

# On the Energy of Vaidya Space-time

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## Abstract

In this paper we calculate the energy distribution of six cases of Vaidya-solutions in the Møller prescription. Except the energy complex of Møller for the monopole solution vanishes everywhere, for other solutions have non-zero energy component, only the energy distributions of the de Sitter and anti-de Sitter solution are independent on  $v$ . For the radiating dyon solution, the difference in energy complex between Møller's and Einstein's prescription is like the case of Reissner-Nordström solution.

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A non-static spherically symmetric solution of Einstein's equations for an imploding (exploding) null dust fluid is found by Vaidya in 1951 [1]. In various contexts this "null dust" may be interpreted as a high-frequency electromagnetic or gravitational wave, incoherent superposition of aligned waves with random phases and polarisations, or as massless scalar particles or neutrinos. Since then, the solution has been intensively studied in gravitational collapse [2]. In particular, Papapetrou [3] firstly showed that this solution can give rise to the formation of naked singularities, and thus provided one of the earlier counterexamples to the cosmic censorship conjecture [4]. Furthermore, the solution was generalized to the charged case [5], and the charged Vaidya solution has been studied soon in various situations. It had been used to study the thermodynamics of black holes by Sullivan and Israel [6] and be a classical model for the geometry of evaporating charged black holes by Kaminga [7]. In the meantime, Lake and Zannias [8] studied the self-similar case and found that, similar to the uncharged case, naked singularities can be also formed from gravitational collapse. Otherwise, Chamorro and Virbhadra [9] obtained the dyonic case, in which both with the electric and magnetic charge, and Husian [10] further generalized the Vaidya solution to a null fluid with a particular equation of state. Husian's solutions have been lately used as the formation of black holes with short hair [11].

In this article, we would to consider the energy of several well-known cases of Vaidya-type solution. The energy in curved space-time has been a subject of extensive research since the early days of general relativity. Here, to evaluate the energy of a system in general relativity is for two physical reasons. First, the total energy of a system must be a conserved quantity and could play an important role in solving the equation of motion. Second, the energy distribution must be positive everywhere if attractive force exists only. Early energy-momentum investigations for gravitating systems gave reference-frame-dependent energy-momentum complexes, and many physicists such as Einstein [12], Tolman [13], Landau and Lifshitz [14], Papapetrou [15], Bergmann and Thompson [16], Weinberg [17], and Møller [18], had given different definitions for the energy-momentum complex. The Møller energy-momentum complex is an efficient tool for the energy-momentum localization and al-

lows obtaining satisfactory results for the energy and momentum distributions in the case of a general relativistic system. The Møller energy-momentum complex in a four-dimensional background is given as [18]

$$\Theta_{\nu}^{\mu} = \frac{1}{8\pi} \frac{\partial \chi_{\nu}^{\mu\sigma}}{\partial x^{\sigma}}, \quad (1)$$

where

$$\chi_{\nu}^{\mu\sigma} = \sqrt{-g} \left( \frac{\partial g_{\nu\alpha}}{\partial x^{\beta}} - \frac{\partial g_{\nu\beta}}{\partial x^{\alpha}} \right) g^{\mu\beta} g^{\sigma\alpha} \quad (2)$$

is the Møller's superpotential, a quantity antisymmetric in the indices  $\mu, \sigma$ . According to the definition of the Møller energy-momentum complex, the energy component is given as

$$E = \int \Theta_0^0 d^3x = \frac{1}{8\pi} \int \frac{\partial \chi_0^{0k}}{\partial x^k} d^3x, \quad (3)$$

and the Latin index takes values from 1 to 3.

The line element, in terms of the standard spherical coordinates, of the non-static spherically symmetric type D solution [19] can be expressed as

$$ds^2 = e^{2\psi(v,r)} \left[ 1 - \frac{2m(v,r)}{r} \right] dv^2 - 2\epsilon e^{\psi(v,r)} dvdr - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (4)$$

note that  $m(v, r)$  is usually called the mass function and relates to the gravitational energy within a given radius  $r$ . When  $\epsilon = +1$ , the null coordinate  $v$  represents the Eddington advanced time, in which  $r$  is decreasing towards the future along a ray  $v = \text{constant}$  (ingoing). On the other hand, when  $\epsilon = -1$ , it represents the Eddington retarded time, in which  $r$  is increasing towards the future along a ray  $v = \text{constant}$  (outgoing). In the following, we shall consider the particular case where  $\psi(v, r) = 0$ , and the stress-energy tensor of radiation and null dust fluid in Vaidya solution is of the form [19]

$$T_{\mu\nu} = (\rho + P)(l_{\mu}n_{\nu} + l_{\nu}n_{\mu}) + Pg_{\mu\nu} + \mu l_{\mu}l_{\nu}, \quad (5)$$

where

$$\mu = \frac{\epsilon \dot{m}(v, r)}{4\pi r^2}, \quad \rho = \frac{m'(v, r)}{4\pi r^2}, \quad P = -\frac{m''(v, r)}{8\pi r}$$

and

$$\dot{m}(v, r) \equiv \frac{\partial m(v, r)}{\partial v}, \quad m'(v, r) \equiv \frac{\partial m(v, r)}{\partial r}.$$

Here  $l_\mu, n_\mu$  are two null vector,

$$l_\mu = \delta_\mu^0, \quad n_\mu = \frac{1}{2} \left( 1 - \frac{2m(u, r)}{r} \right) \delta_\mu^0 - \delta_\mu^1,$$

$$l_\lambda l^\lambda = n_\lambda n^\lambda = 0, \quad l_\lambda n^\lambda = -1.$$

Therefore, the nonvanishing component of Møller's superpotential is

$$\chi_0^{01} = \frac{\sin \theta}{\epsilon} (2m - 2m'r). \quad (6)$$

Applying the Gauss theorem to (3) and using (6), we evaluate the integral over the surface of a sphere with radius  $r$ , and find that the energy distribution is as the form

$$E = \frac{1}{\epsilon} (m - m'r). \quad (7)$$

Next, The following cases include six known Vaidya-type solutions of the Einstein field equations with spherical symmetry:

(i) The monopole solution [20]: The mass function of the monopole solution is given as

$$m(v, r) = \frac{ar}{2}, \quad (8)$$

where  $a$  is an arbitrary constant. The corresponding solution can be identified as representing the gravitational field of a monopole. After the previous computation, all components of Møller's superpotential is zero, and the energy component of Møller energy-momentum complex is obtained with

$$E = 0. \quad (9)$$

(ii) The de Sitter and Anti-de Sitter solutions [19]: The well-known de Sitter and Anti-de Sitter solutions in the Vaidya's radiation coordinates show that the mass function is

$$m(v, r) = \frac{\Lambda}{6}r^3, \quad (10)$$

where  $\Lambda$  is the cosmological constant. This corresponds to the de Sitter solution for  $\Lambda > 0$  and to anti-de Sitter solution for  $\Lambda < 0$ . The required nonvanishing component of  $\chi_0^{0k}$  is

$$\chi_0^{01} = -\frac{\sin\theta}{\epsilon} \frac{2}{3} \Lambda r^3, \quad (11)$$

and the obtained energy component is

$$E = -\frac{\Lambda}{3\epsilon} r^3. \quad (12)$$

(iii) The charged Vaidya solution [19]: For the charged Vaidya solution, the mass function is found as

$$m(v, r) = f(v) - \frac{q^2(v)}{2r}, \quad (13)$$

where the two arbitrary functions  $f(v)$  and  $q(v)$  represent, respectively, the mass and electric charge at the advanced (retarded) time  $v$ . The non-zero component of Møller's superpotential only is obtained as

$$\chi_0^{01} = \frac{\sin\theta}{\epsilon} \left[ 2f(v) - \frac{2q^2(v)}{r} \right], \quad (14)$$

and the energy component of Møller energy-momentum complex is evaluated with

$$E = \frac{1}{\epsilon} \left[ f(v) - \frac{q^2(v)}{r} \right]. \quad (15)$$

(iv) The monopole-de Sitter-charged Vaidya solutions [19]: According to the monopole-de Sitter-charged Vaidya solutions, its mass function is represented as

$$m(v, r) = \frac{ar}{2} + \frac{\Lambda}{6}r^3 + f(v) - \frac{q^2(v)}{2r}. \quad (16)$$

Using the definition of Møller energy-momentum complex, the nonvanishing component of Møller's superpotential is

$$\chi_0^{01} = \frac{\sin\theta}{\epsilon} \left[ -\frac{2}{3} \Lambda r^3 + 2f(v) - \frac{2q^2(v)}{r} \right], \quad (17)$$

and the energy component of Møller energy-momentum complex is

$$E = \frac{1}{\epsilon} \left[ -\frac{\Lambda}{3} r^3 + f(v) - \frac{q^2(v)}{r} \right]. \quad (18)$$

(v) The radiating dyon solution [9]: The metric, which describes the gravitational field of non-rotating massive radiating dyon, is found by Chamorro and Virbhadra, and its mass function is

$$m(v, r) = M(v) - \frac{q_e^2(v) + q_m^2(v)}{r}. \quad (19)$$

With the energy-momentum pseudotensor of Møller, we obtain the nonvanishing component of Møller's superpotential is

$$\chi_0^{01} = \frac{\sin\theta}{\epsilon} \left[ 2M(v) - \frac{2q_e^2(v) + 2q_m^2(v)}{r} \right], \quad (20)$$

and the energy is calculated as

$$E = \frac{1}{\epsilon} \left[ M(v) - \frac{q_e^2(v) + q_m^2(v)}{r} \right]. \quad (21)$$

(vi) The Husain solutions [10]: This solution found by Husian with imposing the equation of state  $P = k\rho$  have the mass function defined as

$$m(v, r) = f(v) - \frac{g(v)}{(2k-1)r^{2k-1}}, \quad (22)$$

where  $f(v)$  and  $g(v)$  are two arbitrary functions, and  $k$  is a constant. Then, by the definition of Møller energy-momentum complex, the required non-zero component of Møller's superpotential is

$$\chi_0^{01} = \frac{\sin\theta}{\epsilon} \left[ 2f(v) + \frac{2k-2}{2k-1} \frac{2g(v)}{r^{2k-1}} \right], \quad (23)$$

and the energy component of Møller energy-momentum complex is

$$E = \frac{1}{\epsilon} \left[ f(v) + \frac{2k-2}{2k-1} \frac{2g(v)}{r^{2k-1}} \right]. \quad (24)$$

Energy-momentum complexes provide the same acceptable energy-momentum distribution for some systems. However, for other systems, these prescriptions are disagreed. Based

on some analysis of the results known with many prescriptions for energy distribution (including some well-known quasi-local mass definitions) in a given space-time, Virbhadra [21] remarked that the formulation by Einstein is still the best one. In his previous paper, which motivated us to study this further, Lessner [22] argued that the Møller energy-momentum expression is a powerful concept of energy and momentum in general relativity. There is no consensus as to which is the best. Hamiltonian's principle helps to solve this enigma [23]. Each expression has a geometrically and physically clear significance associated with the boundary conditions. In this work, we briefly present the energy component of Møller energy-momentum complex for six Vaidya-like solutions. Except the energy complex of Møller for the monopole solution vanishes everywhere, for the de Sitter, anti-de Sitter, charged Vaidya, monopole-de Sitter-charged Vaidya, radiating dyon and Husain solution have non-zero energy component. Here only the energy distributions of the de Sitter and anti-de Sitter solution are independent on  $v$ . On the case of radiating dyon solution, the difference in energy complex between Møller's and Einstein's prescription is like the case of Reissner-Nordström solution. For the Reissner-Nordström space-time  $E_{\text{Ein}} = M - e^2/2r$  (the seminal Penrose quasi-local mass definition also yields the same result agreeing with linear theory) whereas  $E_{\text{Møl}} = M - e^2/r$ . Similarly, the energy of radiating dyon solution is  $E_{\text{Møl}} = M(v) - [q_e^2(v) + q_m^2(v)]/r$  whereas  $E_{\text{Ein}} = M(v) - [q_e^2(v) + q_m^2(v)]/2r$  [9]. There is a difference of a factor 2 in the second term.

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