Non-commutative fuzzy structures and pairs of weak negations

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Abstract

Weak pseudo BL-algebras (WPBL-algebras) are non-commutative fuzzy structures which arise from pseudo-t-norms (i.e. the non-commutative versions of triangular norms). In this paper, we study the pairs of weak negations on WPBL-algebras, extending the case of weak negations on Esteva–Godo MTL-algebras. A geometrical characterization of the pairs of weak negations in bounded chains is provided.

Our main result characterizes the pairs of weak negations compatible with the multiplication of totally ordered WPBL-algebras giving a way to obtain new examples of WPBL-algebras. We use this to identify all the compatible pairs of weak negations on finite MV-chains.

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1. Introduction

The notion of triangular norm (t-norm) (i.e. a binary operation on [0, 1] that is associative, commutative, with identity 1 and non-decreasing in both arguments) is central in fuzzy logic and set theory, modeling a general form of propositional conjunction. The three most relevant fuzzy conjunctions on [0, 1] (Łukasiewicz, Gödel and product) are t-norms and, in addition, are continuous. Continuous t-norms, combining some very general and natural properties of logical conjunction with continuous variation of the truth degree that is specific to fuzzy logic (see [2]), were taken in [12] as starting point in the development of the so-called basic logic (BL), the tightest generalization of

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Łukasiewicz, Gödel and product logics. BL-algebras [12], abstract, algebraic forms of continuous t-norms, play an important role (similar to the one of Boolean algebras for classical logic) in the study of fuzzy logic.

Recently, the continuity hypothesis was weakened, by replacing it with a condition that is necessary and sufficient for a t-norm to have a residuum (implication)—namely, left continuity, obtaining, as corresponding logic the monoidal t-norm-based logic (MTL) [6]. Consequently, MTL-algebras, generalizations of BL-algebras by dropping down the divisibility condition, were considered. Giving examples of non-continuous left-continuous t-norms was a very challenging problem, and the papers [4,7,10,15,17] provide a whole variety of such specific t-norms. Among these papers, [4] introduces weak negations as a way of constructing new left-continuous t-norms from existing ones. The structure of weak negations is studied extensively. After a geometrical characterization of weak negations on the unit interval \([0, 1]\), there are given necessary and sufficient conditions for a weak negation being compatible with a left-continuous t-norm. Also, the converse step is taken: there are studied the left-continuous t-norms which are compatible with a given weak negation on \([0, 1]\). The paper [6], which is mainly concerned in developing the MTL, also generalizes some results from [4] to the more abstract level of totally ordered sets and MTL-algebras.

All the above-mentioned work was done on commutative fuzzy logic, that is considering the conjunction to be commutative. On the other hand, fuzzy logical systems with non-commutative conjunction have been quite studied lately. An important “non-commutative success” was the generalization of Mundici’s result about the categorical equivalence between MV-algebras (structures corresponding to Łukasiewicz logic [3]) and commutative l-groups with strong unit [19] (which connected “truth degree” with “measurability”) to the non-commutative case [5], result that imposed pseudo-MV-algebras [11] as viable truth structures for non-commutative fuzzy logic and also provided plenty of examples of such structures, with the help of the very well-developed theory of l-groups. A logic (and also a completeness theorem) corresponding to pseudo-MV-algebras is given in [18]. The normal step to take was the generalization of BL to the non-commutative case, problem that is discussed in [13,14]. The corresponding truth structures (weak pseudo-BL-algebras, or pseudo-MTL-algebras) (in particular, on \([0, 1]\), we have the left-continuous pseudo t-norms, which are t-norms without the commutativity condition) do not exhibit yet a wide variety of specific examples. However, as this paper will show, the technique from [4] and [6] of producing new structures using compatible weak negations (see also [2]) works with undiminished power if we drop commutativity and thus use, instead of one weak negation, what we shall call here a pair of weak negations.

This paper mainly offers a tool of producing non-commutative fuzzy structures—the notion of compatible pair of weak negations. A difference from the commutative case is that a left-continuous pseudo-t-norm \(*\) on \([0, 1]\) induces two residua, then two negations (see [9]). Hence, if we want to extend the situation of [4] to the non-commutative case, then we must consider pairs of negations.

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2 A more proper statement is that BL-algebras generalize the residuated structures on \([0, 1]\) induced by continuous t-norms.

3 The same structures were independently introduced in [9] under the name of weak BL-algebras.

4 The importance of replacing continuity by left-continuity becomes even bigger when we deal with pseudo-t-norms instead of t-norms—it is known that all the continuous pseudo-t-norms are commutative, so we do not have continuous non-commutative conjunctions on \([0, 1]\).
A characterization of the pairs of weak negations compatible with a pseudo-t-norm is provided, but the framework is (in the style of [6]) more general—the algebraic setting given by weak pseudo-BL-algebras (WPBL-algebras for short), structures corresponding to pseudo-t-norms [9]. Not only that this is a more general and more elegant approach for treating the pairs of weak negations, but we are able to analyze some interesting cases of finite WPBL-algebras.

In Section 2, we state some basic facts about these structures.

Section 3 is introducing the notion of pair of weak negations on a bounded partially ordered set. We give some necessary and sufficient conditions for a pair of functions to be a pair of weak negations. The geometrical characterization of pairs of weak negations on a bounded chain (Proposition 3.3) generalizes the corresponding results from [4,6], making in addition the “intuitive meaning” of the symmetry with respect to the identity function (already pointed out in [4]) formally explicit by means of converse graphs (relations).

The main theorem of Section 4 characterizes the pairs of weak negations compatible with the multiplication in a WPBL-algebra.

The above-mentioned theorem turns out to be a useful tool for treating pairs of compatible weak negations on some particular classes of WPBL-algebras—Section 5 determines the pairs of weak negations on the MV-algebras $L_{m+1}$, turning out that all of them are just weak negations. Interestingly, unlike in the case of the MV-algebra $[0,1]$, where we can find only one compatible weak negation (the negation itself) [4], in $L_{m+1}$ their number is $m−1$.

In Section 6, we consider some examples of finite WPBL-algebras and show that generating new non-commutative structures by using compatible pairs of weak negations also works when the starting structure is commutative.

2. Preliminaries

In this section, there are discussed some technical properties of WPBL-algebras, structures introduced in [9].

**Definition 2.1.** A *pseudo-t-norm* is a binary operation on $[0,1]$ such that the following conditions are fulfilled for all $x, y, z \in [0,1]$:

(i) $x \ast (y \ast z) = (x \ast y) \ast z$;
(ii) If $x \leq y$ then $x \ast z \leq y \ast z$ and $z \ast x \leq z \ast y$;
(iii) $x \ast 1 = 1 \ast x = x$.

If $\ast$ is a pseudo-t-norm, then $x \ast 0 = 0 \ast x = 0$ for any $x \in [0,1]$. With any pseudo-t-norm $\ast$ left-continuous in the first argument and the second argument, one can associate two binary operations (called the left and right residuations of $\ast$) on $[0,1]$:

$$x \rightarrow_\ast y = \bigvee \{ z \in [0,1] | z \ast x \leq y \},$$

$$x_\ast \rightarrow y = \bigvee \{ z \in [0,1] | x \ast z \leq y \}.$$
Definition 2.2. A weak-pseudo-BL-algebra \(^5\) (WPBL-algebra for short) is a structure \((A, \lor, \land, *, 
rightarrow, \nRightarrow, 0, 1)\) which satisfies the following axioms for all \(x, y, z \in A:\)

(A1) \((A, \lor, \land, 0, 1)\) is a bounded lattice;
(A2) \((A, *, 1)\) is a monoid;
(A3) \(x \ast y \leq z\) iff \(x \leq y \nrightarrow z\) iff \(y \leq x \nRightarrow z\);
(A4) \((x \rightarrow y) \lor (y \rightarrow x) = (x \nRightarrow y) \lor (y \nRightarrow x) = 1.\)

Remark 2.1. The definition of weak-pseudo-BL-algebras in [9] includes another axiom, namely

\[(x \rightarrow y) \ast x \leq x \land y\] and \(x \ast (x \nrightarrow y) \leq x \land y.\]

As we shall see in the next proposition, we do not need this axiom, since it follows from (A1)–(A4).

If \(\ast\) is a pseudo-t-norm, left-continuous in the first and the second argument, then \([0, 1], \lor, \land, *, 
\nrightarrow_{\ast*}, \nRightarrow_{\ast*}, 0, 1\) is a WPBL-algebra [9].

Proposition 2.1. If \(A\) is a WPBL-algebra, then the following properties hold for all \(x, y, z \in A:\)

(a) \((x \rightarrow y) \ast x \leq y; x \ast (x \nrightarrow y) \leq y;\)
(b) \(x \leq y \rightarrow (x \ast y); x \leq y \nrightarrow (y \ast x);\)
(c) \(x \leq y\) implies \(x \ast z \leq y \ast z\) and \(z \ast x \leq z \ast y;\)
(d) \((x \rightarrow y) \ast x \leq x \land y\) and \(x \ast (x \nrightarrow y) \leq x \land y.\)

Proof. (a) From \(x \rightarrow y \leq x \rightarrow y\) we get, by (A3), \((x \rightarrow y) \ast x \leq y\). From \(x \nrightarrow y \leq x \nrightarrow y\) we get, by (A3), \(x \ast (x \nrightarrow y) \leq y.\)

(b) From \(x \ast y \leq x \ast y\) we get, by (A3), \(x \leq y \rightarrow (x \ast y)\). From \(y \ast x \leq y \ast x\) we get, by (A3), \(x \leq y \nrightarrow (y \ast x)\).

(c) Let \(x \leq y.\)

From (b), we have \(y \leq z \rightarrow (y \ast z)\), so \(x \leq z \rightarrow (y \ast z)\). Applying (A3), we get \(x \ast z \leq y \ast z.\)

Now, from (b), we have \(y \leq z \nrightarrow (z \ast y)\), so \(x \leq z \nrightarrow (z \ast y)\). Applying (A3), we get \(z \ast x \leq z \ast y.\)

(d) Since \(x \rightarrow y \leq 1\), we have, by (c), \((x \rightarrow y) \ast x \leq x \ast 1 = x.\) On the other hand, by (a), we have \((x \rightarrow y) \ast x \leq y.\)

It follows that \((x \rightarrow y) \ast x \leq x \land y.\) Since \(x \nrightarrow y \leq 1\), we have, by (c), \((x \nrightarrow y) \ast x \leq 1 \ast x = x.\) On the other hand, by (a), we have \(x \ast (x \nrightarrow y) \leq y.\) It follows that \((x \rightarrow y) \ast x \leq x \land y.\)

Proving (d) in the last proposition, we have shown that the definition of a WPBL-algebra from [9] is practically the same as ours. This also means that WBL-algebras, as defined in [9], are particular cases of the structures discussed here. What we knew was that MTL-algebras from [6] coincide with WBL-algebras from [9], the axiom of weak divisibility stated in [9] being redundant. (d) shows that

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\(^5\)These algebras, thus defined, are obviously a straight generalisation, for the non-commutative case, of the structures that, in [6], are named MTL-algebras; thus, the word pseudo-MTL-algebra is a good candidate for naming these structures. Nevertheless, as shown below, they also are the same thing as weak-pseudo-BL-algebras from [9], from which we took the name. However, beyond these naming matters, the reader should know that MTL-algebras coincide with WBL-algebras and pseudo-MTL-algebras coincide with WPBL-algebras.
this redundancy is preserved by moving to the non-commutative case. The properties in the following proposition, giving more information about a WPBL-algebra, are, except (h), proved in [9]:

Proposition 2.2. If $A$ is a WPBL-algebra, then the following properties hold for all $x, y, z \in A$:

(a) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \leftarrow y = 1$;
(b) $x \rightarrow x = x \leftarrow x = 1$;
(c) $1 \rightarrow x = 1 \leftarrow x = x$;
(d) $x \leq (x \rightarrow y) \land (x \leftarrow y)$;
(e) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $z \leftarrow x \leq z \leftarrow y$;
(f) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $y \leftarrow z \leq x \leftarrow z$;
(g) for any $\{x_i\}_{i \in I} \subseteq A$ we have

\[
x \leftarrow \left( \bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x \leftarrow x_i),
\]

\[
\left( \bigvee_{i \in I} x_i \right) ^\ast x = \bigvee_{i \in I} (x_i ^\ast x),
\]

whenever the arbitrary suprema exist.

(h) $x \ast 0 = 0 \ast x = 0$.

We shall only prove (h).

From (b), we have $0 \rightarrow 0 = 0 \leftarrow 0 = 1$, so $x \leq 0 \rightarrow 0$ and $x \leq 0 \leftarrow 0$. Applying (A3), we get $x \ast 0 \leq 0$ and $0 \ast x \leq 0$, which means that $x \ast 0 = 0$ and $0 \ast x = 0$.

In any WPBL-algebra, one can define two negations, $-$ and $\sim$:

\[
\overline{x} = x \rightarrow 0 \quad \text{and} \quad \overline{x} = x \leftarrow 0.
\]

Proposition 2.3. For all $x, y \in A$, the following properties hold:

(a) $\overline{1} = 1 \ast 0 = 0 \ast 1 = 1$;
(b) $\overline{x} \ast x = 0$, $x \ast \overline{x} = 0$;
(c) $x \leq \overline{y}$ iff $x \ast y = 0$;
(d) $x \leq y$ iff $y \ast x = 0$;
(e) $x \leq \overline{x}$ and $x \leq \overline{\overline{x}}$;
(f) $x \rightarrow \overline{y} = (x \ast y) \ast y$ and $x \leftarrow \overline{y} = (y \ast x)$;
(g) $x \leq \overline{\overline{y}}$ iff $y \leq x$.

Proof. (a) From Proposition 2.2(c), taking $x = 0$, we get $\overline{1} = 1 \rightarrow 0 = 0$ and $\overline{1} = 1 \leftarrow 0 = 0$. From Proposition 2.2(b), taking $x = 0$, we get $0 = 0 \rightarrow 0 = 1$ and $0 = 0 \leftarrow 0 = 1$.

(b) From $x \rightarrow 0 \leq x \rightarrow 0$, we get, by (A3), $(x \rightarrow 0) \ast x \leq 0$, which means $x \ast x = 0$. From $x \leftarrow 0 \leq x \leftarrow 0$, we get, by (A3), $x \ast (x \leftarrow 0) \leq 0$, which means $x \ast \overline{x} = 0$.

(c) From (A3), we get $x \leq y \rightarrow 0$ iff $x \ast y \leq 0$. So $x \leq \overline{y}$ iff $x \leq y \rightarrow 0$ iff $x \ast y \leq 0$ iff $x \ast y = 0$. 

(d) From (A3), we get \( x \leq y \bowtie 0 \) iff \( y \ast x \leq 0 \). So
\[
x \leq \tilde{y} \text{ iff } x \leq y \bowtie 0 \text{ iff } y \ast x \leq 0 \text{ iff } y \ast x = 0.
\]

(e) By (b), \( \tilde{x} \ast x = 0 \), so \( \tilde{x} \ast x \leq 0 \). Applying (A3), we get \( x \leq \tilde{x} \bowtie 0 \), i.e. \( x \leq \tilde{x} \).

By (b), \( x \ast \tilde{x} = 0 \), so \( x \ast \tilde{x} \leq 0 \). Applying (A3), we get \( x \leq \tilde{x} \rightarrow 0 \), i.e. \( x \leq \tilde{x} \).

(f) Let \( a \in A \). From (A3), we have \( a \leq x \rightarrow \tilde{y} \) iff \( a \ast x \leq \tilde{y} \) and \( a \ast x \ast y \leq 0 \) iff \( a \leq (x \ast y) \rightarrow 0 \).

So
\[
\begin{align*}
a & \leq x \rightarrow \tilde{y} \text{ iff } a \ast x \leq \tilde{y} \text{ iff } a \ast x \leq y \rightarrow 0 \text{ iff } \\
& \text{iff } a \ast x \ast y \leq 0 \text{ iff } a \leq (x \ast y) \rightarrow 0 \text{ iff } a \leq (x \ast y)\!.
\end{align*}
\]

We have that, for every \( a \in A \), \( a \leq x \rightarrow \tilde{y} \) iff \( a \leq (x \ast y)\). In particular, \( x \rightarrow \tilde{y} \leq (x \ast y)\) and \( (x \ast y) \leq x \rightarrow \tilde{y} \). So \( x \rightarrow \tilde{y} = (x \ast y)\).

Let again \( a \in A \).

From (A3), we have \( a \leq x \bowtie \tilde{y} \) iff \( x \ast a \leq \tilde{y} \) and \( y \ast x \ast a \leq 0 \) iff \( a \leq (y \ast x) \bowtie 0 \). So
\[
\begin{align*}
a & \leq x \bowtie \tilde{y} \text{ iff } x \ast a \leq \tilde{y} \text{ iff } x \ast a \leq y \bowtie 0 \text{ iff } \\
& \text{iff } y \ast x \ast a \leq 0 \text{ iff } a \leq (y \ast x) \rightarrow 0 \text{ iff } a \leq (y \ast x)\!.
\end{align*}
\]

We have that, for every \( a \in A \), \( a \leq x \bowtie \tilde{y} \) iff \( a \leq (y \ast x)\). In particular, \( x \bowtie \tilde{y} \leq (y \ast x)\) and \( (y \ast x) \leq x \bowtie \tilde{y} \). So \( x \bowtie \tilde{y} = (y \ast x)\).

(g) Applying (A3) twice (first part and then second part), we get
\[
x \leq \tilde{y} \text{ iff } x \leq y \rightarrow 0 \text{ iff } x \ast y \leq 0 \text{ iff } y \leq x \bowtie 0 \text{ iff } y \leq \tilde{x}. \quad \Box
\]

3. Pairs of weak negations

In this section, we introduce the notion of pair of weak negations on a bounded partially ordered set and, generalizing the results from [6], we characterize the pairs of weak negations on a bounded chain.

**Definition 3.1.** Let \( (C, \leq, 0, 1) \) be a bounded partially ordered set and two functions \( n_1 : C \rightarrow C \), \( n_2 : C \rightarrow C \). \((n_1, n_2)\) is said to be a pair of weak negations on \( C \) if the following conditions hold for all \( x, y \in C \):

1. \((N1)\) \( n_1(1) = n_2(1) = 0; \)
2. \((N2)\) If \( x \leq y \) then \( n_1(y) \leq n_1(x) \) and \( n_2(y) \leq n_2(x); \)
3. \((N3)\) \( x \leq n_1 n_2(x) \) and \( x \leq n_2 n_1(x). \)

If, for every \( x \in C \), \( x = n_1 n_2(x) = n_2 n_1(x) \), then \((n_1, n_2)\) is said to be a pair of strong negations. Notice that, in fact, a pair of weak negations is a Galois connection satisfying (N1). Also, if we force \( n_1 \) to be equal to \( n_2 \), we obtain the notion of weak negation from [4].

Throughout this section, \( (C, \leq, 0, 1) \) will be a fixed, but arbitrary, bounded chain. Although some of the results hold without the totally order assumption, we are only interested in this particular case.
Proposition 3.1. If \( (n_1, n_2) \) is a pair of weak negations, then the following properties hold:

(a) \( n_1(0) = n_2(0) = 1 \);
(b) \( n_1 n_2 n_1 = n_1 \) and \( n_2 n_1 n_2 = n_2 \);
(c) for any \( x, y \in C \), we have
   \[
   n_1(x \land y) = n_1(x) \lor n_1(y); \quad n_2(x \land y) = n_2(x) \lor n_2(y),
   \]
   \[
   n_1(x \lor y) = n_1(x) \land n_1(y); \quad n_2(x \lor y) = n_2(x) \land n_2(y);
   \]
(d) for any \( \{x_i\}_{i \in I} \subseteq A \),
   \[\begin{array}{ll}
   &\circ\text{ if there exist } \bigvee_{i \in I} x_i \text{ and } \bigwedge_{i \in I} n_1(x_i), \text{ then} \\
   &n_1 \left( \bigvee_{i \in I} x_i \right) = \bigwedge_{i \in I} n_1(x_i),
   \\
   &\circ\text{ if there exist } \bigvee_{i \in I} x_i \text{ and } \bigwedge_{i \in I} n_2(x_i), \text{ then} \\
   &n_2 \left( \bigvee_{i \in I} x_i \right) = \bigwedge_{i \in I} n_2(x_i);
   \end{array}\]
(e) \( n_1 n_2 \) and \( n_2 n_1 \) are closure operators on \( C \);
(f) for any \( x, y \in C \), we have
   \[x \leq n_2(y)\text{ iff } y \leq n_1(x).\]

**Proof.** For each point of the proposition, we shall prove only one-half of it—the other half, perfectly symmetrical to the first, has a symmetrical proof.

(a) From (N3), we get \( 1 \leq n_1 n_2(1) \), so \( 1 = n_1 n_2(1) \). But, from (N1), \( n_2(1) = 0 \). It follows that \( n_1(0) = 1 \).

(b) Let \( x \in C \). By (N3), \( x \leq n_2 n_1(x) \), so, by (N2), \( n_1 n_2 n_1(x) \leq n_1(x) \). On the other hand, applying (N3) for \( n_1(x) \), we get \( n_1(x) \leq n_1 n_2 n_1(x) \). So \( n_1 n_2 n_1(x) = n_1(x) \).

(c) Let \( x, y \in C \).

If \( x \leq y \), then \( x \lor y = y \) and \( x \land y = x \). By (N2), \( n_1(y) \leq n_1(x) \), so \( n_1(x) \lor n_1(y) = n_1(x) \) and \( n_1(x) \land n_1(y) = n_1(y) \). It follows that \( n_1(x \lor y) = n_1(x \land y) \) and \( n_1(x \lor y) = n_1(x) \lor n_1(y) \).

If \( y \leq x \), then \( y \leq x \) and, similarly, we get \( n_1(x \lor y) = n_1(x) \lor n_1(y) \) and \( n_1(x \lor y) = n_1(y) \).

(d) Suppose that \( \{x_i\}_{i \in I} \subseteq A \) such that there exist \( \bigvee_{i \in I} x_i \) and \( \bigwedge_{i \in I} n_1(x_i) \). We shall prove that
   \[
   n_1 \left( \bigvee_{i \in I} x_i \right) = \bigwedge_{i \in I} n_1(x_i).
   \]

It is obvious that
   \[
   n_1 \left( \bigvee_{i \in I} x_i \right) \leq \bigwedge_{i \in I} n_1(x_i).
   \]
In order to prove the converse inequality, we remark that \( n_2(y_j) \leq n_2\left(\bigwedge_{i \in J} y_i\right) \) for any \( \{y_i\}_{i \in J} \subseteq A \) and any \( j \in J \). Therefore, by (N3)

\[
x_j \leq n_2n_1(x_j) \leq n_2\left(\bigwedge_{i \in I} n_1(x_i)\right)
\]

for any \( j \in I \).

Thus

\[
\bigvee_{i \in I} x_i \leq n_2\left(\bigwedge_{i \in I} n_1(x_i)\right).
\]

In accordance to (N3) and (N2), this implies

\[
\bigwedge_{i \in I} n_1(x_i) \leq n_1n_2\left(\bigwedge_{i \in I} n_1(x_i)\right) \leq n_1\left(\bigvee_{i \in I} x_i\right).
\]

(e) Since \( n_1 \) and \( n_2 \) are both non-increasing, we have that \( n_1n_2 \) and \( n_2n_1 \) are non-decreasing. (N3) postulates the extensivity of \( n_1n_2 \) and \( n_1n_2 \), while \( (n_1n_2)^2 = n_1n_2 \) and \( (n_2n_1)^2 = n_2n_1 \) follow from (b).

(f) Suppose \( y \leq n_1(x) \). By (N2), \( n_2(n_1(x)) \leq n_2(y) \). But, according to (N3), \( x \leq n_2(n_1(x)) \). So \( x \leq n_2(y) \).

Conversely, suppose \( x \leq n_2(y) \). We have, from (N2), that \( n_1(n_2(y)) \leq n_1(x) \) and, from (N3), that \( y \leq n_1(n_2(y)) \). So \( y \leq n_1(x) \).  

Remark 3.1. If \( (n_1, n_2) \) is a pair of strong negations, then the duals of the relations from (d) also hold (as an easy consequence of \( n_1 \) and \( n_2 \) being invertible):

- if there exist \( \bigwedge_{i \in I} x_i \) and \( \bigvee_{i \in I} n_1(x_i) \), then

\[
n_1\left(\bigwedge_{i \in I} x_i\right) = \bigvee_{i \in I} n_1(x_i);
\]

- if there exist \( \bigwedge_{i \in I} x_i \) and \( \bigvee_{i \in I} n_2(x_i) \), then

\[
n_2\left(\bigwedge_{i \in I} x_i\right) = \bigvee_{i \in I} n_2(x_i);
\]

Definition 3.2. Let \( M \subseteq C \). \( M \) is said to satisfy the minimum-condition if, for any \( x \in C \), there exists \( \min\{y \in M \mid x \leq y\} \).

Proposition 3.2. Let \( C_1 \) and \( C_2 \) be two subsets of \( C \) which contain 0 and 1 and satisfy the minimum-condition. Then there exist as many pairs of weak negations \( (n_1, n_2) \) such that \( n_1(C) = C_2 \) and \( n_2(C) = C_1 \) as pairs of functions \( (C_1 \xrightarrow{m_1} C_2, C_2 \xrightarrow{m_2} C_1) \) non-increasing, invertible and inverse to each other.

Proof. Let \( (n_1, n_2) \) be a pair of weak negations on \( C \) such that \( n_1(C) = C_2 \) and \( n_2(C) = C_1 \). Define \( m_1 = n_1|_{C_1} \) and \( m_2 = n_2|_{C_2} \). If \( x \in C_1 \), then \( n_2n_1(x) = x \) because \( n_2n_1 = n_2 \). Similarly, if \( x \in C_2 \), then...
\( n_1n_2(x) = x \). This means that \( m_1 \) and \( m_2 \) are invertible and inverse to each other. Moreover, since \( n_1 \) and \( n_2 \) are non-increasing, \( m_1 \) and \( m_2 \) are non-increasing.

On the other hand, let \((C_1 \xrightarrow{m_1} C_2, C_2 \xrightarrow{m_2} C_1)\) be a pair of functions that are non-increasing, invertible and inverse to each other. Define \( n_1, n_2 : C \longrightarrow C \) as follows:

\[
\begin{align*}
n_1(x) &= m_1(\min\{c_1 \in C_1|x \leq c_1\}), \\
n_2(x) &= m_2(\min\{c_2 \in C_2|x \leq c_2\}).
\end{align*}
\]

If \( x \leq x' \), then \( \min\{c_1 \in C_1|x \leq c_1\} \leq \min\{c_1 \in C_1|x' \leq c_1\} \), so, since \( m_1 \) is non-increasing,

\[
m_1(\min\{c_1 \in C_1|x' \leq c_1\}) \leq m_1(\min\{c_1 \in C_1|x \leq c_1\}),
\]

that is \( n_1(x') \leq n_1(x) \). Similarly, we get \( n_2(x') \leq n_2(x) \). So \( n_1 \) and \( n_2 \) are non-increasing.

Because \( m_1 \) and \( m_2 \) are bijective and non-increasing and \( 0, 1 \in C_1, C_2 \), we obviously have \( m_1(0) = m_2(0) = 1 \) and \( m_1(1) = m_2(1) = 0 \). Consequently,

\[
\begin{align*}
n_1(1) &= m_1(\min\{c_1 \in C_1|1 \leq c_1\}) = m_1(1) = 0, \\
n_2(1) &= m_2(\min\{c_2 \in C_2|1 \leq c_2\}) = m_2(1) = 0.
\end{align*}
\]

Let us show now that \( x \leq n_2n_1(x) \) for any \( x \in C \). Because \( n_1(x) = m_1(\ldots), n_1(x) \in C_2 \). It follows that

\[
n_1(x) = \min\{c_2 \in C_2|n_1(x) \leq c_2\},
\]

so

\[
n_2n_1(x) = m_2(\min\{c_2 \in C_2|n_1(x) \leq c_2\}) = m_2(n_1(x)).
\]

Since \( x \leq \min\{c_1 \in C_1|x \leq c_1\} \), we have

\[
x \leq \min\{c_1 \in C_1|x \leq c_1\} = m_2m_1(\min\{c_1 \in C_1|x \leq c_1\}) = m_2n_1(x) = n_2n_1(x).
\]

Similarly, \( x \leq n_1n_2(x) \). We have shown that \((n_1, n_2)\) is a pair of weak negations.

We shall prove that the two correspondences defined above are bijective and inverse to each other. Let \((n_1, n_2)\) be a pair of weak negations on \( C \) such that \( n_1(C) = C_2 \) and \( n_2(C) = C_1 \). For any \( x \in C \),

\[
\begin{align*}
n_1(x) &= n_1(\min\{c_1 \in C_1|x \leq c_1\}), \\
n_2(x) &= n_2(\min\{c_2 \in C_2|x \leq c_2\}).
\end{align*}
\]

Let us prove the first relation. Consider \( x \in C \). Since \( x \leq \min\{c_1 \in C_1|x \leq c_1\} \), we get \( n_1(\min\{c_1 \in C_1|x \leq c_1\}) \leq n_1(x) \). For the converse inequality, take \( x' = n_2n_1(x) \). We have \( n_1(x) = n_1(x') \). Moreover, \( x' \in C_1 \) and \( x \leq x' \), so

\[
n_1(x) = n_1(x') \leq n_1(\min\{c_1 \in C_1|x \leq c_1\}).
\]

The second relation has a similar proof.

Finally, let \((C_1 \xrightarrow{m_1} C_2, C_2 \xrightarrow{m_2} C_1)\) be a pair of functions that are non-increasing, invertible and inverse to each other.
If \( x \in C_1 \), then \( m_1(x) = m_1(\min\{c_1 \in C_1 \mid x \leq c_1\}) \).
If \( x \in C_2 \), then \( m_2(x) = m_2(\min\{c_2 \in C_2 \mid x \leq c_2\}) \).

Indeed, for any \( x \in C_1 \), \( x = \min\{c_1 \in C_1 \mid x \leq c_1\} \) and, for any \( x \in C_2 \), \( x = \min\{c_2 \in C_2 \mid x \leq c_2\} \) and the upper sentences are true. \( \Box \)

In what follows, we regard functions from \( C \) to \( C \) as relations (i.e. subsets of \( C \times C \)) and, for each \( R \subseteq C \times C \), we denote \( R^\rightarrow = \{(a,b)/(b,a) \in R\} \), the converse of \( R \).

**Definition 3.3.** A pair \((n_1,n_2)\) of non-increasing functions defined on \( C \) and taking values from \( C \) is said to be *symmetric with respect to the identity function* if there exist, for \( i \in \{1,2\} \), \( v_i : C \setminus n_i(C) \rightarrow C \) such that:

- for any \( i \in \{1,2\} \), \( y \in C \), \( x \in C \setminus n_i(C) \), \( [n_i(y) < x \iff v_i(x) < y] \) and \( [n_i(y) > x \iff v_i(x) > y] \);
- \( (n_1 \cup \{(v_1(x),x)/x \in C \setminus n_1(C)\})^\rightarrow = n_2 \cup \{(v_2(x),x)/x \in C \setminus n_2(C)\} \).

The functions \( v_1 \) and \( v_2 \) from the above definition represent possible ways to “complete” the graphs of \( n_1 \) and \( n_2 \) (which we identified with the functions themselves) in such a way that they become surjective relations (i.e. with the whole \( C \) as second projection), but still keeping the non-decreasingness property—when containing \((x,y)\) and \((x',y')\), \([x \leq x' \iff y \leq y']\); moreover, the “completions” are required:

- to be minimal, that is to add pairs only where they must (only with the second component being an element out of the range) and
- to be made “downwards”, that is, whenever we have \((a,b_1)\) in the graph and we add \((a,b_2)\) (and one can easily see that this is the only case), always \( b_2 < b_1 \) (notice that all we add to \( n_i \) are pairs \((v_i(x),x)\), with \( x \in C \setminus n_i(C) \) and the only possible “clash” is with a pair \((y,n_i(y))\), where \( y = v_i(x) \); since \( x = n_i(y) \) is not possible, because it would mean \((v_i(x),x) = (y,n_i(y))\), and \( n_i(y) < x \) is excluded by the first point of the definition, it remains \( x < n_i(y) \)).

The definition asks that there exist such completions for \( n_1 \) and \( n_2 \) so that they become symmetric (converse) as relations. \( \ast \) A perhaps more suitable term in Definition 3.3 would be “downwards virtually symmetric (to each other)”, but we adopted the term from [4]. Also, instead of asking that there exist such symmetric completions, we could have asked, with the same effect, that the relations \{\((x,y)/y \leq n_i(x)\), \( i \in \{1,2\}\)\} are converse to each other.

**Proposition 3.3.** Let \((C \xrightarrow{n_1} C, C \xrightarrow{n_2} C)\) be a pair of non-increasing functions such that \( n_1(1) = n_2(1) = 0 \). The following are equivalent:

(a) \((n_1,n_2)\) is a pair of weak negations;
(b) suppose \( \{i,j\} = \{1,2\} \). For all \( x \in C \),
   - if \( x \in n_i(C) \), then \( n_j(x) = y \implies x = n_i(y) \);
   - if \( x \notin n_i(C) \), then, for any \( y \in C \), \( n_j(y) < x \implies n_i(x) < y \);
(c) \((n_1,n_2)\) is symmetric w.r.t. the identity function.

\( \ast \) This completions correspond to the “vertical lines” added on the graph of the weak negation \( n \) on \([0,1]\) above its discontinuity points [4].
Proof. (a) implies (b): Let \((n_1, n_2)\) be a pair of weak negations on \(C\).

If \(x \in n_2(C)\), then \(x = n_2(z)\), where \(z \in C\) and, knowing that \(n_2n_1n_2 = n_2\), we get \(n_2n_1(x) = n_2n_1n_2(z) = n_2(z) = x\). So, if \(y = n_1(x)\), then \(x = n_2(y)\). Analogously, \(y \in n_1(C)\) implies \(n_1n_2(y) = y\), so \(y = n_1(x)\) whenever \(n_2(y) = x\).

Suppose now that \(x \notin n_2(C)\). Let \(y\) be an arbitrary element of \(C\). Then, according to Proposition 3.1(f), we have that \(y \leq n_1(x)\) implies \(x \leq n_2(y)\), which means, \(C\) being a chain, that \(n_2(y) < x\) implies \(n_1(x) < y\).

The proofs for the two properties from Definition 3.3 for \(i = 2\) take place in a similar manner.

(b) implies (a): Consider \((n_1, n_2)\) to be a pair of functions symmetric with respect to the identity function and such that \(n_1, n_2\) are non-increasing and \(n_1(1) = n_2(1) = 0\). Since \((N1)\) and \((N2)\) hold by hypothesis, it remains to prove \((N3)\).

Let \(x \in C\).

If \(x \in n_2(C)\), then, taking \(y = n_1(x)\), we get \(n_2n_1(x) = n_2(y) = x\), so \(x \leq n_2n_1(x)\).

If \(x \notin n_2(C)\), let \(y = n_1(x)\). We know that \(n_2(y) < x\) implies \(n_1(x) < y\), so, since \(C\) is a chain, \(y \leq n_1(x)\) implies \(x \leq n_2(y)\). Because it is true that \(n_1(x) \leq n_1(1)\), it follows that \(x \leq n_2(y) = n_2n_1(x)\).

In both cases, \(x \leq n_2n_1(x)\). Similarly, \(x \leq n_1n_2(x)\).

[(a) and (b)] implies (c): Define, for each \(i \in \{1, 2\}\) and \(x \in C \setminus n_i(C)\), \(v_i(x) = n_3_{-i}(x)\). Let \(x \in C \setminus n_i(C)\) and \(y \in C\). The fact that \(n_i(y) < x\) implies \(v_i(x) < y\) follows from (b). Suppose now that \(n_i(y) > x\). Then, by \((N1)\) and \((N3)\), \(y \leq n_{3-i}n_i(y) \leq n_{3-i}(x) = v_i(x)\).

Denote \(R_i = n_i \cup \{(v_i(x), x) / x \in C \setminus n_i(C)\} = n_i \cup \{(n_{3-i}(x), x) / x \in C \setminus n_i(C)\}, \ i \in \{1, 2\}\). Consider a fixed \(i \in \{1, 2\}\) and let \((a, b) \in R_i\). We have two cases:

1. \((a, b) = (v_i(y), y)\), with \(y \in C \setminus v_i(C)\), so \((a, b) = (n_{3-i}(y), y) \in R_{3-i}^-\);
2. \((a, b) = (x, n_i(x))\), with \(x \in C\). If \(x = n_{3-i}(y)\), then, by (b), \(y = n_i(x)\) and so \((a, b) = (n_{3-i}(y), y) \in R_{3-i}^-\). But if \(x \notin n_{3-i}(C)\), then, by the definition of \(v_{3-i}\), \((a, b) = (x, v_{3-i}(x)) \in R_{3-i}^-\).

We obtained \(R_i \subseteq R_{3-i}^-\), \(i \in \{1, 2\}\), that is \(R_1 \subseteq R_2^-\) and \(R_2 \subseteq R_1^-\). But this means \(R_1^- = R_2\).

(c) implies (b): Denote \(R_i\), \(i \in \{1, 2\}\) as above. Let \(\{i, j\} = \{1, 2\}\) and \(x \in n_i(C)\). Also, let \(y = n_j(x)\). Then \((x, y) \in R_j\), hence \((y, x) \in R_i\). Now, \((y, x)\) cannot be of the form \((v_i(z), z)\), because we would have \(x = z \notin n_i(C)\), contradicting the choice of \(x\). So the only chance is \((y, x) \in n_i\), that is \(x = n_i(y)\).

Finally, let \(x \in C \setminus n_i(C)\) and let \(y \in C\) such that \(n_i(y) < x\). Since \((x, n_i(x)) \in R_j\), it follows that \((n_j(x), x) \in R_i\). But, by definition, \(R_i\) contains only one pair with second component \(x\), namely \((v_i(x), x)\), hence \(n_j(x) = v_i(x)\). The fact that \(n_j(x) < y\) follows now from Definition 3.3.

4. Compatible pairs of weak negations

Combining the structure of WPBL-algebra with a pair of weak negations compatible (in some sense) with the multiplication, we get a way of constructing new WPBL-algebras from a fixed one, which can be, for example, a BL-algebra or any other more particular structure. Then we characterize the pairs of weak negations compatible with a WPBL-chain. It is also obtained a generalization of a result from [6] regarding the existence, for every pair of weak negations on a chain, of a WPBL-structure for which they become the two negations. Throughout this section, \((A, \forall, \land, *, \rightarrow, \sim, 0, 1)\) will denote a totally ordered WPBL-algebra.
With any pair of weak negations \((n_1,n_2)\) on \(A\), one can associate a binary operation \(\otimes\):

\[
x \otimes y = \begin{cases} 
x \ast y & \text{if } x \nleq n_1(y), \\
0 & \text{if } x \nleq n_1(y).
\end{cases}
\]

Because \(x \nleq n_1(y)\) iff \(y \nleq n_2(x)\), and \(A\) is a chain, an alternative definition of \(\otimes\) would be

\[
x \otimes y = \begin{cases} 
x \ast y & \text{if } y > n_2(x), \\
0 & \text{if } y \nleq n_2(x).
\end{cases}
\]

**Lemma 4.1.** For any \(x,y,z \in A\), the following properties hold:

(a) \(x \leq y\) implies \(x \otimes z \leq y \otimes z\) and \(z \otimes x \leq z \otimes y\);
(b) \(x \otimes 1 = 1 \otimes x = x\);
(c) \(x \otimes 0 = 0 \otimes x = 0\).

**Proof.**

(a) Suppose \(x \leq y\).

If \(x \leq n_1(z)\), then \(x \otimes z = 0 \leq y \otimes z\); if \(x > n_1(z)\), then also \(y > n_1(z)\), so \(x \otimes z = x \ast z \leq y \ast z = y \otimes z\). If \(z \leq n_1(x)\), then \(z \otimes x = 0 \leq z \otimes y\); if \(z > n_1(x)\), then also \(z > n_1(y)\), because \(n_1(y) \leq n_1(x)\). So \(z \otimes x = z \ast x \leq z \ast y = z \otimes y\).

(b) \(x \otimes 1 = x \ast 1 = x\) if \(x > 0 = n_1(1)\) and \(x \otimes 1 = 0 = x\) if \(x = 0\). \(1 \otimes x = 1 \ast x = x\) if \(x > 0 = n_2(1)\) and \(1 \otimes x = 0 = x\) if \(x = 0\).

(c) \(x \otimes 0 = 0\) because \(x \leq 1 = n_1(0)\). \(0 \otimes x = x\) if because \(x \leq 1 = n_2(0)\). \(\square\)

Now let us define two binary operations on \(A\):

\[
x \Rightarrow y = \begin{cases} 
1 & \text{if } x \leq y, \\
n_1(x) \lor (x \rightarrow y) & \text{if } x > y,
\end{cases}
\]

\[
x \Rightarrow y = \begin{cases} 
1 & \text{if } x \leq y, \\
n_2(x) \lor (x \sim y) & \text{if } x > y.
\end{cases}
\]

**Proposition 4.1.** For any \(x,y,z \in A\), the following equivalences hold:

(i) \(z \leq x \Rightarrow y\) iff \(z \otimes x \leq y\);
(ii) \(z \leq x \Rightarrow y\) iff \(x \otimes z \leq y\).

**Proof.**

First we shall establish the inequality

\((x \Rightarrow y) \otimes x \leq y\) for all \(x,y \in A\). (*)

For \(x \leq y\), the inequality follows immediately, hence we assume that \(y < x\). We have two cases:

(a) \(x \rightarrow y \leq n_1(x)\). Thus, \(x \Rightarrow y = n_1(x) \lor (x \rightarrow y) \leq n_1(x)\), then \((x \Rightarrow y) \otimes x = 0 \leq y\).

(b) \(x \rightarrow y > n_1(x)\). In this case \(x \Rightarrow y = n_1(x) \lor (x \rightarrow y) > n_1(x)\), then

\((x \Rightarrow y) \otimes x = (x \rightarrow y) \ast x \leq x \land y \leq y\).
In accordance to (*), $z \leq x \Rightarrow y$ implies $z \otimes x \leq (x \Rightarrow y) \otimes x \leq y$. Conversely, assume that $z \otimes x \leq y$.

We shall distinguish two cases:

1. $z \leq n_1(x)$. Thus $z \leq n_1(x) \lor (x \rightarrow y) = x \Rightarrow y$.
2. $z > n_1(x)$. Thus $z \ast x = z \otimes x \leq y$, hence $z \leq x \rightarrow y \leq x \Rightarrow y$.

(ii) For any $x, y \in A$ one can prove the inequality $x \otimes (x \Rightarrow y) \leq y$, then the proof of (ii) follows the same steps as in the case of (i). 

**Lemma 4.2.** For any $x \in A$, the following equivalences hold:

(i) $n_1(x) = x \Rightarrow 0$ iff $n_1(x) \geq x \rightarrow 0 = \bar{x}$;

(ii) $n_2(x) = x \Rightarrow 0$ iff $n_2(x) \geq x \rightarrow 0 = \bar{x}$.

**Proof.** (i) We have that

$$x \Rightarrow 0 = \begin{cases} 1 & \text{if } x = 0, \\ n_1(x) \lor (x \rightarrow 0) & \text{if } x > 0. \end{cases}$$

If $x = 0$, then $x \Rightarrow 0 = 1 = n_1(x)$ and also $n_1(x) = 1 \geq \bar{x}$.

If $x > 0$, then $x \Rightarrow 0 = n_1(x) \lor (x \rightarrow 0)$, so

$$n_1(x) = x \Rightarrow 0 \text{ iff } n_1(x) \geq x \rightarrow 0 = \bar{x}.$$ 

The proof of (ii) goes analogously.

**Corollary 4.1.** For any $\{x_i\}_{i \in I} \subseteq A$ and $x \in A$, we have

$$x \otimes \left( \bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x \otimes x_i),$$

$$\left( \bigvee_{i \in I} x_i \right) \otimes x = \bigvee_{i \in I} (x_i \otimes x),$$

whenever the arbitrary suprema exist.

**Proof.** Applying Proposition 3.1(d), we get

$$x \geq n_1(x_i) \text{ for any } i \in I \text{ iff } x \leq \bigwedge_{i \in I} n_1(x_i) = n_1 \left( \bigvee_{i \in I} x_i \right).$$

It follows that

$$\bigvee_{i \in I} (x \otimes x_i) = \bigvee_{i \in I} (x \ast x_i) \text{ iff } x \otimes \left( \bigvee_{i \in I} x_i \right) = x \ast \left( \bigvee_{i \in I} x_i \right).$$

It remains to apply Proposition 2.2(g) to get the first equality. The second equality follows in the same manner.
Definition 4.1. The pair of weak negations \((n_1,n_2)\) is said to be compatible with \(*\) (or, simply, compatible, when the structure is understood) if the following conditions hold:

(i) \(\otimes\) is an associative operation;
(ii) \(n_1(x) = x \Rightarrow 0\) for all \(x \in A\);
(iii) \(n_2(x) = x \Leftrightarrow 0\) for all \(x \in A\).

Lemma 4.3. If \((n_1,n_2) = (−, ∼)\), then \(\otimes = ∗\).

Proof. Since \(x \leqslant y\) iff \(x ∗ y = 0\), we have

\[x \otimes y = \begin{cases} x ∗ y & \text{if } x > y, \\ 0 & \text{if } x \leqslant y.\end{cases}\]

Therefore \((- , ∼)\) is compatible with \(*\).

Proposition 4.2. If \((n_1,n_2)\) is compatible with \(*\), then \((A, ∨, ∧, ⊗, ⇒, ≈, 0, 1)\) is a WPBL-algebra.

Proof. It is clear that the axioms (A1) and (A2) hold. (A3) follows by Proposition 4.1. It remains to show (A4). From \(x → y \leqslant x \Rightarrow y\), and \(x \sim y \leqslant x \Leftrightarrow y\) for all \(x, y\), we get

\[(x \Rightarrow y) \lor (y ⇒ x) \geqslant (x → y) \lor (y → x) = 1,\]

\[(x \sim > y) \lor (y ≈ > x) \geqslant (x → y) \lor (y → x) = 1.\]  

Theorem 4.1. A pair of weak negations \((n_1,n_2)\) on \(A\) is compatible with \(*\) iff the following conditions hold for any \(x, y, z ∈ A\):

(i) \(\tilde{x} \leqslant n_1(x)\) and \(\tilde{x} \leqslant n_2(x)\);
(ii) \(y > n_1(z)\) implies \(n_1(y * z) = n_1(y) \lor (y → n_1(z))\);
(iii) \(y > n_2(z)\) implies \(n_2(z * y) = n_2(y) \lor (y → n_2(z))\).

Proof. Assume that \((n_1,n_2)\) is compatible with \(*\). Then condition (i) follows by Lemma 4.2. We shall prove (ii) and (iii).

(ii) If \(y > n_1(z)\), then \(y \otimes z = y * z\) and, by Propositions 4.2 and 2.3(f)

\[n_1(y * z) = n_1(y \otimes z) = y \Rightarrow n_1(z) = n_1(y) \lor (y → n_1(z)).\]

(iii) If \(y > n_2(z)\), then \(z \otimes y = z * y\) and, hence, similarly,

\[n_2(z * y) = n_2(z \otimes y) = y \Rightarrow n_2(z) = n_2(y) \lor (y → n_2(z)).\]

Conversely, suppose that the conditions (i)–(iii) hold. We must only prove that \(⊗\) is associative.

Let \(x, y, z \in A\). If \(y \leqslant n_2(x)\), then \(x \otimes y = 0\), hence \((x \otimes y) \otimes z = 0\). Since \(y \otimes z \leqslant y\), we have \(x \otimes (y \otimes z) = 0\).
Assume now that \( n_2(x) < y \), hence \( x \otimes y = x \cdot y \). We have two cases:

(a) \( z \leq n_2(x \cdot y) \). Then \((x \cdot y) \otimes z = (x \cdot y) \otimes z = 0 \). Since \( n_2(x) < y \), it follows that \( z \leq n_2(x \cdot y) = n_2(y) \vee (y \sim n_2(z)) \) in accordance to the hypothesis (iii). Therefore \( z \leq n_2(y) \) or \( z \leq y \sim n_2(z) \). If \( z \leq n_2(y) \), then \( y \otimes z = 0 \), hence \( x \otimes (y \otimes z) = 0 \); if \( z \leq y \sim n_2(z) \), then \( y \cdot z \leq n_2(x) \), hence \( x \otimes (y \otimes z) = 0 \).

(b) \( n_2(x \cdot y) < z \). In this case \((x \cdot y) \otimes z = (x \cdot y) \cdot z \), hence \((x \cdot y) \otimes z = (x \cdot y) \cdot z \). By (iii), \( n_2(y) \vee (y \sim n_2(x)) < z \), therefore \( n_2(y) < z \) and \( y \sim n_2(x) < z \). The inequality \( y \sim n_2(x) < z \) implies \( z \leq y \sim n_2(x) \), so \( y \cdot z \leq n_2(x) \). Therefore \( n_2(x) < y \cdot z \), hence \( x \otimes (y \cdot z) = x \cdot (y \cdot z) \). Then \( x \otimes (y \otimes z) = x \otimes (y \cdot z) \) follows from the associativity of \( \cdot \).

Let \((C, \leq, 0, 1)\) be a bounded chain. One can naturally define on \( C \) a structure of WPBL-algebra, which is in fact a BL-algebra and, even more, a Gödel-algebra: For any \( x, y \in C \),

\[
    x \cdot y = x \land y,
    \quad x \rightarrow y = x \sim y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y. \end{cases}
\]

Indeed, it is obvious that \((C, \lor, \land, *, \rightarrow, \sim, 0, 1)\) becomes a WPBL-algebra. The two corresponding negations are

\[
    \bar{0} = 1 \quad \text{and} \quad \bar{0} = 1,
    \quad \bar{x} = x \rightarrow 0 = 0 \quad \text{and} \quad \bar{x} = x \sim 0 = 0, \quad \text{if } x \neq 0.
\]

**Lemma 4.4.** Any pair of weak negations on \( C \) is compatible with the meet operation \((\cdot = \land)\).

**Proof.** Consider \((n_1, n_2)\) a pair of weak negations on \( C \). Let us check the conditions (i)—(iii) from Theorem 4.1.

Let \( x \in C \). If \( x \neq 0 \), then \( n_1(x) \geq 0 = \bar{x} \) and \( n_2(x) \geq 0 = \bar{x} \); moreover, \( n_1(0) = 1 \geq \bar{0} \) and \( n_2(0) = 1 \geq \bar{0} \).

Let \( y, z \in C \) such that \( y > n_1(z) \). Then \( y \rightarrow n_1(z) = n_1(z) \), and, according to Proposition 3.1(c),

\[
    n_1(y \cdot z) = n_1(y \land z) = n_1(y) \lor n_1(z) = n_1(y) \lor (y \rightarrow n_1(z)).
\]

Let \( y, z \in C \) such that \( y > n_2(z) \). Then \( y \sim n_2(z) = n_2(z) \), and, from Proposition 3.1(c), we get

\[
    n_2(y \cdot z) = n_2(y \land z) = n_2(y) \lor n_2(z) = n_2(y) \lor (y \sim n_2(z)).
\]

**Remark 4.1.** (1) Assume that \( * \) is the Gödel t-norm on \([0, 1]\):

\[
    x \cdot y = x \land y \quad \text{for all } x, y \in [0, 1].
\]

Let \((n_1, n_2)\) be a pair of weak negations on \([0, 1]\). From Lemma 4.4, \((n_1, n_2)\) is compatible with \( * \). So, the operation \( \otimes \) defined by

\[
    x \otimes y = \begin{cases} x \land y & \text{if } x > n_1(y), \\ 0 & \text{if } x \leq n_1(y) \end{cases}
    = \begin{cases} x \land y & \text{if } y > n_2(x), \\ 0 & \text{if } y \leq n_2(x). \end{cases}
\]

is a pseudo-t-norm which extends the Fodor t-norm ([10]).
(II) Let $0 < a < b < 1$. Consider the functions
\[ n_1 : [0, 1] \rightarrow [0, 1], n_2 : [0, 1] \rightarrow [0, 1] \]
defined by
\[
\begin{align*}
n_1(x) &= \begin{cases} 
1 & \text{if } x = 0, \\
a & \text{if } 0 < x \leq b, \\
0 & \text{if } x > b,
\end{cases} \\
n_2(x) &= \begin{cases} 
1 & \text{if } x = 0, \\
b & \text{if } 0 < x \leq a, \\
0 & \text{if } x > a.
\end{cases}
\end{align*}
\]
It is easy to see that $(n_1, n_2)$ is a pair of weak negations with respect to the natural order. Applying again Lemma 4.4, $(n_1, n_2)$ is compatible with the min-operation $\land$ and the associated pseudo-t-norm $\otimes$ has the form
\[
x \otimes y = \begin{cases} 
x \land y & \text{if } y \geq n_1(y) \\
0 & \text{if } y < n_1(y)
\end{cases} = \begin{cases} 
0 & \text{if } 0 \leq x \leq a \text{ and } 0 \leq y \leq b, \\
x \land y & \text{otherwise.}
\end{cases}
\]
Therefore, one obtains the pseudo-t-norm $T_{01}$ of [9].

Proposition 4.3. Let $(C, \leq, 0, 1)$ be a bounded chain and $(n_1, n_2)$ be a pair of weak negations on $C$. Then one can define the operations $\otimes, \Rightarrow, \bowtie$ such that $(C, \lor, \land, \otimes, \Rightarrow, \bowtie, 0, 1)$ becomes a WPBL-algebra for which $n_1$ and $n_2$ are the two negations.

Proof. According to Lemma 4.4, any pair of weak negations is compatible with $*$ in the structure $(C, \lor, \land, * , \rightarrow, \bowtie, 0, 1)$, where $* = \land$ and
\[
x \rightarrow y = x \bowtie y = \begin{cases} 
1 & \text{if } x \leq y, \\
y & \text{if } x > y.
\end{cases}
\]
We define the operations $\otimes, \Rightarrow, \bowtie$:
\[
x \otimes y = \begin{cases} 
x * y & \text{if } y > n_2(x), \\
0 & \text{if } y \leq n_2(x).
\end{cases}
\]
\[
x \Rightarrow y = \begin{cases} 
1 & \text{if } x \leq y, \\
n_1(x) \lor (x \rightarrow y) & \text{if } x > y,
\end{cases}
\]
\[
x \bowtie y = \begin{cases} 
1 & \text{if } x \leq y, \\
n_2(x) \lor (x \bowtie y) & \text{if } x > y.
\end{cases}
\]
Because \((n_1, n_2)\) is compatible with \(*\), by Proposition 4.2, it follows that \((C, \lor, \land, \otimes, \Rightarrow, \bowtie, 0, 1)\) is a WPBL-algebra and \(n_1\) and \(n_2\) are its negations.

5. Pairs of weak negations on \(L_{m+1}\)

This section characterizes the compatible pairs of weak negations on the MV-algebra \(L_{m+1}\). It happens that each pair has the form \((n, n)\) where \(n\) is a weak negation, as defined in [6] for MTL-algebras. This characterization is a step towards a goal that would be relevant both for commutative and non-commutative situations: finding all the compatible pairs of weak negations (hence all the compatible weak negations) on an arbitrary finite BL-chain. The result from [1], saying that BL-chains are ordinal sums of Wajsberg chains (that is, modulo a termwise equivalence, MV-chains) could be very helpful for this, provided that one studies the behavior of pairs of weak negations to ordinal sum.

Let \(m \geq 1\). We consider the structure \((L_{m+1}, \lor, \land, *, \rightarrow, \bowtie, 0, 1)\), where

\[
L_{m+1} = \left\{ 0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1 \right\},
\]

\(\land = \min, \ \lor = \max,\)

\[
x \ast y = 0 \lor (x + y - 1) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ x + y - 1 & \text{otherwise}, \end{cases}
\]

\[
x \rightarrow y = x \bowtie y = 1 \land (1 - x + y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - x + y & \text{otherwise}. \end{cases}
\]

The corresponding negations are

\[
\tilde{x} = \tilde{x} = 1 - x.
\]

Notice that \(L_{m+1}\) is a very particular case of WPBL-algebra, namely an MV-algebra.

We intend to characterize the pairs of weak negations on \(L_{m+1}\) compatible with \(*\). According to Theorem 4.1, any pair \((n_1, n_2)\) of compatible weak negations satisfies the following conditions, for any \(x, y, z \in L_{m+1}:\)

(i) \(\tilde{x} \leq n_1(x)\) and \(\tilde{x} \leq n_2(x);\)

(ii) \(y > n_1(z)\) implies \(n_1(y \ast z) = n_1(y) \lor (y \rightarrow n_1(z));\)

(iii) \(y > n_2(z)\) implies \(n_2(z \ast y) = n_2(y) \lor (y \bowtie n_2(z)).\)

So let \(n_1, n_2 : L_{m+1} \longrightarrow L_{m+1}\) be two functions. We shall establish first some necessary conditions for \((n_1, n_2)\) to be a pair of compatible weak negations.

We have that \(n_1(1) = n_2(1) = 0\) and \(n_1(0) = n_2(0) = 1\) from the “pair of weak negations” property. If \(z \notin \{0, 1\}\), from (i), we get

\[
n_1(z) \geq 1 - z > 0, \quad n_2(z) \geq 1 - z > 0.
\]
Again, if \( z \not\in \{0, 1\} \), we get \( n_1(z) < 1 \), because otherwise we would have \( n_2(n_1(z)) = n_2(1) = 0 < z \), impossible; \( n_2(z) < 1 \), because otherwise we would have \( n_1(n_2(z)) = n_1(1) = 0 < z \), impossible.

So \( n_1(0) = n_2(0) = 1 \), \( n_1(1) = n_2(1) = 0 \) and

\[
n_1(L_{m+1} - \{0, 1\}) \subseteq L_{m+1} - \{0, 1\}, \quad n_2(L_{m+1} - \{0, 1\}) \subseteq L_{m+1} - \{0, 1\}.
\]

Moreover, since \( n_1(1/m), n_1(1/m) \geq (m + 1)/m \), we have

\[
n_1 \left( \frac{1}{m} \right) = n_2 \left( \frac{1}{m} \right) = \frac{m + 1}{m}.
\]

Notice that, because of the commutativity of \( * \) and the identity between \( \to \) and \( \sim \), (ii) and (iii) are in fact the same condition, for \( n_1 \) and \( n_2 \), respectively.

Let us take \( y = (m - 1)/m \). Condition (ii) becomes

\[
n_1(z) < \frac{m - 1}{m} \quad \text{implies} \quad n_1 \left( \frac{m - 1}{m} * z \right) = n_1 \left( \frac{m - 1}{m} \right) \lor \left( \frac{m - 1}{m} \to n_1(z) \right).
\]

Further, considering the fact that \( n_1(z) < (m - 1)/m \) implies \( (m - 1)/m \to n_1(z) = 1 - (m - 1)/m + n_1(z) = n_1(z) + 1/m \), the condition becomes

\[
n_1(z) < \frac{m - 1}{m} \quad \text{implies} \quad n_1 \left( \frac{m - 1}{m} * z \right) = n_1 \left( \frac{m - 1}{m} \right) \lor \left( n_1(z) + \frac{1}{m} \right).
\]

Let \( z \not\in \{0, 1\} \) such that \( n_1(z) < (m - 1)/m \). Then

\[
n_1 \left( \frac{m - 1}{m} + z - 1 \right) = n_1 \left( \frac{m - 1}{m} * z \right) = n_1 \left( \frac{m - 1}{m} \right) \lor \left( n_1(z) + \frac{1}{m} \right),
\]

that is

\[
n_1 \left( z - \frac{1}{m} \right) = n_1 \left( \frac{m - 1}{m} \right) \lor \left( n_1(z) + \frac{1}{m} \right).
\]

Because \( z \leq (m - 1)/m \), we have that \( n_1(z) + 1/m \geq n_1(z) \geq n_1((m - 1)/m) \), so \( n_1(z - 1/m) = n_1(z) + 1/m \).

Notice that, if \( z \not\in \{0, 1\} \) and \( n_1(z) < (m - 1)/m \), then any \( y \) placed between \( z \) and \( (m - 1)/m \) has the property that \( y \not\in \{0, 1\} \) and \( n_1(y) < (m - 1)/m \), so \( n_1(y - 1/m) = n_1(y) + 1/m \).

Inductively, knowing that \( n_1(0) = 1 \), \( n_1(1/m) = (m - 1)/m \) and \( n_1(1) = 0 \), we get the following situation (A):

- \( n_1(0) = 1 \);
- there exists \( k \in \{1, \ldots, m - 1\} \) such that
  - for any \( i \in \{1, \ldots, k\} \), \( n_1(i/m) = (m - 1)/m \);
  - for any \( i \in \{k + 1, \ldots, m - 1\} \), \( n_1(i/m) = ((m - 1) - i + k)/m \);
- \( n_1(1) = 0 \).

**Remark.** If \( i = k \), then \( n_1(i/m) = (m - 1)/m = ((m - 1) - i + k)/m \), so we can say that \( n_1(i/m) = ((m - 1) - i + k)/m \) is true for any \( i \in \{k, \ldots, m - 1\} \).
In a perfectly analogous manner, one can obtain the situation above for $n_2$, replacing $n_1$ by $n_2$.

Let us show now that $n_1$ is necessarily equal to $n_2$.

Suppose that $i \in \{1, \ldots, m-1\}$ and $n_1(i/m) = (m-1)/m$. If we had $n_2(i/m) < (m-1)/m$, we would get

$$n_2\left(\frac{i+1}{m}\right) < \frac{m-1}{m} - \frac{1}{m} = \frac{m-1}{m}, \ldots, n_2\left(\frac{m-1}{m}\right) < \frac{i}{m}.$$  

So $n_2n_1(i/m) = n_2((m-1)/m) < i/m$, which contradicts the fact that $n_2n_1(x) \geq x$ for any $x$.

So we have that, for any $i \in \{1, \ldots, m-1\}$,

$$n_1\left(\frac{i}{m}\right) = \frac{m-1}{m} \implies n_2\left(\frac{i}{m}\right) = \frac{m-1}{m}.$$  

Similarly,  

$$n_2\left(\frac{i}{m}\right) = \frac{m-1}{m} \implies n_1\left(\frac{i}{m}\right) = \frac{m-1}{m}.$$  

Now, from the form (A) of the functions $n_1$ and $n_2$, it follows immediately that $n_1 = n_2$.

Thus we have shown that any pair $(n_1, n_2)$ of weak negations on $L_{m+1}$ which is compatible with $*$ has the property that $n_1 = n_2$ and $n_1$ has the form (A).

We shall now prove that any pair of functions $(n_1, n_1)$, where $n_1$ has the form (A), is a pair of weak negations compatible with $*$.

$n_1$ is obviously non-increasing and $n_1(1) = 0$.

Let us show that $n_1^2(x) \geq x$ for any $x \in L_{m+1}$. $n_1^2(0) = n_1(1) = 0$ and $n_1^2(1) = n_1(0) = 1$. If $i \in \{1, \ldots, k\}$,

$$n_1^2\left(\frac{i}{m}\right) = n_1\left(\frac{m-1}{m}\right) = \frac{k}{m} \geq \frac{i}{m}.$$  

If $i \in \{k, \ldots, m-1\}$, then $(m-1) - i + k \in \{k, \ldots, m-1\}$, so

$$n_1^2\left(\frac{i}{m}\right) = n_1\left(\frac{(m-1) - i + k}{m}\right) = \frac{(m-1) - (m-1) - i + k + k}{m} = \frac{i}{m}.$$  

Thus $(n_1, n_1)$ is a pair of weak negations.

It remains to show the compatibility with $*$. We use again Theorem 4.1, checking conditions (i)–(iii).

(i) $n_1(0) = 1 \geq 1 - 0$; $n_1(1) = 0 \geq 1 - 1$.

Let $i \in \{1, \ldots, m-1\}$. If $i \leq k$, then

$$n_1\left(\frac{i}{m}\right) = \frac{m-1}{m} \geq \frac{m-i}{m} = 1 - \frac{i}{m}.$$  

If $i > k$, then

$$n_1\left(\frac{i}{m}\right) = \frac{(m-1) - i + k}{m} = \frac{m-i}{m} + \frac{k-1}{m} \geq \frac{m-i}{m} = 1 - \frac{i}{m}.$$
(ii) and (iii) Let \( y, z \in L_{m+1} \) such that \( y > n_1(z) \). We have four cases:

(I) \( y = 0 \) or \( z = 0 \): \( y > n_1(z) \) implies anything.

(II) \( y = 1 \): in this case, the condition becomes

\[
1 > n_1(z) \implies n_1(z) = n_1(1) \lor (1 \rightarrow n_1(z)),
\]

that is

\[
1 > n_1(z) \implies n_1(z) = 0 \lor n_1(z),
\]

which is of course true.

(III) \( z = 1 \): the condition becomes

\[
y > 0 \implies n_1(y) = n_1(y) \lor (y \rightarrow 0),
\]

which is also true because \( n_1(y) \geq \bar{y} \).

(IV) \( y = (m - j)/m \), with a fixed \( j \in \{1, \ldots, m - 1\} \) and \( z \in \{1/m, \ldots, (m - 1)/m \} \). The condition becomes

\[
n_1(z) < \frac{m - j}{m} \text{ and } z \not\in \{0, 1\} \implies n_1 \left( \frac{m - j}{m} \right) \lor \left( \frac{m - j}{m} \rightarrow n_1(z) \right),
\]

that is

\[
n_1(z) < \frac{m - j}{m} \text{ and } z \not\in \{0, 1\} \implies n_1 \left( \frac{0 \lor (z - j/m)}{m} \right)
\]

\[
= n_1 \left( \frac{m - j}{m} \right) \lor \left( \frac{m - j}{m} \rightarrow n_1(z) \right).
\]

If \( z \leq j/m \), the implication is trivially true, since \( n_1(z) \geq 1 - z \geq 1 - j/m = (m - j)/m \), so \( n_1(z) < (m - j)/m \). So let us assume that \( z > j/m \), \( z \neq 1 \), \( n_1(z) < (m - j)/m \) and let us prove that \( n_1(z - j/m) = n_1((m - j)/m) \lor (n_1(z) + j/m) \). From \( z - j/m < (m - j)/m \), we get \( n_1(z - j/m) \geq n_1((m - j)/m) \), so it will be sufficient to prove that \( n_1(z - j/m) = n_1(z + j/m) \).

We know that \((k \text{ is the number from (A)})\)

\[
n_1 \left( \frac{1}{m} \right) = \cdots = n_1 \left( \frac{k - 1}{m} \right) = \frac{m - 1}{m},
\]

\[
n_1 \left( \frac{k}{m} \right) = \frac{m - 1}{m}, \ldots, n_1 \left( \frac{m - 1}{m} \right) = \frac{k}{m}.
\]

Because \( n_1(z) < (m - j)/m \), we have that \( z \geq (k + j)/m \) and \( z - j/m \geq k/m \). Thus \( z - j/m = i/m \), where \( i \in \{k, \ldots, m - 1 - j\} \). So

\[
n_1 \left( \frac{z - j}{m} \right) = n_1 \left( \frac{i}{m} \right) = \frac{(m - 1) - i + k}{m},
\]

\[
n_1 \left( \frac{z + j}{m} \right) = n_1 \left( \frac{i}{m} \right) = \frac{(m - 1) - i + k}{m},
\]

\[
n_1 \left( \frac{k}{m} \right) = \frac{m - 1}{m}, \ldots, n_1 \left( \frac{m - 1}{m} \right) = \frac{k}{m}.
\]
\[ n_1(z) = n_1 \left( \frac{i + j}{m} \right) = \frac{(m - 1) - (i + j) + k}{m} = \frac{(m - 1) - i + k}{m} - \frac{j}{m}. \]

Consequently,
\[ n_1 \left( z - \frac{j}{m} \right) = n_1(z) + \frac{j}{m}. \]

This ends our discussion of the four cases. Thus, we have proven the following.

**Theorem 5.1.** There is only one compatible pair of weak negations on \( L_2 \), the trivial one. If \( m \geq 2 \), \((n_1, n_2)\) is a compatible pair of weak negations on \( L_{m+1} \) if and only if \( n_1 = n_2 \) and there exist \( k \in \{1, \ldots, m - 1\} \) such that

- \( n_1(0) = 1 \),
- for any \( i \in \{1, \ldots, k\} \), \( n_1(i/m) = (m - 1)/m \),
- for any \( i \in \{k + 1, \ldots, m - 1\} \), \( n_1(i/m) = ((m - 1) - i + k)/m \),
- \( n_1(1) = 0 \).

So all the new structures of WPBL-algebra obtained on \( \{0, 1/m, \ldots, (m - 1)/m, 1\} \) from the pairs of compatible negations on \( L_{m+1} \) have the operation \( \otimes \) commutative and \( \Rightarrow = \approx \approx \approx \Rightarrow = \rightarrow \). Let us illustrate these structures for \( L_4 \) and \( L_5 \): (at \( \otimes \) and \( \Rightarrow \), the values of the first argument are displayed vertically and the values of the second argument horizontally):

(a) \( L_4 \)
1. \( n_1 = n_2 = - \),
\[ \otimes = * \Rightarrow = \approx \Rightarrow = \rightarrow . \]
2. \[ \begin{array}{cccc}
  x & 0 & 1/3 & 2/3 & 1 \\
  n_1(x) = n_2(x) & 1 & 2/3 & 2/3 & 0 \\
\end{array} \]

\[ \begin{array}{ccccccc}
  \otimes & 0 & 1/3 & 2/3 & 1 & \Rightarrow = \approx \Rightarrow & 0 & 1/3 & 2/3 & 1 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
  1/3 & 0 & 0 & 0 & 1/3 & 1/3 & 2/3 & 2/3 & 1 & 1 \\
  2/3 & 0 & 0 & 0 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1 \\
  1 & 0 & 1/3 & 2/3 & 1 & 1 & 0 & 1/3 & 2/3 & 1 \\
\end{array} \]

(b) \( L_5 \)
1. \( n_1 = n_2 = - \),
\[ \otimes = * \Rightarrow = \approx \Rightarrow = \rightarrow . \]
2. \[ \begin{array}{cccc}
  x & 0 & 1/4 & 2/4 & 3/4 & 1 \\
  n_1(x) = n_2(x) & 1 & 3/4 & 3/4 & 2/4 & 0 \\
\end{array} \]
Thus, we have not found on $L_{m+1}$ proper pairs (i.e. that are not identical pairs) of weak negations, but $m - 1$ weak negations. In [4], it is shown that the only compatible weak negation on the MV structure with support $[0, 1]$ is the MV negation itself, $x \mapsto 1 - x$. This discrepancy between the discrete and continuous cases comes from the fact that a weak negation on $[0, 1]$ is forced to be a strong negation (i.e. an involution), hence a non-increasing bijection, hence a continuous bijection. The examples of weak negations on $L_{m+1}$ different from $x \mapsto 1 - x$ are precisely the non-strong ones.

By Proposition 4.3, all the pairs of weak negations on $[0, 1]$ are compatible with the Gödel t-norm. Finding all the pairs of weak negations compatible with the other two canonical BL structures (MV and product) on $[0, 1]$ remains an open problem which looks, unless one proves that they are just identical pairs, quite difficult. Although each member of the pair is still left continuous, the higher degree of liberty in the adjunction condition doesn’t allow us to reduce the problem to strong negations like in the commutative case [4]. The discovering of non-trivial pairs of weak negations might lead us to interesting WPBL-algebras on $[0, 1]$.

6. Putting the pairs of weak negations to work—some examples

In what follows, we shall consider two structures of WPBL-algebra, one of them commutative and the other not commutative showing that the commutativity of the starting structure and the equality between the two components of any compatible pair $(n_1, n_2)$ of weak negations are two independent facts.

Firstly, let $A = \{0, a, b, c\}$ with $0 < a < b < 1$ and the following binary operations on $A$ (as usual, the values of the first argument are displayed vertically and the values of the second argument

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<td>1</td>
</tr>
<tr>
<td>3/4</td>
<td>0</td>
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<tr>
<td>1</td>
<td>0</td>
<td>1/4</td>
<td>2/4</td>
<td>3/4</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus, we have not found on $L_{m+1}$ proper pairs (i.e. that are not identical pairs) of weak negations, but $m - 1$ weak negations. In [4], it is shown that the only compatible weak negation on the MV structure with support $[0, 1]$ is the MV negation itself, $x \mapsto 1 - x$. This discrepancy between the discrete and continuous cases comes from the fact that a weak negation on $[0, 1]$ is forced to be a strong negation (i.e. an involution), hence a non-increasing bijection, hence a continuous bijection. The examples of weak negations on $L_{m+1}$ different from $x \mapsto 1 - x$ are precisely the non-strong ones.

By Proposition 4.3, all the pairs of weak negations on $[0, 1]$ are compatible with the Gödel t-norm. Finding all the pairs of weak negations compatible with the other two canonical BL structures (MV and product) on $[0, 1]$ remains an open problem which looks, unless one proves that they are just identical pairs, quite difficult. Although each member of the pair is still left continuous, the higher degree of liberty in the adjunction condition doesn’t allow us to reduce the problem to strong negations like in the commutative case [4]. The discovering of non-trivial pairs of weak negations might lead us to interesting WPBL-algebras on $[0, 1]$.

6. Putting the pairs of weak negations to work—some examples

In what follows, we shall consider two structures of WPBL-algebra, one of them commutative and the other not commutative showing that the commutativity of the starting structure and the equality between the two components of any compatible pair $(n_1, n_2)$ of weak negations are two independent facts.

Firstly, let $A = \{0, a, b, c\}$ with $0 < a < b < 1$ and the following binary operations on $A$ (as usual, the values of the first argument are displayed vertically and the values of the second argument

<table>
<thead>
<tr>
<th>$\otimes$</th>
<th>0</th>
<th>1/4</th>
<th>2/4</th>
<th>3/4</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/4</td>
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<td>0</td>
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<td>0</td>
<td>1/4</td>
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<tr>
<td>2/4</td>
<td>0</td>
<td>0</td>
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<td>2/4</td>
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</tr>
<tr>
<td>3/4</td>
<td>0</td>
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<td>2/4</td>
<td>3/4</td>
<td>0</td>
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<td>0</td>
<td>1/4</td>
<td>2/4</td>
<td>3/4</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Rightarrow=\Rightarrow$</th>
<th>0</th>
<th>1/4</th>
<th>2/4</th>
<th>3/4</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1/4</td>
<td>0</td>
<td>3/4</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2/4</td>
<td>0</td>
<td>3/4</td>
<td>0</td>
<td>2/4</td>
<td>1</td>
</tr>
<tr>
<td>3/4</td>
<td>0</td>
<td>3/4</td>
<td>3/4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/4</td>
<td>2/4</td>
<td>3/4</td>
<td>1</td>
</tr>
</tbody>
</table>
horizontally):

\[
\begin{array}{ccc|ccc|ccc}
* & 0 & a & b & 1 & \rightarrow & 0 & a & b & 1 & \sim & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
a & 0 & 0 & 0 & a & a & 1 & 1 & 1 & 1 & a & b & 1 & 1 \\
b & 0 & a & b & b & a & a & 1 & 1 & b & 0 & a & 1 & 1 \\
1 & 0 & a & b & b & 1 & 0 & a & b & 1 & 1 & 0 & a & b & 1 \\
\end{array}
\]

By Flondor [8], \((A, \lor, \land, *, \rightarrow, \sim, 0, 1)\) is a WPBL-algebra. The negations \(-, \sim \) of \(A\) are given by

\[
\begin{array}{ccc|ccc|ccc}
x & 0 & a & b & 1 \\
\bar{x} & 1 & a & a & 0 \\
\bar{x} & 1 & b & 0 & 0 \\
\end{array}
\]

We shall determine the compatible pairs of weak negations on \(A\).

Let \((n_1, n_2)\) be such a pair. In accordance to condition (i) of Theorem 4.1,

\[ n_1(a) \geq \bar{\bar{a}} = a, \quad n_1(b) \geq \bar{\bar{b}} = a, \quad n_2(a) \geq \bar{\bar{a}} = b, \quad n_2(b) \geq \bar{\bar{b}} = 0, \]

hence \(n_1(a) \in \{a, b, 1\}, \quad n_1(b) \in \{a, b, 1\}, \quad n_2(a) \in \{b, 1\}, \quad n_2(b) \in \{0, a, b, 1\}\).

If \(n_1(a) = 1\), then \(n_2n_1(a) = n_2(1) = 0 < a\); so, by axiom (N2), \(n_1(a) \in \{a, b\}\) and, similarly, \(n_1(b) \in \{a, b\}, \quad n_2(a) = b \quad \text{and} \quad n_2(b) \in \{0, a, b\}\).

(I) Assume that \(n_1(a) = a\). In accordance to Theorem 4.1(ii), \(b > a = n_1(a)\) implies \(n_1(b*a) = n_1(b) \lor (b \rightarrow n_1(a))\). But \(b*a = a\), hence \(a = n_1(b) \lor (b \rightarrow a) = n_1(b) \lor a\); so \(n_1(b) \leq a\). It follows that \(n_1(b) = a\).

Thus, we have the following possibilities:

\begin{enumerate}
\item[(x1)] \(n_2(a) = b; \quad n_2(b) = 0\);
\item[(x2)] \(n_2(a) = b; \quad n_2(b) = a\) (impossible, because it leads to \(n_1n_2(b) = n_1(a) = a < b\));
\item[(x3)] \(n_2(a) = b; \quad n_2(b) = b\) (impossible, because it would imply \(n_1n_2(b) = n_1(b) = a < b\)).
\end{enumerate}

(II) Assume \(n_1(a) = b\). We must consider two cases:

\begin{enumerate}
\item[(\beta)] \(n_1(b) = a\);
\item[(\gamma)] \(n_1(b) = b\).
\end{enumerate}

For (\beta) we have the following subcases:

\begin{enumerate}
\item[(\beta1)] \(n_2(b) = 0\) (impossible, because we would have \(n_2n_1(a) = n_2(b) = 0 < a\));
\item[(\beta2)] \(n_2(b) = a\);
\item[(\beta3)] \(n_2(b) = b\) (impossible, because in this case \(n_1n_2(b) = n_1(b) = a < b\)).
\end{enumerate}

For (\gamma) we have the following subcases:

\begin{enumerate}
\item[(\gamma1)] \(n_2(b) = 0\) (impossible, because of \(n_2n_1(b) = n_2(b) = 0 < b\));
\item[(\gamma2)] \(n_2(b) = a\) (impossible, because in this case \(n_2n_1(b) = n_2(b) = a < b\));
\item[(\gamma3)] \(n_2(b) = b\).
\end{enumerate}

Now we shall analyze the cases (x1), (\beta2) and (\gamma3).

Case (x1): In this case \((n_1, n_2) = (\sim, \sim)\).
Case ($\beta_2$):

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1(x)$</td>
<td>1</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
<tr>
<td>$n_2(x)$</td>
<td>1</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

The pair $(n_1, n_2)$ verifies conditions (ii) and (iii) of Theorem 4.1. For $b > n_1(b) = a$ we have

$$n_1(b) \lor (1 \rightarrow n_1(b)) = a \lor (1 \rightarrow a) = a = n_1(b \ast b),$$

because $b \ast b = b$. The other cases are obvious. Therefore, $(n_1, n_2)$ is a pair of weak negations compatible with $\ast$. The operations $\otimes$, $\Rightarrow$ and $\approx$ are the following:

<table>
<thead>
<tr>
<th>$\otimes$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
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</table>

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>a</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>1</td>
</tr>
</tbody>
</table>

Case ($\gamma_3$):

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1(x)$</td>
<td>1</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
<tr>
<td>$n_2(x)$</td>
<td>1</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

Conditions (ii) and (iii) of Theorem 4.1 are easily verified, hence $(n_1, n_2)$ is a pair of weak negation compatible with $\ast$. Here are the tables for $\otimes$, $\Rightarrow$ and $\approx$:

<table>
<thead>
<tr>
<th>$\otimes$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus, all the non-trivial (i.e. different from the pair of negations themselves) compatible pairs of weak negations $(n_1, n_2)$ have $n_1 = n_2$ and also, the generated new structures are commutative, although the starting structure was not commutative.

Next, consider $B = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$. The operations $\ast$, $\rightarrow$ and $\sim$ are given in the following tables (since $\ast$ is commutative, we have $\rightarrow = \sim$):

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>a</td>
<td>a</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>a</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>
One can easily see that $B$ is a WPBL-algebra (actually, it is even an MTL-algebra). Using a computer, we discover that we have 19 compatible pairs of weak negations $(n_1, n_2)$. Seven of them have the property that $n_1 = n_2$ (hence the generated structures are commutative, i.e. MTL-algebras), while 12 have $n_1$ different from $n_2$ and also, the generated structures proper (i.e. non-commutative) WPBL-algebras. For instance, a compatible pair of weak negations is

\[
\begin{array}{c|cccc}
    x & 0 & a & b & c \\
    \hline
    n_1(x) & 1 & c & c & a \\
    n_2(x) & 1 & c & b & b \\
\end{array}
\]

The corresponding new WPBL structure on $B$ is given by

\[
\begin{array}{c|cccc}
    & 0 & a & b & c \\
\hline
    0 & 0 & 0 & 0 & 0 \\
    a & 0 & 0 & 0 & a \\
    b & 0 & 0 & b & b \\
    c & 0 & 0 & c & c \\
    1 & 0 & a & b & c \\
\end{array}
\]

\[
\begin{array}{c|cccc}
    & 0 & a & b & c \\
\hline
    0 & 1 & 1 & 1 & 1 \\
    a & c & 1 & 1 & 1 \\
    b & c & c & 1 & 1 \\
    c & a & a & b & 1 \\
    1 & 0 & a & b & c \\
\end{array}
\]

\[
\begin{array}{c|cccc}
    & 0 & a & b & c \\
\hline
    0 & 1 & 1 & 1 & 1 \\
    a & c & 1 & 1 & 1 \\
    b & b & b & 1 & 1 \\
    c & b & b & 1 & 1 \\
    1 & 0 & a & b & c \\
\end{array}
\]

Hence, from a commutative WPBL-algebra (MTL-algebra), using compatible pairs of weak negations, we obtain new examples of non-commutative proper WPBL-algebras.

An interesting open problem is the following. Let $X$ be a set. For $\mathcal{A}$ a class of WPBL-chains on $X$, we can consider its closure $\mathcal{A}'$ w.r.t. compatible pairs of weak negations, consisting of all the WPBL-chains obtained by iteratively applying the construction described in Section 4 and exemplified in this section. We may search for minimal classes $\mathcal{A}$ such that $\mathcal{A}'$ is the whole set of WPBL-chains on $X$. In particular, when $X$ is finite, it might be the case that a particular well-known WPBL-algebra (perhaps a commutative one) iteratively generates by this procedure all the possible structures, obtaining some sort of characterization for finite WPBL-chains. Notice that, because residuation follows from completeness, the finite WPBL-chains are precisely compatible totally ordered monoids for which the identity is the greatest element and the lowest element is annihilator. Thus, a characterization by means of pairs of weak negations starting from a “standard” structure would be a useful result of non-commutative algebra, not envolving the inconvenient axiom of associativity.

7. Concluding remarks

It is worth noticing that compatible pairs of weak negations, as an instrument for building new examples of proper WPBL-algebras, might also work if the starting structure is a commutative one, because commutativity does not force, as shown by the second example of Section 6, the two weak negations to coincide. A topic for future research would be the extension to the non-commutative case of the apparatus developed in [15–17], where there are searched “reasonable” truth structures, with disjunction being dual to the conjunction, fact that is achieved by working with a strong negation.
Theorem 4.1 (a generalization of Theorem 1 from [4]) gives a criterion for the fact that a couple of functions forms a pair of weak negations and it is used in Section 5 to study compatible pairs of weak negations on a well-known finite structure, \( L_{m+1} \). Our initial aim was to regularly obtain a class of finite proper WPBL-algebras for an arbitrary number of elements. Unfortunately, we discovered that all the pairs were identical pairs, hence weak negations, so we got a “commutative” result: we have \( m - 1 \) weak negations on \( L_{m+1} \), complementing the result from [4] which says that, in the canonical MV-algebra on \([0,1]\), there is only one weak negation. As shown by Section 5, finding the compatible pairs of weak negations, already in the case of some well-known structures like \( L_{m+1} \), is a quite non-trivial matter. It remains an open problem to find them for more general finite WPBL-algebras then MV-algebras, for the BL-algebras on \([0,1]\] and the non-commutative WPBL-algebras on \([0,1]\] from [9].

Pairs of weak negations are also interesting to be considered independently from the fuzzy structures, only dealing with a partial order. At this very general level (apart from any “logical” meaning of weak negations), a pair of weak negations (actually forming a Galois connection) seems to appear more naturally then a weak negation that “makes pair with itself”. The geometrical characterization from Proposition 3.3 gives a very “visual” meaning of pairs of weak negations (and, without assumption (N1), of Galois connections), provided the order is total. This shows that the total order assumption makes Galois-connection-like structures much more accessible to the intuition and hence easier to handle.

Acknowledgements

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References