarrow{\partial} \hat{k}^G_{q-1}(EG^+ \wedge \underline{Z}) \longrightarrow \cdots .$$

Carlsson assumes inductively that the Segal conjecture holds for all proper subquotients of G and proves that both $\hat{\pi}^G_*(\tilde{E}G \wedge \underline{Z}) = 0$ and $\hat{\pi}^G_*(EG^+ \wedge \underline{Z}) = 0$ if G is not elementary Abelian. This implies that $\hat{\pi}^G_*(\underline{Z}) = 0$ and thus that the Segal conjecture holds for G. Carlsson observes that his vanishing theorems fail when G is elementary Abelian and suggests the possibility of a direct proof that the connecting homomorphism ∂ is an isomorphism in this case. Priddy and I provide such a proof.

The generalization to k^*_G requires us to introduce a bitheory k^J_* associated to any subquotient J of G. Here $J = N/H$, where H is normal in N. For J-CW complexes X and Y and $q = r-s$ (as above), we define

$$k^J_q(X;Y) = [\Sigma^r X, \operatorname*{colim}_{V \supset R^s} \Omega^{V^H - \mathbb{R}^s}(Y \wedge (k_G V)^H)]_J$$

and specialize as before to obtain $k^J_*(Y)$ and $k^*_J(X)$. Everything said so far works equally well with k^G_* replaced by k^J_*. It is vital to recognize that, in general, k^J_* depends on H and N and not just on J. In particular, we write h_* for the nonequivariant bitheory obtained by taking $H = N$ (and thus $J = e$) in the above definition; we write k_* and $k^\natural_*$ for the nonequivariant bitheories so obtained from $H = e$ and $H = G$, respectively. An example may clarify the definition.

Example 11. For a G-space X, let $S_G X$ be the G-prespectrum with V^{th} space $\Sigma^V X$ and with $\sigma\colon \Sigma^{W-V}\Sigma^V X \longrightarrow \Sigma^W X$ the evident identification. The theory represented by $S_G X$ is split if there exists a map $\zeta\colon X \to X^G$ whose composite with the inclusion $X^G \to X$ is homotopic to the identity. Since every representation of J occurs in V^H for some indexing G-space V, we conclude by cofinality that $(S_G X)^J_*$ is just a copy of the bitheory $(S_J X^H)_*$. Note the particular case $X = S^0$.

Now Carlsson's vanishing theorems generalize as follows.

Theorem 12. If G is not elementary Abelian and the completion conjecture holds for k^*_J for all proper subquotients J of G, then $\hat{k}^G_*(\tilde{E}G \wedge \underline{Z}) = 0$.

Theorem 13. If G is not elementary Abelian and k_* is bounded below, then $k^G_*(EG^+ \wedge \underline{Z}) = 0$.

We have stated these differently since the proof of the second is direct rather than inductive. The same results are valid if we start with some k_*^J as ambient theory, and we deduce the following generalization of Carlsson's theorem by induction. Remember that all theories in sight are assumed to be split and of finite type.

Theorem 14. If G is not elementary Abelian, all h_* are bounded below, and the completion conjecture holds for k_J^* for all elementary Abelian subquotients of G, then the completion conjecture holds for k_G^* and all other k_J^*.

Before discussing the proofs of Theorems 12 and 13, we describe what happens in the elementary Abelian case. Let M_n denote the free $\hat{Z}_p$-module on $p^{n(n-1)/2}$ generators, where $\hat{Z}_p$ denotes the p-adic integers; we take tensor products over $\hat{Z}_p$ below. In the case of stable homotopy, the following theorem is more or less implicit in Carlsson's work. The general case is due to Caruso and myself.

Theorem 15. If $G = (Z_p)^n$ and the completion conjecture holds for k_J^* for all proper subquotients J of G, then

$$\hat{k}_*^G(\tilde{E}G \wedge \underline{Z}) \cong M_n \otimes \Sigma^{1-n}\, \hat{k}_*'(S^0).$$

The following theorem is due to Priddy and myself, although most of the work is in an Ext calculation due to others and discussed below.

Theorem 16. If $G = (Z_p)^n$ and k_* is bounded below and cohomologically bounded above, then

$$\hat{k}_*^G(EG^+ \wedge \underline{Z}) \cong M_n \otimes \Sigma^{-n}\, \hat{k}_* S^0.$$

In the absence of the bounded above hypothesis, there is an inverse limit of Adams spectral sequences such that

$$E_2 = \mathrm{Ext}_A(H^*(BG)[\alpha(Z)^{-1}] \otimes H^*(k), Z_p)$$

and $\{E_r\}$ converges to $\hat{k}_*^G(EG^+ \wedge \underline{Z})$.

Here the Euler class $\alpha(Z) \in H^{2(p^n-1)}(BG)$ has an obvious explicit description.

At this point, the virtues of naturality manifest themselves. Suppose that k_*^G is a split ring theory with unit $e_*\colon \pi_*^G \to k_*^G$. We have the following commutative square.

$$\begin{array}{ccc} \hat{\pi}^G_*(\tilde{E}G \wedge \underline{Z}) & \xrightarrow{\ \partial\ } & \hat{\pi}^G_*(EG^+ \wedge \underline{Z}) \\ \downarrow{\scriptstyle e_*} & & \downarrow{\scriptstyle e_*} \\ \hat{k}^G_*(\tilde{E}G \wedge \underline{Z}) & \xrightarrow{\ \partial\ } & \hat{k}^G_*(EG^+ \wedge \underline{Z}) \end{array}$$

To deduce the Segal conjecture for $G = (Z_p)^n$, it suffices to find a theory k^*_G for which the completion conjecture holds and $e_*: \hat{\pi}_0(S^0) \to \hat{k}'_0(S^0)$ is non-trivial mod p. Indeed, ∂ on the top is a morphism of $\hat{\pi}_0(S^0)$-modules whose domain and target are each freely generated by a copy of the free $\hat{Z}_p$-module M_n. Given k^*_G, Corollary 10, a bit of calculation along the lines of Theorem 15, and a chase of the diagram show that ∂ restricts to an isomorphism between the respective copies of M_n and is therefore an isomorphism. In [20], we thought that equivariant K-theory would do for k^*_G, but Costenoble has since proven the astonishing fact that in this case $\hat{k}'_*(S^0) = 0$. A choice which does work is specified by $k^*_G(X) = H^*_G(EG \times X; \underline{Z}_p)$. The completion conjecture holds trivially for theories so constructed by crossing with EG.

With k^G_* a split ring theory, the exact sequence $(*)$ is one of $\hat{k}_*(S^0)$-modules and the isomorphisms of Theorems 15 and 16 are isomorphisms of $\hat{k}_*(S^0)$-modules. The general case of the diagram and the now established truth of the Segal conjecture lead to the second part of the following result, the first part being evident.

<u>Theorem 17</u>. If $G = (Z_p)^n$, k_* is bounded below and cohomologically bounded above, and the completion conjecture holds for k^*_J for all proper subquotients J of G, then the following conclusions hold.

(i) If $\hat{k}_*(S^0)$ and $\hat{k}^!_*(S^0)$ are not isomorphic as $\hat{Z}_p$-modules, then the completion conjecture fails for k^*_G.

(ii) If k^G_* is a split ring theory and $\hat{k}_*(S^0)$ and $\hat{k}^!_*(S^0)$ are isomorphic as $\hat{k}_*(S^0)$-modules, then the completion conjecture holds for k^*_G.

By Example 11 and part (i), we conclude that the completion conjecture generally fails for the theories $(S_G X)^*$ when X is a finite G-CW complex with non-trivial action by G. We shall later point out an important class of infinite G-CW complexes for which the completion conjecture holds. In these examples, $\hat{k}_*(S^0)$ and $\hat{k}^!_*(S^0)$ are not isomorphic, so we cannot expect the isomorphism of Theorem 16 to hold without the bounded above hypothesis.

We have another rather startling example of the failure of the isomorphism of Theorem 16. For any k^*_G, there is an associated connective G-cohomology theory for which all of the associated nonequivariant theories h^* are connective, $h_q(S^0) = 0$ for $q < 0$. When $G = (Z_p)^n$ and k_G^* is connective, $\hat{k}_q^G(\tilde{E}G \wedge \underline{Z}) = 0$ for $q < 1-n$.

Counterexample 18. The completion conjecture fails for real and complex connective equivariant K-theory kO_G^* and kU_G^* when $G = Z_2$. Here the domain of ∂ is zero in negative degrees, but Davis and Mahowald [11] have calculated its target groups and shown that they are periodic.

The completion conjecture for equivariant cobordism theories is under investigation.

We turn to the proofs of the four calculational theorems above. The starting point for Theorems 12 and 15 is Carlsson's observation that elementary obstruction theory implies a natural isomorphism

$$[X, \tilde{E}G \wedge Y]_G = [TX, Y]_G \ ,$$

where X is a finite G-CW complex, TX is its singular set, and Y is any G-space. Thus $\hat{k}_q^G(\tilde{E}G \wedge \underline{Z})$ is the inverse limit over j of the p-adic completions of the colimits over $V \supset \mathbb{R}^s$ of the groups

$$(**) \qquad [T(\Sigma^r S^{jZ} \wedge S^{V-\mathbb{R}^s}), k_G V]_G$$

The hard work in Carlsson's preprint is his analysis of TX. Following up Gunawardena's insistence that Quillen's work on posets ought to be relevant, Caruso and I found an almost trivial way of carrying out essentially the same analysis.

Let $\mathcal{A}$ be the poset of non-trivial elementary Abelian subgroups of G. As observed by Quillen [25], $B\mathcal{A}$ is contractible. In fact, $\mathcal{A}$ is a G-category via conjugation of subgroups and $B\mathcal{A}$ is G-contractible. We construct a topological G-category $\mathcal{A}[X]$ by parametrizing $\mathcal{A}$ by fixed points of X; the objects of $\mathcal{A}[X]$ are pairs (A,x), where A is an elementary Abelian subgroup of G and $x \in X^A$. There is an evident projection functor from $\mathcal{A}[X]$ to TX (regarded as a trivial topological G-category), and the induced map $B\mathcal{A}[X] \to TX$ is a G-homotopy equivalence by a simple application of Quillen's theorem A [24] to fixed point categories. It is convenient to factor out the contractible G-space $B\mathcal{A}[*]$ since it maps trivially to TX. Thus set $AX = B\mathcal{A}[X]/B\mathcal{A}[*]$. We have a natural G-homotopy equivalence $AX \to TX$. Now AX comes with an evident natural finite filtration. If we set $B_m X = F_m AX/F_{m-1}AX$, then we find by immediate inspection of definitions that

$$B_m X = \bigvee_{[\omega]} G^+ \wedge_{N(\omega)} \Sigma^m X^{A(\omega)}.$$

Here $[\omega]$ runs through the G-orbits of strictly ascending chains $\omega = (A_0 \subset \cdots \subset A_m)$

of non-trivial elementary Abelian subgroups of G, $A(\omega) = A_m$, and $N(\omega)$ is the isotropy group of ω under the action of G (of which $A(\omega)$ is a normal subgroup).

We plug this analysis into the description of $\hat{k}^G_q(\tilde{E}G \wedge \underline{Z})$ above. We may replace the functor T by the functor A in (**) and then pass to colimits, completion, and inverse limits. The filtration of A gives a finite sequence of long exact sequences in which the third terms come by substitution of B_m for T in (**). Inspection of definitions shows that the $[\omega]^{\underline{th}}$ wedge summand of the functor B_m contributes a copy of $\hat{k}^{J(\omega)}_{q+m}(\underline{Z}^{A(\omega)})$, where $J(\omega) = N(\omega)/A(\omega)$. If $J(\omega) \neq e$, then these groups are zero by Proposition 9 and the induction hypothesis. If $J(\omega) = e$ and $A(\omega) \neq G$, then these groups are zero since $Z^{A(\omega)}$ contains a copy of the trivial representation, so that the maps $e: S^0 \to S^{Z^{A(\omega)}}$ which give the inverse limit system are nonequivariantly null homotopic. Theorem 12 follows immediately. For Theorem 15, we are still left with those ω such that $A(\omega) = G = (Z_p)^n$. Since $Z^G = \{0\}$, a check of definitions shows that the $\omega^{\underline{th}}$ wedge summand contributes $\hat{k}'_{q+m}(S^0)$. Here we view our finite sequence of long exact sequences as an exact couple and obtain a spectral sequence converging to $\hat{k}_*(\tilde{E}G \wedge \underline{Z})$. Its E^1-term is the direct sum of a (reindexed) copy of $k^{\wedge}_*(S^0)$ for each ω with $A(\omega) = G$. Recall that the Tits building Tits(G) is the classifying space of the poset of non-trivial proper subgroups of G. We regard ω as a chain of Tits(G) by forgetting $A(\omega) = G$ and obtain an isomorphism

$$E^1_{m,q-m} = C^+_{m-1} \otimes k'_{q+m}(S^0),$$

where C^+_* denotes the augmented simplicial chains of Tits(G). The isomorphism carries d^1 to $d \otimes 1$ and Theorem 15 now follows from standard facts about Tits buildings [25,32].

The proofs of Theorems 13 and 16 are based on the existence of certain (nonequivariant) spectra with good properties.

<u>Theorem 19</u>. There exist spectra BG^{-V} and maps $f: BG^{-W} \to BG^{-V}$ for $V \subset W$ which satisfy the following properties.

(1) $k_*(BG^{-V})$ is isomorphic to $k^G_*(S^V;EG^+)$.

(2) The following diagram commutes.

$$\begin{array}{ccc} k_*(BG^{-W}) & \xrightarrow{\quad f_* \quad} & k_*(BG^{-V}) \\ \cong & & \cong \\ k^G_*(S^W;EG^+) & \xrightarrow{(1\wedge e)_*} k^G_*(S^V \wedge S^{W-V};EG^+ \wedge S^{W-V}) \cong & k^G_*(S^V;EG^+) \end{array}$$

(3) $H^*(BG^{-V})$ is a free $H^*(BG)$-module on one generator ι_V of degree $-\dim V$.

(4) $f^*: H^*(BG^{-V}) \to H^*(BG^{-W})$ is the morphism of $H^*(BG)$-modules determined by $f^*(\iota_V) = \alpha(W-V)\iota_W$, where $\alpha(W-V)$ is the Euler class of $W-V$.

Carlsson constructed such spectra by a kind of double dualization argument based on certain assumed facts about equivariant Spanier-Whitehead duality, proofs of which are given by Adams [2] and by Lewis and myself [17] in different stable contexts. A more conceptual, but also more technically difficult, construction is due to Lewis, Steinberger, and myself. Some years ago, we used our stable category of G-spectra to construct a spectrum level generalization of the familiar twisted half-smash product construction on spaces. The desired spectra may be specified by

$$BG^{-V} = EG \ltimes_G S^{-V}.$$

The required properties are then immediate consequences of spectrum level generalizations of familiar space level properties of twisted half smash products. By use of equivariant Thom spectra, Lewis and I have shown that BG^{-V} can also be described as the Thom spectrum of the virtual bundle $-V$ over BG. With this description, the cohomological properties in Theorem 19 are consequences of the Thom isomorphism.

Properties (1) and (2) of Theorem 19 give that

$$\hat{k}^G_*(EG^+ \wedge \underline{Z}) = \lim \hat{k}_*(BG^{-jZ}).$$

Properties (3) and (4) give that

$$\operatorname{colim} H^*(BG^{-jZ}) = H^*(BG)[L^{-1}],\ L = \alpha(Z).$$

It is true quite generally that passage to inverse limits from an inverse sequence of convergent Adams spectral sequences gives a convergent spectral sequence [20]. In particular, passage to inverse limits from the Adams spectral sequences of the spectra $BG^{-jZ} \wedge k$, where k represents k^*, gives a spectral sequence converging from

$$E_2 = \operatorname{Ext}_A(H^*(BG)[L^{-1}] \otimes H^*(k), Z_p)$$

to $\hat{k}^G_*(EG^+ \wedge \underline{Z})$. For Theorem 13, the localization is zero by the nilpotency of $\alpha(Z)$, hence $E_2 = 0$ and thus $\hat{k}^G_*(EG^+ \wedge \underline{Z}) = 0$. The first statement of Theorem 16 follows by convergence and an easy comparison of spectral sequences argument from the following homological calculation of Adams, Gunawardena, and Miller.

Theorem 20. If $G = (Z_p)^n$ and K is an A-module which is bounded above (and not in general otherwise), then

$$\mathrm{Ext}_A(H^*(BG)[L^{-1}] \otimes K, Z_p) \cong \mathrm{Ext}_A(N_n \otimes K, Z_p)$$

where N_n is a free Z_p-module on $p^{n(n-1)/2}$ generators which is concentrated in degree $-n$ and has trivial A-action.

The general case follows from the case $K = Z_p$ and the fact that $\mathrm{Ext}_A(Q, Z_p) = 0$ implies $\mathrm{Ext}_A(Q \otimes K, Z_p) = 0$ when K is bounded above. We sketch very briefly the key steps in the proof for $K = Z_p$. Singer (and Li) [30,31] introduced a basic construction R_+ on A-modules. For an A-module M, there is an augmentation $\varepsilon: R_+M \to M$. Adams, Gunawardena, and Miller [3,13] proved that $\mathrm{Ext}_A(\varepsilon, 1)$ is always an isomorphism. It follows inductively that there is an Ext isomorphism $(\Sigma^{-1}R_+)^n(Z_p) \to \Sigma^{-n}Z_p$. Singer and Li [30,31] proved that there is an isomorphism of A-modules

$$(\Sigma^{-1}R_+)(Z_p) \cong (H^*(BG)[L^{-1}])^{B_n}.$$

Here the general linear group $GL(n, Z_p)$ acts on the localization, and B_n denotes the Borel subgroup of upper triangular matrices. To obtain Theorem 20, one climbs up from the invariants to the entire localization to obtain an Ext isomorphism $H^*(BG)[L^{-1}] \to N_n$. This last step is carried out by direct inductive calculation up a chain of parabolic subgroups by Adams, Gunawardena, and Miller [3]. A conceptual, but less elementary, argument which highlights the role played by the Steinberg module is given by Priddy and Wilkerson [23]. (I have oversimplified slightly; when $p > 2$, both [3] and [23] replace R_+ by Gunawardena's enlarged analog [13] which is related to $H^*(BZ_p)$ as R_+ is to $H^*(B\Sigma_p)$.)

To close, we return to the Segal conjecture and describe its nonequivariant equivalent and an implied generalization, following Lewis, May, and McClure [16]. We return to general finite groups G. The coefficient groups $\pi_*^G(S^0)$ have been computed by several people [28,12]. The answer is

$$\text{(A)} \qquad \pi_*^G(S^0) \cong \sum_{(H)} \pi_*(BWH^+).$$

Here the sum ranges over conjugacy classes (H) of subgroups H and $WH = NH/H$, where NH is the normalizer of H in G. It is natural to ask for an interpretation of the Segal conjecture in terms of this isomorphism. The connection is best explained in terms of spectra and G-spectra [17]. The groups $\pi_G^*(S^0)$ are the homotopy groups of the fixed point spectrum $(S_G)^G$ of the sphere G-spectrum S_G. The groups $\pi_G^*(EG^+)$ are the homotopy groups of the fixed point spectrum of the dual G-spectrum $D_G(EG^+)$. Of course, $D_G(S^0) = S_G$, and $\varepsilon^+: EG^+ \to S^0$ induces a map of spectra

(B) $$\varepsilon^{*}: (S_G)^G \longrightarrow D_G(EG^+)^G \simeq D(BG^+),$$

where the equivalence comes from Lemma 4. The Segal conjecture may be viewed as a statement about this map. In particular, when G is a p-group, this map induces an equivalence upon p-adic completion. The isomorphism (A) comes from an equivalence

(C) $$\xi: \bigvee_{(H)} \Sigma^{\infty}(BWH^+) \longrightarrow (S_G)^G.$$

Tom Dieck's proof of (A) in [12] leads to an explicit description of ξ in terms of which $\varepsilon^{*} \circ \xi$ can be evaluated; see [16, Thms 1 and 8].

Observe that WH is the group of automorphisms of the G-set G/H, so that $G \times WH$ acts on G/H.

<u>Theorem 21</u>. The $H^{\underline{th}}$ component of the composite

$$\varepsilon^{*} \circ \xi: \bigvee_{(H)} \Sigma^{\infty}(BWH^+) \longrightarrow D(BG^+)$$

is the adjoint of the element $\tau(1) \in \pi^0((BG \times BWH)^+)$, where τ is the transfer associated to the natural cover

$$(G/H) \times_{G\times WH}(EG \times EWH) \longrightarrow BG \times BWH.$$

If G is a p-group, then $\varepsilon^{*} \circ \xi$ induces an equivalence upon completion at p.

In view of Theorem 6 and Proposition 7, the last statement is in fact equivalent to the Segal conjecture, and it is this formulation that was studied in special cases in the papers [18, 13, 27, 3, 21], among others.

Thus the Segal conjecture gives a description of the function spectrum

$$D(BG^+) = F(BG^+,S) = F(BG^+,\Sigma^{\infty}S^0).$$

It is natural to ask more generally if there is an analogous description of the function spectrum

$$F(BG^+,\Sigma^{\infty}B\Pi^+) \simeq F(\Sigma^{\infty}BG^+,\Sigma^{\infty}B\Pi^+)$$

for finite groups G and Π. The question was raised by Adams and Miller and answered when G and Π are elementary Abelian by Adams, Gunawardena, and Miller [3]. Lewis, McClure, and I [16] proved that the Segal conjecture implies an answer for arbitrary G and Π.

Let $B(G,\Pi)$ be the classifying G-space for principal (G,Π)-bundles. We have the G-prespectrum $S_G B(G,\Pi)^+$ of Example 11. Its cohomology theory is always split and is a split ring theory if Π is Abelian. The following is the main result of [16].

Theorem 22. The Segal conjecture implies the completion conjecture for the G-cohomology theories $(S_G B(G,\Pi)^+)^*$ for all finite groups G and Π.

Of course, the Segal conjecture itself is the case $\Pi = e$.

Again, the coefficient groups have been computed by tom Dieck [12]; indeed, he computes $\pi_*^G(X^+)$ in nonequivariant terms for any G-space X. When $X = B(G,\Pi)$, his result leads to the following description; see [16, Thm 1 and Prop 5].

$$\text{(A')} \qquad \pi_*^G B(G,\Pi)^+ \cong \sum_{(H)} \sum_{[(\rho)]} \pi_*(BW\rho^+).$$

Here the sums run over conjugacy classes (H) of subgroups H of G and WH-orbits $[(\rho)]$ of Π-conjugacy classes (ρ) of homomorphisms $\rho: H \to \Pi$; the groups are

$$W\rho = N_{G\times\Pi}(\Delta\rho)/\Delta\rho, \quad \text{where} \quad \Delta\rho = \{(h,\rho(h))\,|\,h \in H\} \subset G \times \Pi.$$

Let Σ_G^∞ denote the suspension G-spectrum functor; $\Sigma_G^\infty X$ is the G-spectrum associated to the G-prespectrum $S_G X$, and Theorem 22 may be viewed as a statement about the map of fixed point spectra

$$\text{(B')} \qquad \varepsilon^*: [\Sigma_G^\infty B(G,\Pi)^+]^G \longrightarrow F(EG^+, \Sigma_G^\infty B(G,\Pi)^+)^G \simeq F(BG^+, \Sigma^\infty B\Pi^+),$$

where the equivalence again comes from Lemma 4. The isomorphism (A') comes from an equivalence

$$\text{(C')} \qquad \xi: \bigvee_{(H)} \bigvee_{[(\rho)]} \Sigma^\infty(BW\rho^+) \longrightarrow [\Sigma_G^\infty B(G,\Pi)^+]^G.$$

Theorem 21 generalizes as follows. See [16, Thms 1 and 8].

Theorem 23. The ρ^{th} component of the composite

$$\varepsilon^* \circ \xi: \bigvee_{(H)} \bigvee_{[(\rho)]} \Sigma^\infty(BW\rho^+) \longrightarrow F(BG^+, \Sigma^\infty B\Pi^+)$$

is the adjoint of the following composite.

$$\Sigma^\infty(BG \times BW\rho)^+ \xrightarrow{\ \tau\ } \Sigma^\infty(B\rho^+) \xrightarrow{\ \Sigma^\infty\mu^+\ } \Sigma^\infty(B\Pi^+)$$

Here $B\rho = E\rho/\Pi$, where $E\rho = [(G \times \Pi)/\Delta\rho] \times_{G\times W\rho} (EG \times EW\rho)$, μ is the classifying map of the Π-bundle $E\rho \to B\rho$, and τ is the transfer associated to the cover

$B\rho \to BG \times BW\rho$. If G is a p-group, then $\varepsilon^* \circ \xi$ induces an equivalence upon completion at p.

On the π_0-level, Theorems 22 and 23 lead to a complete description of the group of stable maps $[\Sigma^\infty BG^+, \Sigma^\infty B\Pi^+]$ in terms of purely algebraic Burnside ring level information. Let $A(G,\Pi)$ be the Grothendieck group of Π-free finite $(G \times \Pi)$-sets. It is free Abelian on the set of transitive $(G \times \Pi)$-sets $S = G \times \Pi/\Delta\rho$, $\rho: H \to \Pi$, appearing in the previous theorem, and we associate to S the stable map

$$\alpha(S): \Sigma^\infty(BG^+) \xrightarrow{\ \tau\ } \Sigma^\infty(BH^+) \xrightarrow{\Sigma^\infty(B\rho^+)} \Sigma^\infty(B\Pi^+).$$

If $\iota: BG \to BG \times BW\rho$ is obtained by choosing a basepoint in $BW\rho$, then $\alpha(S) = \Sigma^\infty\mu^+ \circ \tau \circ \Sigma^\infty\iota^+$ by an easy verification. Observe that $A(G,\Pi)$ is an $A(G)$-module and let $\hat{A}(G,\Pi)$ be its completion with respect to the topology given by the augmentation ideal of $A(G)$.

Corollary 24. There is a natural isomorphism

$$\alpha: \hat{A}(G,\Pi) \longrightarrow [\Sigma^\infty BG^+, \Sigma^\infty B\Pi^+].$$

When G is a p-group, we may use completion at p provided we also complete the right-hand side. As observed by Nishida [22], this has the following striking (and easy) consequence.

Corollary 25. If G and Π are finite groups such that BG and $B\Pi$ are stably p-equivalent, then G and Π have isomorphic p-Sylow subgroups.

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