Input-to-state Stabilizing Controller for Nonlinear Systems on Manifolds

Yasuyuki Satoh, Nami Nakamura, and Hisakazu Nakamura

Abstract—In this paper, we propose an input-to-state stabilizing controller for nonlinear systems on differentiable manifolds based on input-to-state locally semiconcave practical control Lyapunov functions (ISS-LS-PCLFs). We show that the proposed controller input-to-state stabilize the desired equilibrium in the sense of sample and hold solution. The effectiveness of the proposed controller is confirmed by a numerical example.

I. INTRODUCTION

Asymptotic stabilization of systems defined on noncontractible manifolds is a difficult control problem (see [1], [2], [3], and [4]); if a state space is noncontractible, a continuous stabilizing state feedback never exists [1]. For the problem, a discontinuous state feedback is one of effective strategies. Malisoff et al. [2] proved that every globally asymptotically controllable system defined on a smooth manifold attains a discontinuous asymptotically stabilizing state feedback in the sense of sample stability. Nakamura et al [3] introduced the framework of a locally semiconcave practical control Lyapunov function (LS-PCLF) and proposed a Sontag-type discontinuous asymptotically stabilizing state feedback controller.

The concept of input-to-state stability (ISS) plays an important role in recent nonlinear control theory (see [5], [6], and [7]). ISS draws a framework for robustness with respect to disturbances. In Euclidean spaces, input-to-state stabilizability is equivalent to existence of an ISS-CLF [6]. Moreover, ISS for systems on manifolds have been also studied in [2], [8], and [9]. In [8] and [9], extended ISS conditions are introduced by using Carathéodory solutions. However, Carathéodory solutions may not exist under discontinuous state feedback. In [2], concept of sample-input-to-stability (s-ISS) is proposed. The s-ISS is a natural extension of sample stability. However, it is still an open problem how to design a sample-input-to-state stabilizing controller.

In this paper, we tackle this problem by CLF-based approach. We introduce the concept of input-to-state locally semiconcave practical control Lyapunov function (ISS-LS-PCLF) and propose an input to state stabilizing controller for systems on differentiable manifolds. The proposed controller is an extended one proposed by Krstić and Li [6].

II. PRELIMINARIES

A. Sample and hold solution and sample stability for nonlinear systems

Let \( \mathbb{R}_{\geq 0} := [0, +\infty) \) and \( \mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\} \). In this paper, \( X \) denotes an \( n \)-dimensional connected \( C^1 \) manifolds

The authors are with the Department of Electrical Engineering, Faculty of Science and Technology, Tokyo University of Science, Yamazaki 2641, Noda, Japan yasuyuki.satoh@gmail.com, namiff@bu.iij4u.or.jp, nakamura@rs.tus.ac.jp

without boundaries, \( T_x X \) the tangent space to \( X \) at \( x \), and \( T^*_x X \) the cotangent space at \( x \). Moreover, \( \langle \cdot, \cdot \rangle \) denotes inner product and \( \| \cdot \| \) a Euclidian norm. For a subset \( W \subset X \), we define \( P_W \) as follows:

\[
P_W := \{ W' | W' \text{ is a precompact open set with } W \subset W' \subset X \}
\]

In this paper, we consider the following nonlinear system:

\[
\dot{x} = f(x) + g(x)u + h(x)\omega
\]

where \( x \in X \) is a state, \( u \in U := \{ u : X \to \mathbb{R}^m \} \) is control input, and \( \omega \in \Omega := \{ \omega : \mathbb{R}_{\geq 0} \to \mathbb{R}^d | \omega \text{ is a continuous bounded map} \} \) is disturbance. We assume \( f(x), g(x), \) and \( h(x) \) are locally Lipschitz continuous and \( f(0) = 0 \), where \( 0 \in X \) is the origin.

Definition 1 (Partition) [10] Any infinite sequence \( \pi = \{ t_i \}_{i \in \mathbb{Z}_{\geq 0}} \) consisting of numbers \( 0 = t_0 < t_1 < t_2 < \cdots \) with \( \lim_{i \to +\infty} t_i = +\infty \) is called a partition of the interval \( \mathbb{R}_{\geq 0} \) and the number \( \text{diam}(\pi) := \sup\{ t_{i+1} - t_i | i \in \mathbb{Z}_{\geq 0} \} \) its diameter. Let \( \text{Par}(\delta) := \{ \pi | \text{diam}(\pi) \leq \delta \} \) for a positive constant \( \delta > 0 \).

Definition 2 (Sample and hold solution) [10] Consider system (2). Let \( x_0 \in X \) be an initial state, \( u \in U \) a control input, \( \omega \in \Omega \) a disturbance input, and \( \pi \) a partition of the interval \( \mathbb{R}_{\geq 0} \). The sample-and-hold solution \( \varphi(t; x_0, u, \omega, \pi) \) for (2) is defined as the mapping such that \( \varphi(t; x_0, u, \omega, \pi) = x(t) \), where \( x(t) \) is a continuous mapping obtained by recursively solving

\[
\dot{x}(t) = f(x(t)) + g(x(t))u(t) + h(x(t))\omega(t),
\]

with \( x(0) = 0 \).

If disturbance \( \omega = 0 \), system (2) becomes

\[
\dot{x} = f(x) + g(x)u.
\]

In this case, the sample-and-hold solution for (4) is rewritten as \( \varphi(t; x_0, u, \pi) \) and the following sample stability is proposed:

Definition 3 (Sample Stability) [2] Consider system (4). The control input \( u \in U \) said to sample stabilize the origin

\[
P_W := \{ W' | W' \text{ is a precompact open set with } W \subset W' \subset X \}
\]
of the system (4) if the following holds for arbitrary sets \(W_1, W_2 \in \mathcal{P}_1\) such that \(W_1 \subseteq W_2\).

1) There exist constants \(\delta > 0, T \geq 0\) and a set \(M \subseteq X\) such that the following holds.

\[
\varphi(t; x_0, u, \pi) \in W_1 \\
\forall t \geq T, \, \forall x_0 \in W_2, \, \forall \pi \in \text{Par}(\delta)
\]  

(5)

\[
\varphi(t; x_0, u, \pi) \in M \\
\forall t \geq 0, \, \forall x_0 \in W_2, \, \forall \pi \in \text{Par}(\delta)
\]  

(6)

2) for each \(E \in \mathcal{P}_1\), there exists a set \(D \in \mathcal{P}_1\) such that if \(W_2 \subseteq D, M\) in the condition 1 can be chosen satisfying \(M \subseteq E\).

B. LS-PCLF and sample stabilizing feedback \([3]\)

To design a sample stabilizing feedback for system (4), Nakamura et al.\([3]\) introduce the concept of locally semiconcave practical control Lyapunov function (LS-PCLF). In this subsection, we show the definition of LS-PCLF and the Nakamura’s sample stabilizing state feedback controller.

Locally semiconcave functions is a class of non-smooth functions defined as follows:

**Definition 4 (Locally semiconcave function)** \([3], [11]\) A continuous function \(V : X \to \mathbb{R}\) is called a locally semiconcave on \(X\) if for any chart \((w, \eta)\) and compact set \(M \subseteq W\) there exists \(0 > C > 0\) such that

\[
V(x) + V(y) - 2V_W\left(\frac{x + y}{2}\right) \leq C\|\eta(x) - \eta(y)\|^2
\]

(7)

for all \(x, y \in M\) satisfying \((\eta(x) + \eta(y))/2 \in \eta(M)\).

Note that the following important property holds:

**Theorem 1** \([3], [11]\) Let \(V : X \to \mathbb{R}\) be a locally semiconcave function. For any compact subset \(W\) with a local chart \((w, \eta)\), there exist a compact set \(S \subseteq \mathbb{R}^n\) and a family of functions \(\{\hat{V}_s\}_{s \in S}\) such that each \(\hat{V}_s : M \to \mathbb{R}\) is \(C^2\) with respect to \(x\) and

\[
V(x) = \min_{s \in S} \hat{V}_s(x), \, \forall x \in M.
\]

(8)

We employ the following disassembled differential as a generalized derivative of locally semiconcave functions:

**Definition 5 (Disassembled differential)** \([3]\) Let \(V : X \to \mathbb{R}\) be a locally semiconcave function. Then, the following set-valued map \(\hat{D}V : X \to 2^\mathbb{R}^n\) is said to be a disassembled differential of \(V\):

\[
\hat{D}V(x) = \left\{ d\hat{V}_s(x) \big| s \in S | V(x) = \hat{V}_s(x) \right\}.
\]

(9)

\(\hat{D}V(x)\) can be rewritten as follows in the local chart \((w, \eta)\) \([3]\):

\[
\hat{D}V_W(x) = \left\{ \frac{\partial \hat{V}_s}{\partial \xi} \big| s \in S | V(x) = \hat{V}_s(x) \right\}.
\]

(10)

By using \(\hat{D}V(x)\), LS-PCLF is defined as follows:

**Definition 6 (LS-PCLF)** \([3]\) A locally semiconcave practical control Lyapunov function (LS-PCLF) for system (4) is a locally semiconcave function \(V : X \to \mathbb{R}_{\geq 0}\) such that the following properties hold:

(A1) \(V\) is proper; that is, the set \(\{x \in X | V(x) \leq L\}\) is compact for every \(L > 0\),

(A2) \(V\) is positive definite; that is, \(V(0) = 0\), and \(\forall x \in X\) \(\{x \in X | V(x) > 0\}\),

(A3) for arbitrary constants \(r_1, r_2\) such that \(r_2 > r_1 > 0\), there exist a compact set \(U \subseteq \mathbb{R}^n\), a constant \(Q > 0\), and a discontinuous mapping \(p : X \to T^*_x X\) such that

\[
x \in \{x \in X | r_1 \leq V(x) \leq r_2\} \Rightarrow \min_{u \in U} \left(\begin{array}{c}
(p(x), f(x) + g(x)u) < -Q.
\end{array}\right)
\]

(11)

Nakamura et al. proposed the following sample stabilizing state feedback for system (4):

**Theorem 2** \([3]\) Let \(V(x)\) be an LS-PCLF for system (4). Then, the following state feedback \(u : X \to U\) sample stabilizes the origin of system (4)

\[
u_i(x) = \begin{cases}
-p_i(x) \sqrt{\|p_i(x)\|^2 + (\sum_{i=1}^m \|p_i(x)\|^2)^2} & \text{if } (p_i(x), S_{i=1}^m \|p_i(x)\|^2) \neq 0 \\
0 & \text{if } \sum_{i=1}^m \|p_i(x)\|^2 = 0
\end{cases}
\]

(12)

where \(p : X \to T^*_x X\) is a discontinuous map satisfying (A3).

C. Sample Input-to-state stability

For the original nonlinear system (2), sample input-to-state stability (s-ISS) is defined by Malisoff et al. \([2]\).

Let \(\Omega_c := \{x \in \Omega | \|\omega(t)\| \leq c \forall t \geq 0\}\) for a constant \(c \geq 0\).

**Definition 7 (Sample Input-to-state Stability)** \([2]\) The control input \(u \in U\) said to sample input-to-state stabilize the origin of system (2) if the following properties hold:

1) for arbitrary set \(W \in \mathcal{P}_0\) and constant \(c \geq 0\), there exists \(W_0 \in \mathcal{P}_W\) such that the following proper holds:

\[a) \text{for any sets } W_1, W_2 \in \mathcal{P}_W, \text{ such that } W_1 \subseteq W_2, \text{ there exist constants } \delta > 0, T \geq 0 \text{ and a set } M \subseteq X \text{ such that}
\]

\[
\varphi(t; x_0, u, \omega, \pi) \in W_1, \\
\forall t \geq T, \forall x_0 \in W_2, \forall \omega \in \Omega_r, \forall \pi \in \text{Par}(\delta),
\]

(13)

\[
\varphi(t; x_0, u, \omega, \pi) \in M, \\
\forall t \geq 0, \forall x_0 \in W_2, \forall \omega \in \Omega_r, \forall \pi \in \text{Par}(\delta).
\]

(14)

b) for each \(E \in \mathcal{P}_W\), there exists a set \(D \in \mathcal{P}_W\) such that if \(W_2 \subseteq D, M\) in a) can be chosen satisfying \(M \subseteq E\).

2) for each set \(W_0 \in \mathcal{P}_0\), there exists a constant \(c \geq 0\) such that the conditions a) and b) holds.
III. ISS-LS-PCLF AND SAMPLE INPUT-TO-STATE STABILIZING STATE FEEDBACK

The aim of this paper is to design a sample-input-to-state stabilizing state feedback for a system. To design the state feedback, we define the following input-to-state locally semiconcave practical control Lyapunov function (ISS-LS-PCLF):

**Definition 8 (ISS Practical Control Lyapunov Function)**
A locally semiconcave function $V: X \to \mathbb{R}_{\geq 0}$ satisfying (A1) and (A2) is an input-to-state locally semiconcave practical control Lyapunov function (ISS-LS-PCLF) if there exist a continuous positive definite proper function $S: X \to \mathbb{R}_{\geq 0}$ and a class $K_0$ function $^1 \alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the following condition (A4) holds:

(A4) for arbitrary constants $r_1$ and $r_2$ such that $r_2 > r_1 > 0$, there exist a compact set $U \subset \mathbb{R}^m$, a constant $Q$, a function $\alpha \in K_0$, and a discontinuous mapping $p: X \to TX$ such that

$x \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}$

and $\omega \in \{ \omega \in \mathbb{R}^q \mid \| \omega \| \leq c \}$

$\Rightarrow \min_{u \in U} \langle p(x), f(x) + g(x)u + h(x)\omega \rangle < -Q$.

The following theorem is the main result of this paper:

**Theorem 3** Let $V(x)$ be an ISS-LS-PCLF for the system (2). Then, the following input $u(x)$ sample input-to-state stabilize the origin of the system (2):

$$u_i(x) = \begin{cases} \frac{a(x) + \sqrt{a^2(x) + (\sum_{j=1}^{m} (p_j, g_i)^2)^2}}{\sum_{j=1}^{m} (p_j, g_i)^2} (p_j, g_i) & (\sum_{j=1}^{m} (p_j, g_i)^2 \neq 0) \\ 0 & (\sum_{j=1}^{m} (p_j, g_i)^2 = 0) \end{cases}$$

where $p: X \to TX$ is a discontinuous mapping satisfying (15) and a function $a: X \to \mathbb{R}$ is defined by

$$a(x) := \langle p, f \rangle + \alpha^{-1}(S(x)) \sum_{i=1}^{\ell} (p, h_i)^2.$$  

IV. PROOF OF THE THEOREM 3

To prove Theorem 3, we introduce the following five lemmas.

**Lemma 1** Let $c \geq 0$ be a constant, $r_1$ and $r_2$ arbitrary constants and $V(x)$ an ISS-LS-PCLF for system (2). Then, there exists a constant $\tilde{c} \geq 0$ such that if $r_2 > r_1 > \tilde{c}$, the following holds:

$$x \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}, \omega \in \{ \omega \in \mathbb{R}^q \mid \| \omega \| \leq c \},$$

$$\Rightarrow S(x) \geq \alpha(\| \omega \|).$$

Moreover, (18) holds for an arbitrary small $\tilde{c}$ by choosing a sufficiently small $c$.

**Proof**: Note that there exist a function $\alpha, \overline{\alpha} \in K_{\infty}$ satisfying

$$\alpha(S(x)) \leq V(x) \leq \overline{\alpha}(S(x)) \quad \forall x \in X.$$  

Let $\tilde{c} := \overline{\alpha} \circ \alpha(c)$. According to (19), the following holds for arbitrary constants $r_1$ and $r_2$ such that $r_2 > r_1 > \tilde{c}$:

$$x \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}, \omega \in \{ \omega \in \mathbb{R}^q \mid \| \omega \| \leq c \}$$

$$\Rightarrow S(x) \geq \overline{\alpha}^{-1} \circ \alpha(V(x)) \geq \alpha^{-1}(\tilde{c}) = \alpha(c)$$

$$\Rightarrow S(x) \geq \alpha(\| \omega \|).$$

Moreover, we can obtain $\tilde{c} = \overline{\alpha} \circ \alpha(c) \to 0$ as $c \to 0$. ■

**Lemma 2** Let $V(x)$ be an ISS-LS-PCLF for system (2), $a(x)$ a function defined by (17), $c \geq 0$ a constant, and $\tilde{c} \geq 0$ a constant satisfying the condition of Lemma 1.

Then, for arbitrary constants $r_1$ and $r_2$ such that $r_2 > r_1 > \tilde{c}$, there exist a compact set $U \subset \mathbb{R}^m$, a constant $Q > 0$, and a discontinuous mapping $p(x) \in \bar{D}(V(x))$ such that the following holds:

$$x \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}, \omega \in \{ \omega \in \mathbb{R}^q \mid \| \omega \| \leq c \}$$

$$\Rightarrow a(x) + \min_{u \in U} \sum_{i=1}^{m} (p_i, g_i) u_i < -Q$$

**Proof**: Disturbance $\omega$ maximizes $\langle p, h \omega \rangle$ under the constraint $\omega \in \{ \omega \in \mathbb{R}^q \mid S(x) \geq \alpha(\| \omega \|) \}$ is obtained as follows:

$$\omega_i = \left\{ \begin{array}{ll} \frac{(p_i, h_i)}{\sqrt{\sum_{j=1}^{\ell} (p_j, h_i)^2}} \alpha^{-1}(S(x)) & (\sum_{j=1}^{\ell} (p_j, h_i)^2 \neq 0) \\ 0 & (\sum_{j=1}^{\ell} (p_j, h_i)^2 = 0) \end{array} \right.$$  

According to (15), (18) and (22), we can confirm that the lemma holds. ■

**Lemma 3** Let $u_i(x)$ be a control input defined in Theorem 3, $c \geq 0$ a constant, $\tilde{c} \geq 0$ a constant satisfying the condition of Lemma 1. Then, for arbitrary constants $r_1$ and $r_2$ such that $r_2 > r_1 > \tilde{c}$, there exist a constant $Q > 0$ and a discontinuous mapping $p(x) \in \bar{D}(V(x))$ such that the following holds:

$$x \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}, \omega \in \{ \omega \in \mathbb{R}^q \mid \| \omega \| \leq c \}$$

$$\Rightarrow a(x) + \sum_{i=1}^{m} (p_i, g_i) u_i(x) < -Q.$$  

**Proof**: According to Lemma 2, we can obtain

$$x \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}$$

$$\omega \in \{ \omega \in \mathbb{R}^q \mid \| \omega \| \leq c \},$$

and $\omega \in \{ \omega \in \mathbb{R}^q \mid \| \omega \| \leq c \}$,

$$\Rightarrow \min_{u \in U} \sum_{i=1}^{m} (p_i, g_i) u_i < -\frac{Q}{2}.$$
Then, there exists a constant \( s > 0 \) satisfying
\[
x \in \left\{ x \in X \mid r_1 \leq V(x) \leq r_2 \text{ and } a(x) \geq -\frac{Q}{2} \right\},
\]
and \( \omega \in \{ \omega \in \mathbb{R}^\ell \mid \|\omega\| \leq c \}, \)
\[
\Rightarrow \sum_{i=1}^{m} \langle p, g_i \rangle^2 > s.
\]
By (16) and (25), the following inequality holds:
\[
x \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}, \ \omega \in \{ \omega \in \mathbb{R}^\ell \mid \|\omega\| \leq c \},
\]
\[
a(x) + \sum_{i=1}^{m} \langle p, g_i \rangle u_i(x) = -\sqrt{a^2(x) + \left( \sum_{i=1}^{m} \langle p, g_i \rangle^2 \right)^2},
\]
\[
< -\min \left\{ \frac{Q}{2}, s \right\}.
\]
Then, the proof is completed. \( \square \)

**Lemma 4** Let \( u_i(x) \) be a control input defined in the Theorem 3, \( c \geq 0 \) a constant, and \( \tilde{c} \geq 0 \) a constant satisfying the condition of Lemma 1.

Then, for arbitrary constants \( r_1, r_2 \) such that \( r_2 > r_1 > \tilde{c} \), there exists a constant \( G > 0 \) such that the following holds:
\[
x \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}, \ \omega \in \{ \omega \in \mathbb{R}^\ell \mid \|\omega\| \leq c \},
\]
\[
\Rightarrow \sum_{i=1}^{m} u_i^2(x) < G.
\]
(27)

**Proof:** Let \( N := \{ 1, 2, 3, \ldots \} \). Since \( \{ x \in X \mid r_1 \leq V(x) \leq r_2 \} \) is a compact set, it is sufficient to prove
\[
\lim_{j \to \infty} \sum_{i=1}^{m} u_i^2(x_j) < +\infty \text{ for any convergent sequence } (x_j)_{j \in \mathbb{N}} \subset \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}.
\]
If \( \lim_{j \to \infty} \sum_{i=1}^{m} \langle p(x_j), g_i(x_j) \rangle^2 = 0 \), we can obtain
\[
\lim_{j \to \infty} \sum_{i=1}^{m} u_i^2(x_j) < +\infty \text{ by (16).}
\]
Then we consider the case
\[
\lim_{j \to \infty} \sum_{i=1}^{m} \langle p(x_j), g_i(x_j) \rangle^2 \neq 0.
\]
According to Lemma 2, there exists a sufficiently large constant \( J \in \mathbb{N} \) such that
\[
a(x_j) < 0 \quad \forall j > J.
\]
(28)

Note that the following inequality holds:
\[
\sqrt{a^2(x) + \left( \sum_{i=1}^{m} \langle p, g_i \rangle^2 \right)^2} \leq |a(x)| + \sum_{i=1}^{m} |\langle p, g_i \rangle|.
\]
(29)

This implies for each \( x \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}, \) there exists a constant \( \Lambda \in [0, 1] \) such that
\[
\sqrt{a^2(x) + \left( \sum_{i=1}^{m} \langle p, g_i \rangle^2 \right)^2} = |a(x)| + \Lambda \sum_{i=1}^{m} \langle p, g_i \rangle^2.
\]
(30)

According to (16) and (30), there exists a function \( \lambda : \{ x \in X \mid r_1 \leq V(x) \leq r_2 \text{ and } a(x) < 0 \} \to [0, 1] \) satisfying the following equation:
\[
u_i(x) = -\lambda(x) \langle p, g_i \rangle.
\]
(31)

Therefore, we can obtain
\[
\lim_{j \to +\infty} \sum_{i=1}^{m} u_i^2(x_j) < +\infty \text{ by (28), (31), \ and semiconcavity of } V(x).
\]

**Lemma 5** Let \( u_i(x) \) be a control input defined in the Theorem 3, \( c \geq 0 \) a constant, and \( \tilde{c} \geq 0 \) a constant satisfying the condition of Lemma 1.

Then, for arbitrary constants \( r_1, r_2 \) such that \( r_2 > r_1 > \tilde{c} \), there exist constants \( \delta, \tilde{Q} > 0 \) such that the following holds:
\[
x_0 \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}, \ \omega \in \Omega_c, \ \pi \in \text{Par}(\delta)
\]
\[
\text{and } t \in [t \geq 0 \mid \varphi(t) \in \{ x \in X \mid r_1 \leq V(x) \leq r_2 \}]
\]
\[
\Rightarrow V(\varphi(t)) - V(x_0) - \langle p(x_0); f(\varphi(t)) + g(\varphi(t))u(x_0) + h(\varphi(t))\pi(t) \rangle + C\|j(\varphi(t)) - j(x_0)\|^2 < -\tilde{Q}t.
\]
(32)

**Proof:** Since \( R := \{ x \in X \mid r_1 \leq V(x) \leq r_2 \} \) is a compact set, there exist finitely many local charts \( (W_j, \eta_j) \) \( (j \in \{ 1, \ldots, J \}) \) such that \( R \subset \bigcup_{j \in \{ 1, \ldots, J \}} W_j \). Since \( f, g, h \) and \( V \) are locally Lipschitz continuous functions, there exist constants \( K, L > 0 \) such that
\[
|V(y) - V(x)| \leq K|\eta_j(y) - \eta_j(x)|, \quad j \in \{ 1, \ldots, J \}, \quad \forall y, x \in W_j \cap R.
\]
(33)

According to Lemma 4, there exists a constant \( M > 0 \) such that
\[
|f_{W_j}(x) + g_{W_j}(x)u(x) + h_{W_j}(x)\omega| < M, \quad j \in \{ 1, \ldots, J \}, \quad \forall x \in W_j \cap R, \forall \omega \in \{ \omega \in \mathbb{R}^\ell \mid \|\omega\| \leq c \},
\]
(35)

According to semiconcavity of \( V \) and Proposition 3.3.1 of [11], the following holds for any \( t \in [0, D/M] \) and \( \omega \in \Omega_c \):
\[
\varphi(t; x_0, u, \omega, \pi) \in W_j.
\]
(37)

According to semiconcavity of \( V \) and Proposition 3.3.1 of [11], the following holds for any \( t \in [0, D/M] \) and \( \omega \in \Omega_c \):
\[
V(\varphi(t)) - V(x_0) \leq \langle p(x_0), f(\varphi(t)) + g(\varphi(t))u(x_0) + h(\varphi(t))\omega(t) \rangle + C\|j(\varphi(t)) - j(x_0)\|^2.
\]
(38)

The mean value inequality [12] implies that there exists \( t^* \in [0, t] \) such that
\[
V(\varphi(t)) - V(x_0) \leq \langle p(x_0), f(\varphi(t^*)) + g(\varphi(t^*))u(x_0) + h(\varphi(t^*))\omega(t^*) \rangle + C\|j(\varphi(t)) - j(x_0)\|^2.
\]
(39)
By the similar discussion of the proof of Lemma 5 in [3], the following holds for any \( t \in [0, D/M] \) and \( \omega \in \Omega_c. \)

\[
V(\varphi(t)) - V(x_0) \\
\leq \langle p(x_0), f(\varphi(t^*)) \rangle + g(\varphi(t^*))u(x_0) + h(\varphi(t^*))\omega(t^*) \\
- p(x_0), f(x_0) + g(x_0)u(x_0) + h(x_0)\omega(t^*) \rangle t \\
+ \langle p(x_0), f(x_0) + g(x_0)u(x_0) + h(x_0)\omega(t^*) \rangle t \\
+ CM^2t^2 \\
\leq \|pw_j(x_0)\| \cdot L\|\eta_j(\varphi(t^*)) - \eta_j(x_0)\| t \\
+ \left( a(x_0) + \sum_{i=1}^m \langle p(x_0), g_i(x_0)u_i(x_0) \rangle \right) t \\
+ CM^2t^2 \\
\leq KLMt^2 - Q t + CM^2t^2. \tag{40}
\]

Therefore, the following holds for any \( t \in [0, \min\{D/M, Q/2M(KL + CM)\}] \) and \( \omega \in \Omega_c. \)

\[
V(\varphi(t)) - V(x_0) \leq -\frac{Q}{2} t. \tag{41}
\]

Now, let us prove the Theorem 3.

**Proof:** [Proof of Theorem 3] Let \( u(x) \) be the control input defined in Theorem 3. Then, we prove all conditions of Definition 6 are satisfied.

Firstly, we prove the condition 1 holds. Note that for any set \( W \in \mathcal{P}(\varnothing) \), there exists an arbitrary large constant \( \bar{c} > 0 \) such that \( W \subset \{x \in X \mid V(x) < \bar{c} \} \). Let \( W \in \mathcal{P}(\varnothing) \) be an arbitrary set, \( \bar{c} > 0 \) an arbitrary constant, \( \bar{c} > 0 \) a constant satisfying \( W \subset \{x \in X \mid V(x) < \bar{c} \} \) and the condition of Lemma 5, and \( W_0 := \{x \in X \mid V(x) < \bar{c} \} \). Then, the condition 1 holds by Lemma 5.

Then we prove the condition 2) holds. Note that for any set \( W_0 \in \mathcal{P}(\varnothing) \), there exists an arbitrary small constant \( \bar{c} > 0 \) such that \( \{x \in X \mid V(x) < \bar{c} \} \subset W_0 \). Let \( W_0 \in \mathcal{P}(\varnothing) \) be an arbitrary set. We choose a constant \( \bar{c} > 0 \) such that there exists a constant \( \bar{c} > 0 \) satisfying \( \{x \in X \mid V(x) < \bar{c} \} \subset W_0 \) and the condition of Lemma 5. Then, the condition 2 holds by Lemma 5.

**V. NUMERICAL EXAMPLE**

In this section, we confirm the effectiveness of the proposed method by the computer simulation. Let us consider a 1-link robot arm without friction and gravity modeled as follows:

\[
\dot{x}_1 = x_2, \\
\dot{x}_2 = \frac{1}{J} \tau + \omega, \tag{42}
\]

where \( x_1 \in S^1 \) and \( x_2 \in \mathbb{R} \) are state variables, \( J > 0 \) is the inertial moment, \( \tau \in \mathbb{R} \) is an input and \( \omega \in \mathbb{R} \) is a disturbance.

The problem here is to design a sample input-to-state stabilizing state feedback \( \tau = u(x) \) at the origin \( (x_1, x_2) = (0, 0) \).

According to the minimum projection method, we can obtain the following LS-PCLF for the system [4]:

\[
V(x) = (x_1 - 2\pi n)^2 + (x_1 - 2\pi n)x_2 + x_2^2, \tag{43}
\]

with \( x_1 \in (-\pi, \pi) \) and \( n \in \mathbb{Z} \) such that \(-2(x_1 + (2n - 1)\pi) < x_2 \leq -2(x_1 + (2n + 1)\pi)\). Note that \( V \) is not differentiable on \( \{(x_1, x_2) \in S^1 \times \mathbb{R} \mid x_2 = -2(x_1 + (2n - 1)\pi), \; n \in \mathbb{Z}\}. \tag{44} \)

We can verify that \( V(x) \) is an ISS-LS-PCLF for the system (42) by choosing

\[
S(x) = \sqrt{x_1^2 + x_2^2}, \tag{45}
\]

\[
\alpha(\vert \omega \vert) = \frac{1}{\beta} \vert \omega \vert, \tag{46}
\]

with \( \beta > 0 \). We can design a discontinuous mapping \( p \) as

\[
p(x) = \begin{pmatrix} 2(x_1 - 2\pi n) + x_2 \\ x_1 - 2\pi n + 2x_2 \end{pmatrix}, \tag{47}
\]

where \( n \) satisfies \(-2(x_1 + (2n - 1)\pi) < x_2 \leq -2(x_1 + (2n + 1)\pi)\). Then, each function appeared in the proposed controller (16) are calculated as follows:

\[
\langle p(x), f(x) \rangle = x_2 \{2(x_1 - 2\pi n) + x_2\}, \tag{48}
\]

\[
\langle p(x), g(x) \rangle = \frac{1}{J}(x_1 - 2\pi n + 2x_2), \tag{49}
\]

\[
\langle p(x), h(x) \rangle = x_1 - 2\pi n + 2x_2, \tag{50}
\]

\[
\alpha^{-1}(S(x)) = \beta \sqrt{x_1^2 + x_2^2}. \tag{51}
\]

Finally, we can construct the controller (16) by using (17) and (48)-(51).

We consider initial condition \( x_0 = (0, 10) \), inertial moment \( J = 1 \) and disturbance \( \omega = 4 \sin(2\pi t) + 4 \sin(8\pi t) \). We show a computer simulation with design parameter \( \beta = 1 \) in Figs. 1 and 2. The variation of \( x_1 \) is normalized to \((-\pi, \pi)\). While we can view discontinuity of \( x_1 \) at \( t = 0.5 \) in Fig. 1, this is due to the normalization; \( x_1 \) smoothly changes with respect to \( t \). This also illustrates the controller does not show an unwinding phenomenon.

Figures 3 and 4 show a simulation result in the case of the conventional controller (12). Note that the conventional controller is designed for systems without disturbance. While both state oscillate due to disturbance \( \omega \), the proposed controller suppresses the oscillation effectively.

**VI. CONCLUSIONS AND FUTURE WORKS**

In this paper, we proposed a sample input-to-state stabilizing controller for nonlinear systems on differentiable manifolds. The proposed controller based on ISS-LS-PCLF and extends one of Krstić and Li [6]. The effectiveness of the proposed controller was confirmed by the numerical example.

Our results confirmed that we can construct discontinuous state feedback controllers achieving robustness with respect to system disturbance by using nonsmooth CLFs. On the
contrary, discontinuous state feedbacks never have robustness with respect to measurement errors if there are no differentiable CLFs [13]. How to construct a controller on manifolds having robustness with respect to measurement errors is a difficult problem. There is the challenge for future work.

REFERENCES