GRAVITOMAGNETIC EFFECTS

G. Schäfer

Friedrich-Schiller-Universität Jena, Theoretisch-Physikalisches Institut
Max-Wien-Pl. 1, 07743 Jena, Germany, Email: gos@tpi.uni-jena.de

ABSTRACT/RESUME

The paper summarizes the most important effects in Einsteinian gravitomagnetic fields related to propagating light rays, moving clocks and atoms, orbiting objects, and precessing spins. Emphasis is put onto the gravitational interaction of spinning objects. The gravitomagnetic field lines of a rotating or spinning object are given in analytic form.

1. INTRODUCTION

In the Einstein theory of gravity the gravitational field is described by ten potential functions $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3,\ g_{\mu\nu} = g_{\nu\mu})$ which at the same time are the metric coefficients of curved spacetime. The potential function g_{00}, sometimes called gravitational redshift potential, is connected with the Newtonian gravitational potential, the six functions g_{ij} $(i, j = 1, 2, 3)$ describe the geometry of the curved three-dimensional spaces defined by the slices of constant time t, or $x^0 = ct$, where c denotes the speed of light, and finally, the three functions g_{0i} describe how the geometry of the three-dimensional slices rotates in going from one slice to another. For weak gravitational fields, the three potential functions g_{0i} behave very much like the electromagnetic 3-potential A_i^{em} on account of which the g_{0i} are often called gravitomagnetic field potentials. Similarly to the generation of the electromagnetic potential A_i^{em} through charge currents is the gravitomagnetic field generated through mass currents (momentum densities) if also by a factor of four more efficient because of an underlying tensor theory (spin-2 field theory versus spin-1 field theory of electrodynamics; implying also a sign difference).
In this paper important effects connected with **weak** gravitomagnetic fields will be discussed. The applied class of gravitomagnetic fields will originate both from spinning and orbiting mass currents. The objects moving or propagating in the gravitomagnetic fields will be non-spinning and spinning objects (celestial bodies, particles, clocks, atoms, black holes, etc.) as well as light rays. The gravitomagnetic field lines of a spinning object will be given in analytic form. In all publications known to the author, the graphs for the gravitomagnetic field lines are not presented in fully exact form.

2. SPINS IN MINKOWSKI SPACE

In the weak-field limit of Einstein's theory of gravity the treatment of spinning objects is closely related to their treatment in Minkowski space. Therefore, in this section, properties of spinning or (rigidly) rotating objects in Minkowski space will be discussed. The most important outcome of the present section is the relation between the canonical position variable of a spinning object and the various centre-of-mass definitions.
Written in canonical variables, the total angular momentum of a spinning object in Minkowski space takes the form

$$\mathbf{J} \ = \ \mathbf{R} \times \mathbf{P} \ + \ \mathbf{S} \,, \tag{1}$$

where $\mathbf{R}$, $\mathbf{P}$, and $\mathbf{S}$ denote the position vector, the linear momentum, and the spin vector of the object, respectively. $\mathbf{R}$ and $\mathbf{P}$ are canonically conjugate varibles which commute with the spin vector $\mathbf{S}$ the components of which fulfil the standard angular momentum commutation relations. The Poincaré algebra tells us that the centre-of-mass constant $\mathbf{K}$ has to take the form [1]

$$\mathbf{K} \ = \ x^0 \mathbf{P} \ - \ \mathbf{R} P^0 \ + \ \frac{1}{P^0 + Mc} \, \mathbf{S} \times \mathbf{P} \,, \tag{2}$$

where M and $cP^0 = H$ are the rest mass and the energy of the object, respectively. The 4-momentum reads $P^\mu = (P^0, \mathbf{P})$ $(P^\mu P_\mu = -M^2 c^2)$ and for the (coordinate) velocity $\mathbf{v}$ the relation $\mathbf{v} = c\mathbf{P}/P^0$ holds. From the Eq. 2 the centre-of-mass coordinate results in the form

$$\hat{\mathbf{R}} \ = \ \mathbf{R} \ - \ \frac{1}{P^0 + Mc} \, \mathbf{S} \times \frac{\mathbf{P}}{P^0} \,, \tag{3}$$

using the standard definition

$$\mathbf{K} \;=\; x^0\mathbf{P} \;-\; \hat{\mathbf{R}}P^0\,. \tag{4}$$

In terms of the antisymmetric spin-4-tensor $S^{\mu\nu}$ the Eq. 4 implies $S^{0i} = 0$ (so-called Corinaldesi-Papapetrou spin supplementary condition, e.g. see [2]). This spin tensor we may call $\hat{S}^{\mu\nu}$ with $\hat{S}^{ij} = \epsilon^{ijk}\hat{S}_k$, where ϵ^{ijk} denotes the total antisymmetric Levi-Civita tensor. Then the total angular momentum 4-tensor $J^{\mu\nu}$ takes the form

$$J^{\mu\nu} \;=\; \hat{X}^\mu P^\nu \;-\; \hat{X}^\nu P^\mu \;+\; \hat{S}^{\mu\nu}, \quad \hat{S}^{0\mu} \;=\; 0\,, \tag{5}$$

with $\hat{X}^\mu = (\hat{X}^0, \hat{\mathbf{R}})$.

Let us call the spin-4-tensor say, $\hat{S}^{\mu\nu}_{\rm rf}$, if the covariant condition holds, $S^{\mu\nu}U_\nu = 0$ (so-called Pirani spin supplementary condition, e.g. see [2]) with $U_\nu = P_\nu/Mc$. The index rf is chosen such as to indicate that the definition of the centre-of-mass is made in the rest frame. Then we get

$$J^{\mu\nu} \;=\; \hat{X}^\mu_{\rm rf}P^\nu \;-\; \hat{X}^\nu_{\rm rf}P^\mu \;+\; \hat{S}^{\mu\nu}_{\rm rf}, \quad \hat{S}^{\mu\nu}_{\rm rf}U_\nu \;=\; 0\,, \tag{6}$$

with $\hat{X}^\mu_{\rm rf} = (\hat{X}^0_{\rm rf}, \hat{\mathbf{R}}_{\rm rf})$ and $\hat{S}^{\mu\nu}_{\rm rf} = \epsilon^{\mu\nu\alpha\beta}U_\alpha\hat{S}_{\beta{\rm rf}}$. Belonging to the same reference frame, we may put $\hat{X}^0 = \hat{X}^0_{\rm rf} = x^0$. The relation between $\hat{\mathbf{R}}$ and $\hat{\mathbf{R}}_{\rm rf}$ is achieved by a Lorentz transformation from the rest frame ($X^\mu_{\rm rf}$) to the moving frame where, for centre-of-mass coordinates, the hat applies ($\hat{X}^\mu_{\rm rf}$). One finds,

$$\hat{\mathbf{R}} \;=\; \hat{\mathbf{R}}_{\rm rf} \;-\; \frac{\mathbf{S}_{\rm rf}}{Mc} \times \frac{\mathbf{P}}{P^0}\,, \tag{7}$$

where $\mathbf{S}_{\rm rf}$ denotes the spin of the object in the rest frame, $S^{ij}_{\rm rf} = \epsilon^{ijk0}S_{k{\rm rf}}$, [3], [2].

Obviously, $\hat{\mathbf{R}} - \hat{\mathbf{R}}_{\rm rf}$, i.e. the difference vector of the centre-of-mass positions, on the one side defined in the reference frame where the object is moving and on the other side defined in the rest frame but (Lorentz) transformed to the system where the object is moving, is orthogonal to $\mathbf{v}$; thus there is no Lorentz contraction involved. Under the assumption of positivity of energy density of the object in all inertial frames it follows that the radius of the **minimum size** of the object, orthogonal to the spin direction, is given by [3], [4],

$$\frac{|\mathbf{S}_{\rm rf}|}{Mc}\,. \tag{8}$$

The Eq. 8 also fits with electrons. The insertion of the spin of the electron results in the Compton wavelength as diameter of the area in question which is consistent with the minimum size of positive frequency wave functions. It is also nice to point out that even the radius of the ring singularity of a Kerr black hole fits with Eq. 8 although in Einstein's theory of gravity Minkowski spaces do exist only locally.

3. GRAVITOMAGNETISM

The line element of 4-dimensional curved spacetime can be decomposed into various forms (the signature of the metric is +2),

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= g_{00}c^2dt^2 + 2g_{0i}cdtdx^i + g_{ij}dx^idx^j \\ &= (-g_{00})[-(cdt - A_idx^i)^2 + (G_{ij} + A_iA_j)dx^idx^j] \\ &= -N^2c^2dt^2 + g_{ij}(dx^i + N^icdt)(dx^j + N^jcdt)\,, \end{aligned} \tag{9}$$

where by definition,

$$A_i \equiv \frac{-g_{0i}}{g_{00}}\,, \quad G_{ij} \equiv \frac{-g_{ij}}{g_{00}}\,, \tag{10}$$

$$N^i \equiv \frac{-g^{0i}}{g^{00}}\,, \quad N^2 \equiv \frac{-1}{g^{00}} = -g_{00} + g_{ij}N^iN^j\,, \tag{11}$$

hold. The functions $g^{\mu\nu}$ define the inverse metric, $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$. N and N^i are called lapse and shift functions, respectively.

The decomposition of the line element into the form where A_i appears is adapted to observers at rest in the given coordinate system; for those observers equal time means: $cdt = A_idx^i$, see Fig. 1. On the other side, the decomposition where the lapse and shift functions appear relates to observers at rest in "absolute spaces", i.e. in the spaces defined by $t =$ const., [5]. Those observers move in the given coordinate system according to $dx^i = -N^icdt$, see Fig. 2.

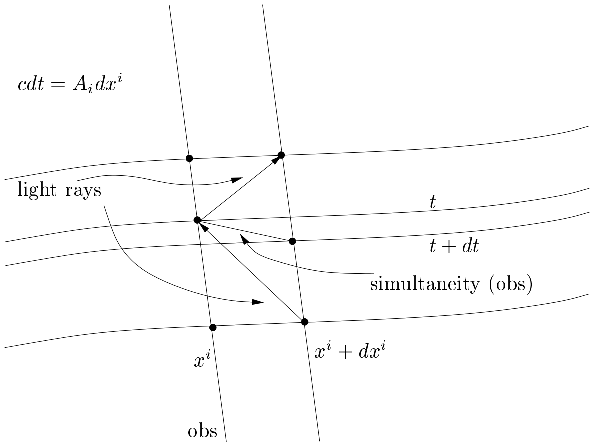

Fig. 1. Shown is infinitesimal simultaneity for an observer located at x^i (Einstein synchronization).

The functions g_{0i}, A_i, and N^i are different representations of the gravitomagnetic field (this field comprises the Coriolis field as well), respectively being denoted by $\mathbf{g}$, $\mathbf{A}$, and $\mathbf{N}$; their relations are: $g_{0i} = -g_{00}A_i = g_{ik}N^k$. Obviously, $N^i = \gamma^{ik}g_{0k}$ is valid, whereby γ^{ik} is the inverse metric to g_{ik}, $\gamma^{il}g_{lk} = \delta^i_k$.

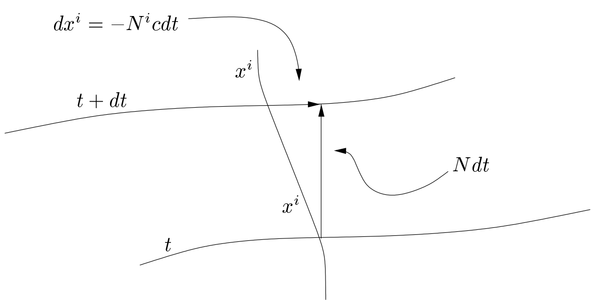

Fig. 2. Shown are the lapse and shift functions. The adapted observer moves along Ndt in the spacetime.

In the following, typical examples of Coriolis and (genuine) gravitomagnetic fields will be presented.

3.1 The Coriolis field

The line element of a rigidly rotating reference frame, in cylindrical coordinates, takes the form,

$$ds^2 \;=\; -\;(1 - \frac{\Omega^2\varrho^2}{c^2})\;c^2dt^2 \;+\; 2\Omega\varrho^2 dt d\phi \\ +\;(d\varrho^2 + \varrho^2 d\phi^2 + dz^2)\,. \qquad (12)$$

One easily reads off, $g_{0\phi} = \Omega\varrho^2/c$, or, in vectorial notation

$$\mathbf{g} \;=\; \frac{1}{c}\;\mathbf{\Omega}\times\mathbf{r}\,. \qquad (13)$$

This is the well-known Coriolis field.
In the rotating reference frame, the velocity $\mathbf{v} = d\mathbf{R}/dt$, with $d\mathbf{R} = dx^i(t)$, of a particle which is at rest in the (global) inertial frame connected with the centre of the rotating frame, is given by

$$\mathbf{v} \;=\; -\;\mathbf{N}c \;=-\;\mathbf{\Omega}\times\mathbf{R}\,. \qquad (14)$$

It should be interesting for the reader to compare this velocity with the corresponding one in the next section.

3.2 Dragging of inertial frames

The line element of the exterior field of a rotating body, in non-rotating cylindrical coordinates, to linear order in G, where G denotes the Newtonian gravitational constant, reads, if the spin vector (proper rotation vector) $\mathbf{S}$ is pointing in z-direction,

$$ds^2 \;=-\;(1 - \frac{2GM}{c^2 r})\;c^2dt^2 \;-\; \frac{4GS\varrho^2}{c^2r^3}dtd\phi \\ +\;(1 + \frac{2GM}{c^2r})(d\varrho^2 + \varrho^2 d\phi^2 + dz^2)\,, \qquad (15)$$

where $r^2 = \varrho^2 + z^2$ and $S^2 = \mathbf{S}^2$ hold. In this case we find $g_{0\phi} = -2GS\varrho^2/c^3r^3$ or,

$$\mathbf{g} \;=\; -\;\frac{2G}{c^3r^3}\;\mathbf{S}\times\mathbf{r}\,. \qquad (16)$$

The velocity of a particle, as measured in the given coordinate system, which freely follows the action of the gravitational field along the azimuthal direction (in radial direction the particle is kept fixed), is

$$\mathbf{v} \;=-\,\mathbf{N}c \;=\; (1+\frac{2GM}{c^2R})^{-1}\frac{2G}{c^2R^3}\,\mathbf{S}\times\mathbf{R}$$
$$=\; \frac{2G}{c^2R^3}\,\mathbf{S}\times\mathbf{R} \;+\; ... \qquad (17)$$

This is the famous frame-dragging effect for particle motion; it results from the gravitomagnetic field only. The particles show the dragging of the inertial frames in the same way as small wood pieces may show the streamlines of flowing water.
For more insight into the "frame-dragging" field, we give the orbital angular velocities of particles in circular motion in the equatorial plane of a spinning object. The velocities read, e.g. see [6],

$$\left(\frac{d\phi}{dt}\right)^2 \;=\; \frac{GM}{R^3} \;\mp\; \frac{2GS}{c^2R^3}\sqrt{\frac{GM}{R^3}}\,. \qquad (18)$$

The naive expectation that the particle moving in the dragging direction is the faster one is not correct, rather the particle on the retrograde orbit is the faster (lower sign in Eq. 18). This is understandable from the fact that the retrograde orbit needs stronger centrifugal force, and thus higher velocity, to exist.

4. SAGNAC EFFECTS

The differential of the phase ψ of a light ray is given by $d\psi = k_\mu dx^\mu$, where $k_\mu = (-\omega/c, \mathbf{k})$ is the wave 4-vector with frequency ω. By the aid of the dispersion relation $k_\mu k^\mu = 0$ one can derive the integral representation

$$\psi(t, x^i; t_0, x_0^i) =$$
$$-\int_{t_0}^{t}\omega dt + \int_{x_0}^{x}\frac{\omega}{c}(\sqrt{(G_{ij}+A_iA_j)dx^idx^j} + \mathbf{A}\cdot d\mathbf{r})\,, \qquad (19)$$

where $\mathbf{A}\cdot d\mathbf{r} \equiv A_i dx^i$. Along the light ray, where $ds^2 = 0$ holds, the phase is constant, $\psi = 0$. Hereof,

$$\int_{t_0}^{t}\omega dt = \int_{x_0}^{x}\frac{\omega}{c}(\sqrt{(G_{ij}+A_iA_j)dx^idx^j} + \mathbf{A}\cdot d\mathbf{r}) \qquad (20)$$

follows. For time independent metric coefficients, the frequency ω is a constant. In this case, the light path results from the simple condition that the right side of Eq. 20 is an extremum in 3-space. The well-known Sagnac effect in optics comes from the last term on the right side of the Eq. 20 applied to two light rays which originate from a specific event and end in another one.

4.1 The Sagnac effect for atoms

For the proper time τ of an object we have $c^2d\tau^2 = -ds^2$. The related action W takes the form,

$$W \;=\; -\,Mc^2\tau\,, \qquad (21)$$

and the quantum phase of the object is given by $W/\hbar$. For the growth of proper time between the coordinate times t_0 and t, one finds,

$$\tau(t;t_0) = \int_{t_0}^{t}\sqrt{N^2 - g_{ij}(N^i+\frac{v^i}{c})(N^j+\frac{v^j}{c})}\;dt\,, \qquad (22)$$

where $v^i = dx^i(t)/dt$. The additive structure of $cN^i + v^i$ nicely reveals the motion of the reference frame with respect to a particle's dragged motion in azimuthal direction, see section 3.2.
The straight comparison with the change of phase in the light propagation case, Eq. 20, is best achieved through the following approximate expression,

$$W = -\,Mc^2\int_{t_0}^{t}\sqrt{-g_{00}}(1-(G_{ij}+A_iA_j)\frac{v^iv^j}{2c^2}+...)dt$$
$$+\,Mc\int_{x(t_0)}^{x(t)}\sqrt{-g_{00}}\;\mathbf{A}\cdot d\mathbf{R}\,, \qquad (23)$$

where the phase of the particle (atom, etc.) is given by $W/\hbar$. The Sagnac effect for atoms results from Eq. 23 in the same way as the Sagnac effect for light rays results from Eq. 20. By the aid of Eqs. 21 and 23 the **clock effects** originating from the gravitomagnetism can be deduced in a straightforward manner.

4.2 Analogy with the electrodynamics

The gravitational Hamiltonian of a point mass reads,

$$H = Nc\,\sqrt{M^2c^2+\gamma^{ij}P_iP_j} - c\mathbf{N}\cdot\mathbf{P}$$
$$= Mc^2 + \frac{1}{2M}(\mathbf{P}-M\mathbf{N}c)^2 + M\Phi \;+\; ...\,, \qquad (24)$$

where $\mathbf{P} = (P_i)$ and where the Newtonian gravitational potential Φ has been introduced, $g_{00} = -1 - 2\Phi/c^2 + ...$ (expansion in powers of $1/c^2$).
The analogous expressions in the electrodynamics are,

$$
\begin{aligned}
H &= c\,\sqrt{M^2c^2 + (\mathbf{P} - \frac{e}{c}\mathbf{A}^{\rm em})^2} + e\Phi^{\rm em} \\
&= Mc^2 + \frac{1}{2M}(\mathbf{P} - \frac{e}{c}\mathbf{A}^{\rm em})^2 + e\Phi^{\rm em} \ + \ ... \qquad (25)
\end{aligned}
$$

Obviously, the analogy, valid in the weak-field slow-motion limit, takes the form: $e\mathbf{A}^{\rm em} \longleftrightarrow M\mathbf{N}c^2$, $e\Phi^{\rm em} \longleftrightarrow M\Phi$. Remember, however, the difference in the field-generation relations: $4M\mathbf{v} \to -\,\mathbf{N}c^3$, $M \to -\,\Phi$, in the harmonic gauge, whereas $e\mathbf{v} \to \mathbf{A}^{\rm em}c$, $e \to \Phi^{\rm em}$, correspondingly in the Lorentz gauge.

5. SPIN DYNAMICS IN GRAVITY

In the framework of the Einstein theory of gravity the gravitational interaction Hamiltonian of two spinning objects, linear in G and linear in each spin (in this approximation, the proper rotation of an extended object can be treated as rigid), is obtained by the substitution, in the momentum and Hamiltonian constraints of the Einstein field equations, of the momentum density $\mathbf{P}_a\delta(\mathbf{x} - \mathbf{x}_a)$, with $a = 1, 2$, through $(\mathbf{P}_a + \frac{1}{2}\mathbf{S}_a \times \boldsymbol{\nabla}_a)\delta(\mathbf{x} - \mathbf{x}_a)$. Hereof, in the rest frame (vanishing total linear momentum), the spin-orbit interaction results in the form, also e.g. see [2],

$$
H_{\rm SO} \ = \ \frac{2G}{c^2R^3}\,(\mathbf{S}\cdot\mathbf{L}) \ + \ \frac{3GM_1M_2}{2c^2R^3}\,(\mathbf{b}\cdot\mathbf{L})\,, \qquad (26)
$$

where

$$
\begin{aligned}
\mathbf{S} \ &\equiv \ \mathbf{S}_1 + \mathbf{S}_2\,, \qquad (27) \\
\mathbf{L} \ &\equiv \ \mathbf{R}\times\mathbf{P}\,, \qquad (28) \\
\mathbf{b} \ &\equiv \ \frac{\mathbf{S}_1}{M_1^2} \ + \ \frac{\mathbf{S}_2}{M_2^2}\,, \qquad (29)
\end{aligned}
$$

and where $\mathbf{P} \equiv \mathbf{P}_1 = -\mathbf{P}_2$, $\ \mathbf{R} \equiv \mathbf{x}_1 - \mathbf{x}_2$, $\ R = |\mathbf{R}|$; and for the spin-spin interaction one obtains

$$
H_{\rm S_1S_2} = \frac{G}{c^2R^3}\left(\frac{3(\mathbf{S}_1\cdot\mathbf{R})(\mathbf{S}_2\cdot\mathbf{R})}{R^2} - (\mathbf{S}_1\cdot\mathbf{S}_2)\right)\,. \qquad (30)
$$

Notice that the coordinate $\mathbf{R}$ is the canonical conjugate to $\mathbf{P}$. It relates to the particle coordinate of section 2 with the same name. Obviously, the total angular momentum

$$
\mathbf{J} \ = \ \mathbf{L} \ + \ \mathbf{S}\,, \qquad (31)
$$

is conserved in time,

$$
\frac{d\mathbf{J}}{dt} \ = \ \{\mathbf{J}, H_{\rm SO} \ + \ H_{\rm S_1S_2}\} \ = \ 0\,, \qquad (32)
$$

where $\{,\}$ denotes the standard Poisson brackets, and also the absolute values of the spin vectors $\mathbf{S}_a$ $(a = 1, 2)$ do not change with time,

$$
\frac{d\mathbf{S}_a^2}{dt} \ = \ \{\mathbf{S}_a^2, H_{\rm SO} \ + \ H_{\rm S_1S_2}\} \ = \ 0\,. \qquad (33)
$$

The spin-spin interaction Hamiltonian, Eq. 30, is identical with the spin(1)-spin(2) Hamiltonian of two Kerr black holes. The latter is part of the Hamiltonian which describes the full spin-spin interaction of two Kerr black holes to linear and quadratic orders in G and $\mathbf{S}$, respectively, [5]

$$
H_{\rm SS}^{\rm Kerr} \ = \ \frac{GM_1M_2}{2c^2R^3}\left(\frac{3(\mathbf{a}\cdot\mathbf{R})(\mathbf{a}\cdot\mathbf{R})}{R^2} - (\mathbf{a}\cdot\mathbf{a})\right)\,, \qquad (34)
$$

where

$$
\mathbf{a} \ \equiv \ \frac{\mathbf{S}_1}{M_1} \ + \ \frac{\mathbf{S}_2}{M_2}\,. \qquad (35)
$$

It is interesting to note that the spin(a)-spin(a) interaction terms $(a = 1, 2)$ are of monopole-quadrupole type whereas the spin(1)-spin(2) interaction is of dipole-dipole type. The latter is independent from the masses M_a, see Eq. 30. The spin-spin interaction Hamiltonian, Eq. 30, as well as the part of the spin-orbit interaction Hamiltonian, Eq. 26, which depends on $\mathbf{S}$ only result from the gravitomagnetic field, the other part in Eq. 26, depending on the masses, would also be present without gravitomagnetic field.
For completeness we give the orbit-orbit interaction Hamiltonian which results from the gravitomagnetic field. In the rest frame, it reads, cf. [7],

$$
H_{\rm O_1O_2}^{\rm gmag} \ = \ - \ \frac{G}{2c^2R}\left(7\mathbf{P}^2 + \frac{(\mathbf{P}\cdot\mathbf{R})^2}{R^2}\right)\,. \qquad (36)
$$

Again, the expression does not depend on the masses.

5.1 The Lense-Thirring effect

The (proper) Lense-Thirring effect is the precession of the orbital plane of an object moving in the gravitomagnetic field of a spinning central object. By the aid of the orbital angular momentum vector $\mathbf{L}$ and the Runge-Lenz-Laplace-Lagrange vector [7] which is defined in the rest frame of the binary system and points from the centre-of-mass position to the periastron of the relative orbit,

$$\mathbf{M} \equiv \mathbf{P} \times \mathbf{L} - \frac{G M_1^2 M_2^2}{M_1 + M_2} \frac{\mathbf{R}}{R}, \qquad (37)$$

($\mathbf{M} \cdot \mathbf{L} = 0$) the precession of the orbit takes the form

$$< \left(\frac{d\mathbf{L}}{dt}\right)_{\mathrm{O}}^{\mathrm{S}} >_t \equiv < \{\mathbf{L}, H_{\mathrm{SO}}\} >_t = \mathbf{\Omega}_{\mathrm{SO}} \times \mathbf{L}, \qquad (38)$$

$$< \left(\frac{d\mathbf{M}}{dt}\right)_{\mathrm{O}}^{\mathrm{S}} >_t \equiv < \{\mathbf{M}, H_{\mathrm{SO}}\} >_t = \mathbf{\Omega}_{\mathrm{SO}} \times \mathbf{M}, \qquad (39)$$

where $<>_t$ denotes orbital averaging and where the precessional frequency vector is given by

$$\mathbf{\Omega}_{\mathrm{SO}} = \frac{2G}{c^2} < \frac{1}{R^3} >_t \left(\mathbf{S}_{\mathrm{eff}} - 3\frac{(\mathbf{L} \cdot \mathbf{S}_{\mathrm{eff}})\mathbf{L}}{L^2}\right), \qquad (40)$$

with

$$\mathbf{S}_{\mathrm{eff}} \equiv \mathbf{S} + \frac{3}{4} M_1 M_2 \mathbf{b}. \qquad (41)$$

If respectively e and a are eccentricity and semimajor axis of the relative orbit, the averaging procedure yields, e.g. see [8], [7],

$$< \frac{1}{R^3} >_t = \frac{1}{a^3 (1 - e^2)^{3/2}}. \qquad (42)$$

For the LAGEOS satellite, $|\mathbf{\Omega}_{\mathrm{SO}}|$ results in 31 mas/yr [9]. In this case, where the index 1 may apply to the non-spinning satellite and the index 2 to the Earth, $\mathbf{S}_{\mathrm{eff}}$ is simply given by $\mathbf{S}_2$.

5.2 The Schiff effect

The Schiff effect (also called Lense-Thirring effect for spin, or frame-dragging effect) is the precession of a spin in the gravitomagnetic field of a spinning central object. It is given by

$$\left(\frac{d\mathbf{S}_1}{dt}\right)_{\mathrm{S}}^{\mathrm{S}} \equiv \{\mathbf{S}_1, H_{\mathrm{S}_1\mathrm{S}_2}\} = \mathbf{\Omega}_{\mathrm{S}_2} \times \mathbf{S}_1, \qquad (43)$$

where the precessional vector reads

$$\mathbf{\Omega}_{\mathrm{S}_2} = \frac{G}{c^2 R^3} \left(3\frac{(\mathbf{R} \cdot \mathbf{S}_2)\mathbf{R}}{R^2} - \mathbf{S}_2\right). \qquad (44)$$

For the gyroscopes of the GP-B mission an orbital-averaged spin precession of 41 mas/yr is predicted [10] (usually quoted as 42 mas/yr, e.g. see Fig. 5 in [11]) to be measured with an accuracy of 0.3%.

5.3 Gravitomagnetic field lines

In the following discussion about gravitomagnetic field lines the object 2 with mass M_2 and spin $\mathbf{S}_2$ will be assumed to be much heavier than the object 1. Then the object 1 can be treated as test object in the gravitational field of object 2. The shift function of object 2 reads

$$\mathbf{N} = \frac{2G}{c^3 r^3} \mathbf{r} \times \mathbf{S}_2. \qquad (45)$$

Hereof the gravitomagnetic field strength $\mathbf{H}$, analogously to the electrodynamics, follows in the form,

$$\mathbf{H} = \nabla \times \mathbf{N} c = \frac{2G}{c^2 r^3} \left(\mathbf{S}_2 - 3\frac{(\mathbf{r} \cdot \mathbf{S}_2)\mathbf{r}}{r^2}\right). \qquad (46)$$

Obviously,

$$\mathbf{\Omega}_{\mathrm{S}_2} = -\frac{1}{2} \mathbf{H} \qquad (47)$$

holds. In the exterior regime of object 2, the gravitomagnetic field strength allows the representation

$$\mathbf{H} = \nabla \lambda \qquad (48)$$

with

$$\lambda = \frac{2G}{c^2 r^3} \mathbf{r} \cdot \mathbf{S_2}. \qquad (49)$$

As orthogonal trajectories of the $\lambda = \text{const.}$ lines, the gravitomagnetic field lines are easily obtained. The equation for them reads,

$$\frac{r_{\max} - r}{r_{\max}} = \frac{(\mathbf{r} \cdot \mathbf{S}_2)^2}{r^2 S_2^2}, \qquad (50)$$

where $r_{\max}$ is the maximum value of r for a given field line (notice: $\boldsymbol{\nabla}\lambda(\mathbf{r}) \cdot \boldsymbol{\nabla} r_{\max}(\mathbf{r}) = 0$). The gravitomagnetic field strength, in the exterior regime of object 2, takes the form,

$$\mathbf{H} = \frac{2GS_2}{c^2 r^3 r_{\max}} \frac{d\mathbf{r}(r(\theta), \theta)}{d\cos\theta}, \qquad (51)$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{S}_2$, i.e. $\mathbf{r} \cdot \mathbf{S}_2 = rS_2\cos\theta$ and where in the function $r(\theta)$ the $r_{\max}$ has to be kept constant.

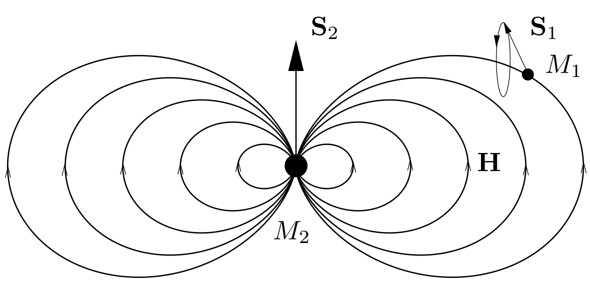

Fig. 3. Shown are the gravitomagnetic field lines of object 2 and the spin precession of object 1 about a field line. The object 2 is assumed to be at rest in the given coordinate system, i.e. $M_2 \gg M_1$.

It should be remarked that the gravitomagnetic field lines in Fig. 3 differ in shape from those in other references, [5], [6], [9], [10], [11], where [5] is closest.

5.4 The de Sitter effect

The de Sitter effect (also called Fokker effect, or geodetic precession) results from

$$\left(\frac{d\mathbf{S}_1}{dt}\right)_{\mathrm{O}}^{\mathrm{S}} \equiv \{\mathbf{S}_1, H_{\mathrm{SO}}\} = \boldsymbol{\Omega}_{\mathrm{SO}}^{\mathrm{s}} \times \mathbf{S}_1, \qquad (52)$$

where

$$\boldsymbol{\Omega}_{\mathrm{SO}}^{\mathrm{s}} = \frac{2G}{c^2 R^3}\left(1 + \frac{3M_2}{4M_1}\right)\mathbf{L}. \qquad (53)$$

The first term on the right side of the Eq. 53 stems from the gravitomagnetic field. For the Earth-Moon gyroscope, i.e. the Earth-Moon binary system is regarded as spinning object, it holds, 19 mas/yr, e.g. see [9], and for the GP-B mission, 6,600 mas/yr are expected, e.g. see [10]. In both cases $M_1 \ll M_2$ applies (mass of Earth-Moon system vs. mass of Sun, resp. mass of satellite vs. mass of Earth), so that only the second term on the right side of the Eq. 53 contributes.

Acknowledgements: I thank G. Faye for help with the Figures and C. Lämmerzahl for useful discussions.

6. REFERENCES

1. Schwinger, J., *Particles, Sources, and Fields I*, Perseus Books, Reading, Massachusetts, 1998.

2. Barker, B. M. and O'Connell, R. F., The Gravitational Interaction: Spin, Rotation, and Quantum Effects – A Review, *General Relativity and Gravitation*, Vol. 11, 149 - 175, 1979.

3. Møller, C., *The Theory of Relativity*, Oxford University Press, Oxford 1969.

4. Misner, C. W., Thorne, K. S., and Wheeler, J. A., *Gravitation*, Freeman, San Francisco 1973.

5. Thorne, K. S., Price, R. H., and Macdonald, D. A., (Eds.), *Black Holes: The Membrane Paradigm*, Yale Univerity Press, New Haven, 1986.

6. Rindler, W., *Relativity*, Oxford University Press, Oxford, 2001.

7. Damour, T. and Schäfer, G., Higher-Order Relativistic Periastron Advances and Binary Pulsars, *Nuovo Cimento B*, Vol. 101, 127 - 176, 1988.

8. Landau, L. D. und Lifschitz, E. M., *Klassische Feldtheorie*, Akademie-Verlag, Berlin, 1981.

9. Ciufolini, I. and Wheeler, J. A., *Gravitation and Inertia*, Princeton University Press, Princeton, 1995.

10. Lämmerzahl, C. and Neugebauer, G., The Lense-Thirring Effect: From the Basic Notions to the Observed Effects, in: *Gyros, Clocks, Interferometers ...: Testing Relativistic Gravity in Space*, Lämmerzahl, C., Everitt, C. W. F., and Hehl, F. W., (Eds.), Springer, Berlin, 2001.

11. Lämmerzahl, C. and Dittus, H., Fundamental Physics in Space: A Guide to Present Projects, *Ann. Phys. (Leipzig)*, Vol. 9, 1 - 57, 2000.