Complexity results for Minimum Sum Edge Coloring*

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Abstract

In the Minimum Sum Edge Coloring problem we have to assign positive integers to the edges of a graph such that adjacent edges receive different integers and the sum of the assigned numbers is minimal. We show that the problem is (a) NP-hard for planar bipartite graphs with maximum degree 3, (b) NP-hard for 3-regular planar graphs, (c) NP-hard for partial 2-trees, and (d) APX-hard for bipartite graphs.

1 Introduction

A vertex coloring of a graph is an assignment of colors to the vertices of a graph such that if two vertices are adjacent, then they are assigned different colors. In this paper, we assume that the colors are the positive integers; a vertex $k$-coloring is a coloring where the color of each vertex is taken from the set \{1, 2, \ldots, k\}. Given a vertex coloring of a graph $G$, the sum of the coloring is the sum of the colors assigned to the vertices. The chromatic sum $\Sigma(G)$ of $G$ is the smallest sum that can be achieved by any proper coloring of $G$. In the Minimum Sum Coloring problem we have to find a coloring of $G$ with sum $\Sigma(G)$.

Minimum Sum Coloring was introduced independently by Kubicka [15] and Supowit [25]. Besides its combinatorial interest, the problem is motivated by applications in scheduling [2, 3, 11] and VLSI design [22, 26]. In [16] it is shown that the problem is NP-hard in general, but polynomial-time solvable for trees. The dynamic programming algorithm for trees can be extended to partial $k$-trees [14]. For further complexity results and approximation algorithms, see [2, 3, 9, 24].

One can analogously define the edge coloring version of Minimum Sum Coloring. Formally, we will investigate the following optimization problem:

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Table 1: Results for Minimum Sum Edge Coloring.

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<th>Class</th>
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<td>Bipartite graphs</td>
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**Minimum Sum Edge Coloring**

*Input:* A graph $G(V, E)$.

*Find:* An edge coloring $\psi : E \rightarrow \mathbb{N}$ such that if $e_1$ and $e_2$ have a common vertex, then $\psi(e_1) \neq \psi(e_2)$.

*Goal:* Minimize $\sum_{e \in E} \psi(e)$, the sum of the coloring.

In this paper we prove complexity results for Minimum Sum Edge Coloring restricted to certain classes of graphs. These results nicely complement the approximation algorithms published in the literature, as they show that the constant-factor approximation algorithms of [11, 2] cannot be improved to a polynomial-time approximation scheme (PTAS), and the approximation schemes of [19] cannot be replaced by a polynomial-time exact algorithm.

Table 1 summarizes the algorithmic and complexity results known for Minimum Sum Edge Coloring. The problem is NP-hard in general (even for bipartite graphs [10]) and trees are the only class of graphs where Minimum Sum Edge Coloring is known to be polynomial-time solvable [10, 24, 28]. Therefore, most of the algorithmic results presented in the literature are approximation algorithms.

For general graphs, a 2-approximation algorithm for Minimum Sum Edge Coloring is presented in [2]. For bipartite graphs better approximation ratio is possible: a 1.796-approximation algorithm follows from [11], and a 1.414-approximation algorithm is given in [8]. It is proved in Section 4 that the problem is APX-hard for bipartite graphs, hence these constant-factor approximations cannot be improved to a PTAS.

For partial $k$-trees (graphs of bounded tree width) and planar graphs, Minimum Sum Edge Coloring admits a PTAS [19]. (In fact, the approximation scheme of [19] works also for the more general multicoloring version of the problem.) We show that a polynomial-time exact algorithm for these classes cannot be expected, as the problem is NP-hard for partial 2-trees (Section 5) and for planar graphs (Section 3).

As noted above, for trees Minimum Sum Edge Coloring can be solved in polynomial time [10, 24, 28] by a dynamic programming algorithm that uses weighted bipartite matching as a subroutine. In most cases, when a problem can be solved in trees by dynamic programming, then this easily generalizes to partial $k$-trees, and a similar dynamic programming approach can solve the problem in partial $k$-trees. For example, that is the case with the vertex coloring version of Minimum Sum Coloring on trees and partial $k$-trees. Other examples include the Maximum Independent Set, Vertex Coloring, and


Vertex Disjoint Paths (see [5, 6, 7] for more information on partial \(k\)-trees). Therefore, it is somewhat surprising that Minimum Sum Edge Coloring is NP-hard for partial 2-trees. There are only two other examples that we are aware of where the algorithm for trees does not generalize to partial 2-trees. The Edge Disjoint Paths problem is trivial for trees, but it becomes NP-hard for partial 2-trees [23]. Furthermore, the Edge Precoloring Extension problem is polynomial-time solvable for trees [18], but NP-hard for partial 2-trees [20].

2 Preliminaries

For the rest of the paper, we consider only edge colorings, hence even if it is not noted explicitly, “coloring” will mean “edge coloring.” We introduce notation and new parameters that turn out to be useful in studying minimum sum edge colorings. Let \(\psi\) be an edge coloring of \(G(V, E)\), and let \(E_v\) be the set of edges incident to vertex \(v\). For every \(v \in V\), let \(\Sigma'_\psi(v) = \sum_{e \in E_v} \psi(e)\) be the sum of \(v\), and for a subset \(V' \subseteq V\), let \(\Sigma'_\psi(V') = \sum_{v \in V'} \Sigma'_\psi(v)\). Clearly, \(\Sigma'_\psi(V) = 2\Sigma'_\psi(G)\); therefore, minimizing \(\Sigma'_\psi(V)\) is equivalent to minimizing \(\Sigma'_\psi(G)\).

The degree of vertex \(v\) is denoted by \(d(v) := |E_v|\). For every vertex \(v\), let \(\ell(v) := \sum_{\ell=1}^{d(v)} i = d(v)(d(v) + 1)/2\), and for a set of vertices \(V' \subseteq V\), let \(\ell(V') := \sum_{v \in V'} \ell(v)\). Since \(\Sigma'_\psi(v)\) is the sum of \(d(v)\) distinct positive integers, \(\Sigma'_\psi(v) \geq \ell(v)\) in every proper coloring \(\psi\). Let \(\epsilon_\psi(v) = \Sigma'_\psi(v) - \ell(v) \geq 0\) be the error of vertex \(v\) in coloring \(\psi\). For \(V' \subseteq V\) we define \(\epsilon_\psi(V') = \sum_{v \in V'} \epsilon_\psi(v)\), and call \(\epsilon_\psi(V)\) the error of coloring \(\psi\). The error is always non-negative: \(\Sigma'_\psi(V) \geq \ell(V)\), hence \(\epsilon_\psi(V) = \Sigma'_\psi(V) - \ell(V) \geq 0\). Notice that \(\epsilon_\psi(V)\) has the same parity for every coloring \(\psi\). Minimizing the error of the coloring is clearly equivalent to minimizing the sum of the coloring. In particular, if \(\psi\) is a zero error coloring, that is, \(\epsilon_\psi(V) = 0\), then \(\psi\) is a minimum sum coloring of \(G\). In a zero error coloring, the edges incident to vertex \(v\) are colored with the colors 1, 2, \ldots, \(d(v)\).

However, in general, \(G\) does not necessarily have a zero error coloring. Deciding whether \(G\) has a zero error coloring is a special case of Minimum Sum Edge Coloring. It might be worth pointing out that finding a zero error coloring is very different from finding a minimum sum coloring: zero error is a local constraint on the coloring (every vertex has to have zero error), while minimizing the sum is a global constraint.

Parallel edges are not allowed for the graphs considered in this paper. However, for convenience we extend the problem by introducing half-loops. A half-loop is a loop that contributes only 1 to the degree of its end vertex. Every vertex has at most one half-loop. If a graph is allowed to have half-loops, then it will be called a quasigraph (the terminology half-loop and quasigraph is borrowed from [17]). In a quasigraph, the sum of an edge coloring is defined to be the sum of the color of the edges plus half the sum of the color of the half-loops; therefore, the sum of a quasigraph is not necessarily an integer. The sum \(\Sigma'_\psi(v)\) is defined to be the integer \(\sum_{e \in E_v} \psi(e)\), as before, thus a half-loop contributes to the sum of exactly one vertex. Thus it remains true that the error of a coloring is always integer and the sum of the vertices is twice the sum of the edges.

The following observation shows that allowing half-loops does not make the problem more difficult, thus any hardness result for quasigraphs immediately implies hardness for ordinary graphs as well. This observation was used in
Proposition 2.1. Given a quasigraph $G$, one can create in polynomial time a graph $G'$ such that $\Sigma'(G') = 2\Sigma'(G)$.

Proof. To obtain $G'$, take two disjoint copies $G_1, G_2$ of $G$ and remove every half-loop. If there was a half-loop at $v$ in $G$, then add an edge $v_1v_2$ to $G'$, where $v_1$ and $v_2$ are the vertices corresponding to $v$ in $G_1$ and $G_2$, respectively. In graph $G'$, give to every edge the color of the corresponding edge in $G$. If the sum of the coloring in $G$ was $S$, then we obtain a coloring in $G'$ with sum $2S$: two edges of $G'$ correspond to every edge of $G$, but only one edge corresponds to every half-loop of $G$.

On the other hand, one can show that if $G'$ has a $k$-coloring with sum $S$, then $G$ has a $k$-coloring with sum at most $S/2$. The edges of $G'$ can be partitioned into three sets $E_1, E_2, E'$: set $E_i$ contains the edges induced by $G_i$ ($i = 1, 2$), and $E'$ contains the edges corresponding to the half-loops. If $\psi$ is an edge coloring of $G'$ with sum $S$, then $S = \Sigma_\psi(E_1) + \Sigma_\psi(E_2) + \Sigma_\psi(E')$. Without loss of generality, it can be assumed that $\Sigma_\psi(E_1) \leq \Sigma_\psi(E_2)$, hence $\Sigma_\psi(E_1) + \Sigma_\psi(E')/2 \leq S/2$. The $k$-coloring of $G_1$ induced by $\psi$ has sum $\Sigma_\psi(E_1) + \Sigma_\psi(E')/2 \leq S/2$, since the edges in $E'$ correspond to half-loops.

Therefore, minimizing the sum of the coloring on $G'$ is the same problem as minimizing the sum on $G$. Notice that if $G$ is bipartite, then $G'$ is bipartite as well. On the other hand, the transformation does not preserve planarity in general. Therefore, quasigraphs will be used only when proving hardness results for bipartite graphs (Section 4), but not in the case of planar graphs (Section 3).

3 Planar graphs

In this section we show that Minimum Sum Edge Coloring is NP-hard for planar bipartite graphs of maximum degree 3, and for planar 3-regular graphs. The proof is by reduction from Edge Precoloring Extension.

In Precoloring Extension a graph $G$ is given with some of the vertices having preassigned colors, and it has to be decided whether this precoloring can be extended to a proper vertex $k$-coloring of the whole graph. One can analogously define the problem Edge Precoloring Extension. It is shown in [20] that Edge Precoloring Extension is NP-complete for 3-regular planar bipartite graphs. For more background on Precoloring Extension and Edge Precoloring Extension, the reader is referred to [27, 4, 12, 13].

In the following theorem, we reduce the NP-complete Edge Precoloring Extension (a problem with local constraints) to deciding whether a graph has a zero error coloring. This proves that Minimum Sum Edge Coloring is NP-hard.

Theorem 3.1. It is NP-hard to decide if a planar bipartite graph with degree at most 3 has a zero error coloring.

Proof. Using simple local replacements, we reduce Edge Precoloring Extension to the problem of finding a zero error coloring, which is a special case of Minimum Sum Edge Coloring. Given a 3-regular graph $G$ with some
of the edges having preassigned colors, construct a graph $G'$ by replacing the precolored edges with the subgraphs shown in Figure 1. If we replace the edge $e = uv$ with such a subgraph, then the two new edges incident to $v$ and $u$ will be called $e_1$ and $e_2$. If $G$ is planar and bipartite, then clearly $G'$ is planar and bipartite as well.

We show that $G'$ has a zero error coloring if and only if $G$ has a precoloring extension with 3 colors. Assume that $\psi$ is a zero error coloring. We show that for every precolored edge $e$, the edges $e_1$ and $e_2$ receive the color of $e$. If $e$ is precolored to 1 (see case a) in Figure 1), then $d(a) = d(b) = 1$, thus $e_1$ and $e_2$ receive color 1 in every zero error coloring. If $e$ has color 2, then edges $ac$ and $bd$ must have color 1, thus edges $e_1$, $e_2$ have color 2 in every zero error coloring. Finally, if $e$ has color 3, then $ac$ and $bd$ have color 1, edges $ax$ and $by$ have color 2, hence $e_1$ and $e_2$ have color 3. Therefore, $\psi$ extends the precoloring of $G$.

The converse is also easy to see: given a precoloring extension of $G$, for each edge $e$ in $G$ we assign the color of $e$ to edges $e_1$ and $e_2$ in $G'$, and extend this coloring the straightforward way. It can be verified that this is a zero error coloring of $G'$, there is no vertex $v$ that is incident to an edge with color greater than $d(v)$ (here we use that $G$ is 3-regular).

As finding a zero error coloring is a special case of Minimum Sum Edge Coloring, we have

**Corollary 3.2.** Minimum Sum Edge Coloring is NP-hard for planar bipartite graphs having degrees at most 3.

It is tempting to try to strengthen Corollary 3.2 by replacing “degree at most 3” with “3-regular.” However, Minimum Sum Edge Coloring becomes polynomial-time solvable for bipartite, regular graphs. In fact, every such graph has a zero error coloring: by the line coloring theorem of König, every bipartite graph $G$ has a $\Delta(G)$-edge-coloring, which has zero error if $G$ is regular. However, if we add the requirement of 3-regularity, but drop the requirement that the graph is bipartite, then the problem remains NP-complete.

**Theorem 3.3.** Minimum Sum Edge Coloring is NP-complete for planar 3-regular graphs.
Proof. The reduction is from zero error coloring of planar graphs with degree at most 3 (Theorem 3.1). We attach certain gadgets to the graph $G$ to make it a 3-regular graph $G'$. The gadgets are attached in such a way that $G$ has a zero error coloring if and only if $G'$ has a coloring with error $K$, where $K$ is an integer determined during the reduction.

Figure 2 shows three gadgets $R_1, R_2, R_3$, each gadget has a pendant edge $e$. We show that gadget $R_i$ has the following property: if its edges are colored in such a way that the total error on the internal vertices is as small as possible, then the pendant edge receives color $i$. The figure shows such a coloring for each gadget, the circled vertices are the vertices where there are errors in the coloring.

Gadget $R_1$ (see Figure 2) has a pendant edge $e$, 5 internal vertices (denoted by $S$), and 7 edges connecting the internal vertices. Since each color can be used at most twice on these 7 edges, they have sum at least $2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 1 \cdot 4 = 16$ in every coloring. Therefore, if a coloring assigns color $i$ to edge $e$, then the vertices in $S$ have sum at least $32 + i$ and error at least $32 + i - \ell(S) = 2 + i$. Thus the error of $S$ is at least 3 and it can be 3 only if the pendant edge $e$ is colored with color 1.

In gadget $R_3$ (second graph on Figure 2), two copies of gadget $R_1$ are attached to vertex $v$. The error on the internal vertices is at least 6 in every coloring: there are at least 3 errors in each of $S_1$ and $S_2$. However, the error is strictly greater than this: at least one of $e_1$ and $e_2$ is colored with a color greater than 1, hence either $S_1$ or $S_2$ has error at least 4. Moreover, if the error of the internal vertices in $R_3$ is 7, then one of $e_1$ and $e_2$ is colored with color 1, the other edge is colored with color 2; therefore, edge $e$ has to be colored with color 3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gadgets.png}
\caption{The gadgets $R_1$, $R_2$, $R_3$. The coloring given on the figure has as few errors on the internal vertices as possible. The circles show the errors on the internal vertices in this coloring.}
\end{figure}

Gadget $R_2$ (third graph on Figure 2) contains a gadget $R_1$ and $R_3$ attached to vertex $v$. It has error at least $3 + 7 = 10$, since the internal vertices of these gadgets have at least that much error in every coloring. Furthermore, if the error on the internal vertices of $R_2$ is exactly 10, then this is only possible if the error in $S_1$ is 3 and the error in $S_2$ is 7. This implies that the edge $e_1$ has color 1 and edge $e_2$ has color 3; therefore, edge $e$ has color 2.

Given a planar graph $G$ with degree at most 3, we attach a gadget $R_2$ and a gadget $R_3$ to every vertex of degree 1. Furthermore, we attach a gadget $R_3$ to
every degree 2 vertex. Clearly, the resulting graph \( G' \) is planar and 3-regular. Let \( n \) be the number of \( R_3 \) gadgets attached, and let \( m \) be the number of \( R_2 \) gadgets. We claim that \( G \) has a zero error coloring if and only if \( G' \) has a coloring with error at most \( K = 7n + 10m \).

Assume first that \( G \) has zero error. This coloring can be extended in such a way that the error on every attached \( R_3 \) (resp., \( R_2 \)) gadget is 7 (resp., 10), and the edge that connects an \( R_2 \) (resp., \( R_3 \)) gadget to \( G \) has color 2 (resp., 3). If \( v \) is a vertex of \( G \) (not an internal vertex of a gadget), then the three colors 1, 2, and 3 appear at \( v \). Therefore, the error of the coloring is the total error of the gadgets, that is, \( K = 7n + 10m \).

Assume now that \( G' \) has a coloring with error at most \( K \). As we have seen, every gadget \( R_3 \) has error at least 7 in every coloring, and every gadget \( R_2 \) has error at least 10; therefore, if the coloring has error \( 7n + 10m \), then every \( R_3 \) gadget has error exactly 7, and every \( R_2 \) gadget has error exactly 10. This means that every edge connecting an \( R_2 \) (resp., \( R_3 \)) gadget to \( G \) has color 2 (resp., 3). Since \( G \) is a subgraph of \( G' \), the coloring of \( G' \) induces a coloring of \( G \). We show that this coloring is a zero error coloring of \( G \). If \( v \) is a degree 1 vertex of \( G \), then two additional edges connect \( v \) to an \( R_2 \) and an \( R_3 \) gadget in \( G' \), and these two edges have colors 2 and 3. The error of \( v \) is zero in the coloring; therefore, the edge incident to \( v \) in \( G \) receives color 1. Similarly, if \( v \) has degree 2 in \( G \), then an additional edge with color 3 is connected to \( v \) in \( G \), and it follows that the two edges incident to \( v \) in \( G \) have the colors 1 and 2, as required.

4 Approximability

A polynomial-time approximation scheme (PTAS) is an approximation algorithm that has an input parameter \( \epsilon \), and for every \( \epsilon > 0 \) it produces a solution with cost at most \( (1 + \epsilon) \) times the optimum. The running time has to be polynomial in the size of the input for every fixed value of \( \epsilon \), i.e., it is of the form \( n^{f(\epsilon)} \). If a problem admits a PTAS, then this means that there is no “best” approximation algorithm: an approximation ratio arbitrarily close to 1 can be achieved. On the other hand, by proving that a problem is APX-hard we can show that the problem does not admit a PTAS (unless P = NP), that is, there is a \( c > 1 \) such that there is no polynomial-time approximation algorithm with approximation ratio better than \( c \). Here we prove that Minimum Sum Edge Coloring is APX-hard, even for bipartite graphs. Therefore, the approximation schemes for partial \( k \)-trees and planar graphs presented in [19] cannot be generalized to arbitrary graphs.

**Theorem 4.1.** Minimum Sum Coloring is APX-hard for graphs with maximum degree 3.

**Proof.** The theorem is proved by an L-reduction from Minimum Vertex Cover for 3-regular graphs, which is shown to be APX-hard in [1]. For every graph \( G(V, E) \) with minimum vertex cover size \( \tau(G) \), a graph \( G'' \) is constructed that has edge chromatic sum \( C = c_1|V| + c_2|E| + \tau(G) \), where \( c_1 \) and \( c_2 \) are constants to be determined later. To see that this is an L-reduction, notice that \( |E| = \frac{3}{2}|V| \) and \( \tau(G) \geq |V|/4 \) follows from the fact that \( G \) is 3-regular. Therefore, \( C \leq 4c_1\tau(G) + 6c_2\tau(G) + \tau(G) = c_3\tau(G) \), as required. Furthermore, we
show that given an edge coloring of $G''$ with sum at most $c_1|V| + c_2|E| + t$, one can find a vertex cover of size $t$. This proves the correctness of the L-reduction.

The graph $G''$ is constructed in two steps: first we create a quasigraph $G'$, then apply the transformation of Proposition 2.1 to obtain the graph $G''$. The graph $G'$ consists of vertex gadgets and edge gadgets. The vertex gadget shown in Figure 3 has 3 pendant edges $e_1, e_2, e_3$, and satisfies the following two properties:

- If a coloring has zero error on the internal vertices of the variable gadget, then it colors all three pendant edges with color 1.
- There is a coloring that colors all three pendant edges with color 2 and has only 1 error on the internal vertices.

Figure 3 shows two possible colorings of the gadget, the two numbers on each edge show the color of the edge in the two colorings. The first coloring is the unique coloring with zero error on the internal vertices. To see this, notice first that an edge incident to a degree 1 internal vertex has to be colored with color 1. Furthermore, if an edge of a degree 2 vertex is colored with color 1, then the other edge has to be colored with color 2. Applying these and similar implications repeatedly, we get the first coloring of Figure 3. In particular, edges $e_1, e_2, e_3$ have color 1, proving the first property. The second coloring has one error (at $v$), and colors $e_1, e_2, e_3$ with color 2, proving the second property.

![Figure 3: The vertex gadget.](image)

The edge gadget shown in Figure 4 has two pendant edges $f$ and $g$. If a coloring has zero error on the internal vertices of the gadget, then clearly $f$ and $g$ have color 1 or 2. There are 4 different ways of coloring $f$ and $g$ with colors 1 or 2. In 3 out of 4 of these combinations, when at least one of $f$ and $g$ is colored with color 2, the coloring can be extended to the whole gadget with zero error (Figure 4 shows these 3 colorings). On the other hand, if both $f$ and $g$ have color 1, then there is at least one error on the internal vertices of the gadget. The reader can verify this by following the implications of coloring $f$ and $g$ with color 1, and requiring that every internal vertex has zero error.

The quasigraph $G'(V', E')$ is constructed as follows. A vertex gadget $S_v$ corresponds to every vertex $v$ of $G$, and an edge gadget $S_e$ corresponds to every
Furthermore, a coloring of $G'$ as in Proposition 2.1. We have that $\Sigma$ is $\ell$ (denote it by $\tau$ for every $v$) pendant edge of $S$. The vertex gadget $S$ pendant edge of means that every edge in $\hat{e}$ constants $\tau$ least $v$ is incident to $v$ is covered, since there is a show that this set is a vertex cover of $G$. Direct the edges of $G$ of $G'$, we can extend the coloring to every edge of $S$ color gadget $S$ internal vertices of gadget $S$ internal vertices of gadget $S$; clearly these sets form a partition of $S$.

We claim that $G'$ has a coloring with error $t$ if and only if $G$ has a vertex cover of size $t$. Assume first that $D \subseteq V$ is a vertex cover of $G$. If $v \in D$, then color gadget $S_v$ such that every pendant edge has color 2 (and there is one error on the internal vertices), otherwise color $S_v$ in such a way that every pendant edge has color 1, and there is no error on the internal vertices. Now consider a gadget $S_v$ for some $e \in E$. The two pendant edges $f$ and $g$ are already colored with colors 1 or 2. However, at least one of these two edges is colored with 2, since at least one end vertex of $e$ is in $D$. Therefore, using one of the three colorings shown in Figure 4, we can extend the coloring to every edge of $S_v$ with zero error on the internal vertices of the gadget. This means that errors appear only on the internal vertices of $S_v$ for $v \in D$, and the total error is $|D|$.

On the other hand, consider a coloring of $G'$ with error $t$. Let $\hat{V} \subseteq V$ be the set of those $v \in V$ for which $V_v$ is colored with error. Similarly, let $\hat{E} \subseteq E$ be the set of those $e \in E$ for which $V_v$ is colored with error. Clearly, the coloring has error at least $|\hat{V}| + |\hat{E}| \leq t$. Let $\hat{V}$ be a set of $|\hat{E}|$ vertices in $G$ that cover every edge in $\hat{E}$. The set of vertices $\hat{V} \cup \hat{V}$ has size at most $|\hat{V}| + |\hat{E}| \leq t$. We show that this set is a vertex cover of $G$. It is clear that every edge $e \in \hat{E}$ is covered, since there is a $v \in \hat{V}$ covering $e$. Now consider an edge $e \notin \hat{E}$, this means that $V_v$ is colored with zero error, thus, as we have observed, at least one pendant edge of $S_v$ is colored with color 2. If this edge is the pendant edge of the vertex gadget $S_v$, then there is at least one error in $V_v$ and $v$ is in $\hat{V}$. If the pendant edge of $S_v$ and $S_v$ is identified in the construction, this means that $e$ is incident to $v$, thus $v \in \hat{V}$ covers $e$.

We have proved that the error of a minimum sum edge coloring of $G'$ is at least $\tau(G)$. Furthermore, $\Sigma'(G') = (c_1/2)|V| + (c_2/2)|E| + \tau(G)/2$ for some constants $c_1$ and $c_2$. To see this, notice that the lower bound $\ell(V_v)$ is the same for every $v \in V$ (denote it by $c_1$), and $\ell(V_v)$ is the same for every $e \in E$ (denote it by $c_2$). Therefore, the sum of the vertices in the optimum coloring is $\ell(V_v) + \tau(G) = c_1|V| + c_2|E| + \tau(G)$. The edge chromatic sum is the half of this value, $(c_1/2)|V| + (c_2/2)|E| + \tau(G)/2$. Now construct graph $G''$ from $G'$ as in Proposition 2.1. We have that $\Sigma'(G'') = 2\Sigma'(G') = c_1|V| + c_2|E| + \tau(G)$. Furthermore, a coloring of $G''$ with sum $c_1|V| + c_2|E| + t$ gives a coloring of

Figure 4: The edge gadget.
$G'$ with sum $(c_1/2)|V| + (c_2/2)|E| + t/2$, that is a coloring with error $t$. It was shown above that given a coloring of $G'$ with error $t$, one can find a vertex cover of $G$ with size at most $t$. This completes the proof of the L-reduction.

Theorem 4.1 can be strengthened: the problem remains APX-hard for bipartite graphs. The graph constructed in the proof of Theorem 4.1 is not bipartite, since the vertex gadget in Figure 3 is not bipartite. However, the vertex gadget can be replaced by the slightly more complex quasigraph shown in Figure 5, which is bipartite and has the same properties. That is, if a coloring has zero error on the internal vertices, then the pendant edges have color 1, and there is a coloring that has error 1 on the internal vertices, and assigns color 2 to the pendant edges. The vertex and edge gadgets are bipartite, and they are connected in a way that ensures that the resulting graph $G'$ is bipartite as well.

**Theorem 4.2.** Minimum sum edge coloring is APX-hard for bipartite graphs with maximum degree 3.

![Figure 5: The bipartite quasigraph version of the vertex gadget.](image)

## 5 Partial $k$-trees

In this section, we show that Minimum Sum Edge Coloring is NP-hard for partial 2-trees. A $k$-tree is a graph defined by the following three rules:

1. A clique of size $k + 1$ is a $k$-tree.
2. If $G$ is a $k$-tree, and $K$ is a clique of size $k$ in $G$, then the graph $G'$ that is obtained by adding a new vertex $v$ and connecting $v$ to every vertex of $K$ is also a $k$-tree.
3. Every $k$-tree can be obtained using 1 and 2.

Another way to define $k$-trees is to say that a graph is a $k$-tree if and only if it is a chordal graph with clique number $k + 1$. A graph is a partial $k$-tree if it is a subgraph of a $k$-tree. The notion of tree width gives an alternate characterization of partial $k$-trees: a graph is partial $k$-tree if and only if it has tree width at most $k$. For more information on the algorithmic and combinatorial significance of partial $k$-trees and tree width, the reader is referred to [7, 6].

Before presenting the proof of NP-completeness, we introduce some gadgets used in the reduction. These gadgets are trees with a single pendant edge, and have the following general property: if a coloring is “cheap,” meaning that it has as small error on the internal vertices as possible, then the color of the pendant edge has to be one of the special allowed colors of the gadget. For the gadget $F_n$, this means that in every such cheap coloring, the pendant edge has color $n$. In the gadget $L_n$, the color of the pendant edge has to be either $n - 1$ or $n + 1$ in such a coloring. In the gadget $A_n$, the color of the pendant edge has to be an odd number not greater than $n$.

The reduction is from 3-SAT; therefore, we need satisfaction testing gadgets and variable setting gadgets. All these gadget are connected to a central vertex $v$. The satisfaction testing gadget has the property that in every cheap coloring the pendant edge (the edge that connects the gadget to $v$) has one of the three preassigned colors. The variable setting gadget $W_n$ is different from the other gadgets. First, it is not a tree, but a partial 2-tree. Moreover, there are two edges connecting it to the central vertex $v$. The crucial property of this gadget is that in every cheap coloring, these two edges either use the colors $n + 1$, $n + 3$, or they use the colors $n + 5$, $n + 7$.

In the following lemmas, we formally define the properties of the gadgets, describe how they are constructed, and prove the required properties.

**Lemma 5.1.** For every $n \geq 2$, there is a tree $F_n$ and an integer $f_n$, such that

1. $F_n$ has one pendant edge $e$,
2. the internal vertices of $F_n$ have error at least $f_n$ in every coloring,
3. if a coloring has error $f_n$ on the internal vertices of $F_n$, then this coloring assigns color $n$ to the pendant edge $e$, and
4. $F_n$ can be constructed in time polynomial in $n$.

**Proof.** The tree $F_n$ is a star with a central vertex $v$, and $n$ leaves $v_1, v_2, \ldots, v_n$. The pendant edge $e$ is the edge $v_n v$, thus the internal vertices are $v, v_1, v_2, \ldots, v_{n-1}$. Let $f_n := (n-1)(n-2)/2$. The $n - 1$ edges $v_1 v, \ldots, v_{n-1} v$ have different colors, hence the sum of the vertices $v_1, \ldots, v_{n-1}$ is at least $\sum_{i=1}^{n-1} i = n(n-1)/2$. Therefore, the error on these vertices is at least $n(n-1)/2 - (n-1) = f_n$. There is equality if and only if the sum of these vertices is exactly $n(n-1)/2$ and there is no error on $v$. This implies that edge $v_n v$ has color $n$, as required. 

**Lemma 5.2.** For every even $n \geq 1$, there is a tree $L_n$ and an integer $k_n$, such that

1. $L_n$ has one pendant edge $e$,
2. The internal vertices of $L_n$ have error at least $k_n$ in every coloring.

3. If a coloring has error $k_n$ on the internal vertices of $L_n$, then this coloring assigns either color $n-1$ or $n+1$ to the pendant edge $e$.

4. There are colorings $\psi_{n-1}$ and $\psi_{n+1}$ of $L_n$ with $\psi_{n-1}(e) = n-1$, $\psi_{n+1}(e) = n+1$, such that they have error $k_n$ on the internal vertices, and

5. $L_n$ can be constructed in time polynomial in $n$.

Proof. The tree $L_n$ is constructed as follows (see Figure 6). The pendant edge $e$ connects external vertex $u$ and internal vertex $v$. A set $V$ of $n-2$ vertices $v_1, v_2, \ldots, v_{n-2}$ are connected to $v$. There are two additional neighbors of $v$: vertices $a$ and $b$. Besides $v$, vertex $a$ has $n-1$ neighbors $a_1, a_2, \ldots, a_{n-1}$, let $A$ be the set containing these $n-1$ vertices. Similarly, vertex $b$ has $n-1$ additional neighbors $B = \{b_1, b_2, \ldots, b_{n-1}\}$.

Since the edges $v_1v, v_2v, \ldots, v_{n-2}v$ have different colors in every coloring of $L_n$, the sum of $V$ is at least $\sum_{i=1}^{n-2} i = (n-2)(n-1)/2$ in every coloring. Therefore, there is error at least $(n-2)(n-1)/2 - \ell(V) = (n-2)(n-1)/2 - (n-2) = (n-2)(n-3)/2$ on $V$ in every coloring. This minimum is reached if and only if the edges $v_1v, \ldots, v_{n-2}v$ have the colors $1, \ldots, n-2$ (in some order). Similarly, there is error at least $(n-1)n/2 - (n-1) = (n-1)(n-2)/2$ on both $A$ and $B$. Therefore, there is error at least $(n-2)(n-3)/2 + 2(n-1)(n-2)/2$ on the internal vertices in every coloring. However, the error is always strictly greater than that. If the error is exactly $(n-1)(n-2)/2$ on both $A$ and $B$, and there is zero error on $a$ and $b$, then edges $va$ and $vb$ both have to receive color $n$. Thus we can conclude that there is error at least $k_n := (n-2)(n-3)/2 + 2(n-1)(n-2)/2 + 1$ in every coloring.

The coloring $\psi_{n-1}$ is defined as

- $\psi_{n-1}(e) = n-1$,
- $\psi_{n-1}(va) = n$,
- $\psi_{n-1}(vb) = n+1$,
- $\psi_{n-1}(v_i v) = i$ for $1 \leq i \leq n-2$, 

![Figure 6: The gadget $L_n$.](image)
\[\psi_{n-1}(a;b) = i \text{ for } 1 \leq i \leq n - 1, \text{ and}\\
\psi_{n-1}(b;b) = i \text{ for } 1 \leq i \leq n - 1.\]

It can be verified that \(\epsilon_{\psi_{n-1}}(V) = (n-2)(n-3)/2\), \(\epsilon_{\psi_{n-1}}(A) = \epsilon_{\psi_{n-1}}(B) = (n-1)(n-2)/2\), \(\epsilon_{\psi_{n-1}}(a) = \epsilon_{\psi_{n-1}}(v) = 0\), and \(\epsilon_{\psi_{n-1}}(b) = 1\); therefore, the error of \(\psi_{n-1}\) on the internal vertices of \(L_n\) is exactly \(k_n\). Coloring \(\psi_{n+1}\) is the same as coloring \(\psi_{n-1}\), except that

\[\psi_{n+1}(e) = n + 1,\\
\psi_{n+1}(vb) = n - 1, \text{ and}\\
\psi_{n+1}(b-b) = n.\]

This change decreases the error on \(b\) to zero, and increases the error on \(b_{n-1}\) to 1. Therefore, \(\psi_{n+1}\) also has error \(k_n\) on the internal vertices, and this proves Property 4.

To show that Property 3 holds, assume that coloring \(\psi\) has error \(k_n\) on the internal vertices of \(L_n\). As we have observed, \(\epsilon_{\psi}(A \cup \{a\}) = (n-1)(n-2)/2\) implies \(\psi(ua) = n\). Similarly, \(\epsilon_{\psi}(B \cup \{b\}) = (n-1)(n-2)/2\) implies \(\psi(vb) = n\); therefore, at least one of \(A \cup \{a\}\) and \(B \cup \{b\}\) have error strictly greater than \((n-1)(n-2)/2\). Assume, without loss of generality, that \(\epsilon_{\psi}(A \cup \{a\}) > (n-1)(n-2)/2\). In this case, the error of \(\psi\) can be \(k_n\) only if \(\epsilon_{\psi}(B \cup \{b\}) = (n-1)(n-2)/2\), \(\epsilon_{\psi}(V) = (n-2)(n-3)/2\), thus \(v\) has zero error. Therefore, color \(n\) is used by edge \(vb\), and the colors \(1, 2, \ldots, n-2\) are used by the edges \(v_1v, v_2v, \ldots, v_{n-2}v\) (not necessarily in this order). Since there is zero error at \(v\), and \(v\) has degree \(n + 1\), edge \(e\) has a color not greater than \(n + 1\). This can be only \(n - 1\) or \(n + 1\), since the other colors are already used by edges incident to \(v\).

\[\square\]

**Lemma 5.3.** For every odd \(n \geq 1\), there is a tree \(A_n\) and an integer \(a_n\) such that

1. \(A_n\) has one pendant edge \(e\),
2. the internal vertices of \(A_n\) have error at least \(a_n\) in every coloring,
3. if a coloring \(\psi\) has error \(a_n\) on the internal vertices of \(A_n\), then \(\psi(e)\) is odd and \(\psi(e) \leq n\),
4. for every odd \(c\) not greater than \(n\), there is a coloring \(\psi_e\) of \(A_n\) such that \(\psi_e(e) = c\) and it has error \(a_n\) on the internal vertices,
5. \(A_n\) can be constructed in time polynomial in \(n\).

**Proof.** The pendant edge \(e\) of \(A_n\) connects external vertex \(u\) and internal vertex \(v\). Attach the pendant edges of the \((n-1)/2\) trees \(F_2, F_4, \ldots, F_{n-1}\) (Lemma 5.1) to vertex \(v\), let the pendant edges of these trees be \(w_2v, w_4v, \ldots, w_{n-1}v\), respectively (see Figure 7). Similarly, attach the pendant edges of the \((n-1)/2\) trees \(L_2, L_4, \ldots, L_{n-1}\) (Lemma 5.2) to \(v\), let the pendant edges of these trees be \(w_2v, w_4v, \ldots, w_{n-1}v\), respectively. Therefore, the degree of \(v\) in \(A_n\) is \(n\).

Let \(a_n = (f_2 + f_4 + \cdots + f_{n-1}) + (k_2 + k_4 + \cdots + k_{n-1})\). Since \(A_n\) contains a copy of the trees \(F_2, F_4, \ldots, F_{n-1}\) and a copy of the trees \(L_2, L_4, \ldots, L_{n-1}\), it...
Figure 7: The gadget $A_5$.

is clear that every coloring of $A_n$ has at least $a_n$ errors on the internal vertices. Moreover, if a coloring $\psi$ has error $a_n$ on the internal vertices, then $\psi(v_i) = i$ for $i = 2, 4, \ldots, n-1$, and the error of $v$ is zero. This implies that $\psi(e) \leq n$ and not even, as required.

The coloring $\psi_c$ required by Property 4 is the following. For every $i = 2, 4, \ldots, n-1$, coloring $\psi_c$ colors the edges of the tree $F_i$ in such a way that the pendant edge $v_i v$ receives color $i$, and there is error $f_i$ on the internal vertices of $F_i$; by Lemma 5.1, such a coloring exists. For every $i = 2, 4, \ldots, c-1$, the tree $L_i$ is colored such that the pendant edge $w_i v$ has color $i - 1$, and the error on the internal vertices of $L_i$ is $k_i$. Similarly, for $i = c + 1, \ldots, n-1$, the tree $L_i$ is colored such that the pendant edge $w_i v$ has color $i + 1$, and there is error $k_i$ on the internal vertices of $L_i$. Coloring $\psi_c$ assigns color $c$ to edge $e$, thus every color $1, 2, \ldots, n$ appears on exactly one edge incident to $v$. Therefore, $v$ has zero error, and the error on the internal vertices of $A_n$ is $a_n$.

**Lemma 5.4 (Satisfaction testing gadget).** For odd integers $x_1 < x_2 < x_3$, there is a tree $S_{x_1, x_2, x_3}$ and an integer $s_{x_1, x_2, x_3}$ such that

1. $S_{x_1, x_2, x_3}$ has one pendant edge $e$,
2. the internal vertices of $S_{x_1, x_2, x_3}$ have error at least $s_{x_1, x_2, x_3}$ in every coloring,
3. if a coloring $\psi$ has error $s_{x_1, x_2, x_3}$ on the internal vertices of $S_{x_1, x_2, x_3}$, then $\psi(e) \in \{x_1, x_2, x_3\}$
4. for $i = 1, 2, 3$, there is a coloring $\psi_i$ of $S_{x_1, x_2, x_3}$ such that $\psi_i(e) = x_i$ and it has error $s_{x_1, x_2, x_3}$ on the internal vertices,
5. $S_{x_1, x_2, x_3}$ can be constructed in time polynomial in $x_3$.

**Proof.** The pendant edge $e$ of $S_{x_1, x_2, x_3}$ connects external vertex $u$ and internal vertex $v$. Attach to vertex $v$ the pendant edges of

- $x_1 - 1$ trees $F_1, F_2, \ldots, F_{x_1-1}$ (Lemma 5.1),
• \(x_2 - x_1 - 1\) trees \(F_{x_1+1}, \ldots, F_{x_2-1}\),
• \(x_3 - x_2 - 1\) trees \(F_{x_2+1}, \ldots, F_{x_3-1}\), and
• 2 copies of the tree \(A_{x_3}\) (Lemma 5.3).

Vertex \(v\) has degree \(x_3\) in \(S_{x_1,x_2,x_3}\). Set \(s_{x_1,x_2,x_3} := f_1 + f_2 + \cdots + f_{x_1-1} + f_{x_1+1} + \cdots + f_{x_2-1} + f_{x_2+1} + \cdots + f_{x_3-1} + 2a_{x_3}\). Because of the way \(S_{x_1,x_2,x_3}\) is constructed, it is clear that every coloring of \(S_{x_1,x_2,x_3}\) has error at least \(s_{x_1,x_2,x_3}\) on the internal vertices. If \(\psi\) has error exactly \(s_{x_1,x_2,x_3}\) on the internal vertices, then \(v\) has zero error and \(\psi(e) \leq d(v) = x_3\). Furthermore, it also follows that the colors \(1, \ldots, x_1-1, x_1+1, \ldots, x_2-1, x_2+1, \ldots, x_3-1\) are used at \(v\) by the pendant edges of the attached trees \(F_1, \ldots, F_{x_1-1}, F_{x_1+1}, \ldots, F_{x_2-1}, F_{x_2+1}, \ldots, F_{x_3-1}\), respectively. Therefore, edge \(e\) has one of the remaining colors \(x_1, x_2, x_3\), proving Property 3.

The colorings \(\psi_1, \psi_2, \psi_3\) required by Property 4 color the \((x_1 - 1) + (x_2 - x_1 - 1) + (x_3 - x_2 - 1)\) trees of type \(F_i\) in the same way: all three colorings color these trees such that there is error \(f_1 + f_2 + \cdots + f_{x_1-1} + f_{x_1+1} + \cdots + f_{x_2-1} + f_{x_2+1} + \cdots + f_{x_3-1}\) on the internal vertices of the trees, and their pendant edges use the colors \(1, \ldots, x_1-1, x_1+1, \ldots, x_2-1, x_2+1, \ldots, x_3-1\) at \(v\), respectively. Coloring \(\psi_i\) assigns color \(x_i\) to the pendant edge \(e\), hence two colors not greater than \(x_3\) remains unused at \(v\): only the colors \(\{x_1, x_2, x_3\} \setminus x_i\) are not yet assigned. These two colors are odd and not greater than \(x_3\), thus by Property 4 of Lemma 5.3, we can color the two copies of \(A_{x_3}\) attached to \(v\) such that their pendant edges have these two colors, and the additional error that we introduce is \(2a_{x_3}\). Since there is zero error on \(v\), the error of this coloring is exactly \(s_{x_1,x_2,x_3}\) on the internal vertices of \(S_{x_1,x_2,x_3}\), as required by Property 4. \(\square\)

**Lemma 5.5 (Variable setting gadget).** For every \(n \geq 0\), there is a partial 2-tree \(W_n\) and an integer \(w_n\) such that

1. \(W_n\) has an external vertex \(v\), and two edges \(e_1\) and \(e_2\) incident to \(v\),
2. every coloring of \(W_n\) has error at least \(w_n\) on the internal vertices of \(W_n\),
3. if a coloring \(\psi\) of \(W_n\) has error \(w_n\) on the internal vertices, then either
   - \(\psi(e_1) = n + 1\), \(\psi(e_2) = n + 3\) or
   - \(\psi(e_1) = n + 5\), \(\psi(e_2) = n + 7\) holds,
4. there are colorings \(\psi_1\) and \(\psi_2\) of \(W_n\) with error \(w_n\) on the internal vertices such that
   - \(\psi_1(e_1) = n + 1\), \(\psi_1(e_2) = n + 3\),
   - \(\psi_2(e_1) = n + 5\), \(\psi_2(e_2) = n + 7\), and
5. \(W_n\) can be constructed in time polynomial in \(n\).

**Proof.** The graph \(W_n\) is constructed as follows (see Figure 8 for the case \(n = 0\)). The external vertex \(v\) is connected to vertex \(v_1\) by edge \(e_1\), and to \(v_2\) by \(e_2\). Vertices \(v_1\) and \(v_2\) are connected by an edge \(e\). We attach several trees to vertices \(v_1\) and \(v_2\):

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Figure 8: The variable setting gadget $W_0$.

- Attach $n$ trees $F_1, F_2, \ldots, F_n$ to $v_1$, let the pendant edges of these trees be $z_1^1v_1, z_2^1v_1, \ldots, z_n^1v_1$, respectively.
- Similarly, attach a copy of these $n$ trees to $v_2$, let the pendant edges be $z_1^2v_2, z_2^2v_2, \ldots, z_n^2v_2$.
- Attach to $v_1$ the trees $F_{n+2}, F_{n+3}, F_{n+4}, F_{n+6}$ with pendant edges $z_1^{n+2}v_1, z_1^{n+4}v_1, z_1^{n+6}v_1$, respectively.
- Attach to $v_1$ a tree $L_{n+6}$ with pendant edge $u_1v_1$.
- Attach to $v_2$ the trees $F_{n+2}, F_{n+4}, F_{n+5}, F_{n+6}$ with pendant edges $z_2^{n+2}v_2, z_2^{n+5}v_2, z_2^{n+6}v_2$, respectively.
- Attach to $v_2$ a tree $L_{n+2}$ with pendant edge $u_2v_2$.

Notice that both $v_1$ and $v_2$ have degree $n + 7$. The graph $W_n$ is a partial 2-tree: it is chordal, and it has clique number 3.

Set $w_n := 2(f_1 + f_2 + \cdots + f_n) + (f_{n+2} + f_{n+3} + f_{n+4} + f_{n+6} + k_{n+6}) + (f_{n+2} + f_{n+4} + f_{n+5} + f_{n+6} + k_{n+2})$. It is clear that every coloring of $W_n$ has error at least $w_n$ on the internal vertices: the combined error in the attached trees is always at least $w_n$. Moreover, if the error of coloring $\psi$ is $w_n$ on the internal vertices, then there has to be zero error on $v_1$ and $v_2$. Furthermore, from Lemma 5.1 and Lemma 5.2, in this case we also have that

- $\psi(z_i^1v_1) = \psi(z_i^2v_2) = i$ for $i = 1, 2, \ldots, n$;
- $\psi(z_i^1v_1) = \psi(z_i^2v_2) = i$ for $i = n + 2, n + 4, n + 6$;
- $\psi(z_i^{n+3}v_1) = n + 3$;
- $\psi(z_i^{n+5}v_2) = n + 5$;
- $\psi(u_1v_1)$ is either $n + 5$ or $n + 7$, and
- $\psi(u_2v_2)$ is either $n + 1$ or $n + 3$.  

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Since the degree of \( v_1 \) is \( n + 7 \) and there is zero error on \( v_1 \), it follows that \( \psi(e) \leq n + 7 \). Moreover, \( \psi(e) \) is either \( n + 1 \) or \( n + 7 \): as shown above, every other color not greater than \( n + 7 \) is already used on at least one of \( v_1 \) or \( v_2 \). Assume first that \( \psi(e) = n + 1 \). In this case \( v_2v_2 \) cannot have color \( n + 1 \); therefore, \( \psi(v_2v_2) = n + 3 \) follows. Now the only unused color not greater than \( n + 7 \) at \( v_2 \) is \( n + 7 \), hence \( \psi(e_2) = n + 7 \). There remains two unused colors at \( v_1 \): color \( n + 5 \) and color \( n + 7 \). However, edge \( e_1 \) cannot have color \( n + 7 \), since edge \( e_2 \) already has this color. Thus we have \( \psi(e_1) = n + 5 \) and \( \psi(e_2) = n + 7 \), required by Property 4. Similarly, assume that \( \psi(e) = n + 7 \), it follows that \( \psi(u_1v_1) = n + 5 \). The only unused color not greater than \( n + 7 \) at \( v_1 \) is \( n + 1 \), hence edge \( e_1 \) has to receive this color. Colors \( n + 3 \) and \( n + 1 \) are the only remaining colors at \( v_2 \); therefore, \( e_2 \) has color \( n + 3 \), since \( n + 1 \) is already used by \( e_1 \). Thus we have \( \psi(e_1) = n + 1 \) and \( \psi(e_2) = n + 3 \), as required.

The two colorings \( \psi_1 \) and \( \psi_2 \) required by Property 4 are given as follows (see Figure 8 for the case \( n = 0 \)). Consider the (partial) coloring \( \psi \) with

- \( \psi(z^1_1v_1) = \psi(z^2_1v_2) = i \) for \( i = 1, 2, \ldots, n \),
- \( \psi(z^1_1v_1) = \psi(z^2_1v_2) = i \) for \( i = n + 2, n + 4, n + 6 \),
- \( \psi(z^1_{n+3}v_1) = n + 3 \) and
- \( \psi(z^2_{n+5}v_2) = n + 5 \).

Both \( \psi_1 \) and \( \psi_2 \) assign the same colors as \( \psi \), but we also have

- \( \psi_1(e_1) = n + 1, \psi_1(e_2) = n + 3, \psi_1(e) = n + 7 \),
- \( \psi_1(u_1v_1) = n + 5 \),
- \( \psi_1(u_2v_2) = n + 1 \).
- \( \psi_2(e_1) = n + 5, \psi_2(e_2) = n + 7, \psi_2(e) = n + 1 \),
- \( \psi_2(u_1v_1) = n + 7 \),
- \( \psi_2(u_2v_2) = n + 3 \).

In these colorings vertices \( v_1 \) and \( v_2 \) have zero error. Furthermore, these colorings can be extended to the attached trees with error \( w_n \): the colors assigned to the pendant edges of the attached trees are compatible with the “best” coloring of the attached trees (see Property 4 of Lemma 5.2 and Property 3 of Lemma 5.1). This gives Property 4 of the lemma being proved.

**Theorem 5.6.** Minimum Sum Edge Coloring is NP-hard for partial 2-trees.

**Proof.** The proof is by reduction from 3-SAT: given a 3-CNF formula \( \varphi \), we construct a partial 2-tree \( G \) and determine an integer \( K \) such that \( \Sigma'(G) \leq K \) if and only if \( \varphi \) is satisfiable.

We assume that every variable occurs exactly twice positively and exactly twice negated in \( \phi \). This can be achieved as follows. It is well-known that 3-SAT remains NP-complete if every variable occurs exactly twice positively, exactly once negated, and every clause contains two or three literals. Let us assume that the number of variables is even, if not, then duplicate every variable and every clause. Let \( x_1, x_2, \ldots, x_n \) be the variables of \( \phi \). We add \( n/2 \) new variables
graphs. as well: joining graphs at a single vertex does not increase the tree width of the graph. Now every variable occurs exactly twice positively and twice negatively. These new clauses are satisfied in every variable assignment, hence the new formula is satisfiable if and only if the original is satisfiable. Furthermore, if there is a clause $ (x \lor y) $ containing only two literals, then add a new variable $ z $, and replace this clause with $ (x \lor z \lor \bar{z}) \land (\bar{z} \lor \bar{z} \lor y) $. It is easy to see that this transformation does not change the satisfiability of the formula.

Let $ x_0, x_1, \ldots, x_{n-1} $ be the $ n $ variables of $ \varphi $. The number of clauses is therefore $ m = 4n/3 $. For every literal of $ \varphi $, there is a corresponding color, as follows:

- color $ 8i + 1 $ corresponds to the first positive occurrence of $ x_i $,
- color $ 8i + 3 $ corresponds to the second positive occurrence of $ x_i $,
- color $ 8i + 5 $ corresponds to the first negated occurrence of $ x_i $, and
- color $ 8i + 7 $ corresponds to the second negated occurrence of $ x_i $.

Notice that these numbers are odd, and every odd number not greater than $ 8n $ corresponds to a literal.

Take a vertex $ v $, we will attach several gadgets to $ v $ to obtain the graph $ G $. Attach $ 4n $ trees $ F_2, F_4, \ldots, F_{8n} $ to $ v $, let the pendant edges of the attached trees be $ u_{i}v, u_{i+1}v, \ldots, u_{8n}v $, respectively. Attach $ n $ variable setting gadgets $ W_{0}, W_{8}, W_{16}, \ldots, W_{8(n-1)} $ to $ v $, let the two edges of $ W_{i} $ incident to $ v $ be called $ w_{i}1v $ and $ w_{i}2v $. For every clause $ C_j $ of $ \varphi $, we attach a satisfaction testing gadget to $ v $ in the following way: if colors $ c_{j,1} < c_{j,2} < c_{j,3} $ correspond to the three literals in clause $ C_j $, then attach a tree $ S_{c_{j,1},c_{j,2},c_{j,3}} $ to $ v $, and let $ s_{i}v $ be its pendant edge. Finally, attach $ m/2 $ copies of the tree $ A_{8n-1} $ to $ v $, let the pendant edges of these trees be $ t_{1}v, t_{2}v, \ldots, t_{16}v $. This completes the description of the graph $ G $. Since every gadget is a partial 2-tree (or even a tree), the graph $ G $ is a partial 2-tree as well: joining graphs at a single vertex does not increase the tree width of the graphs.

Let $ K^{(1)} := f_2 + f_4 + \cdots + f_{8n} $. In every coloring of $ G $ the error is at least $ K^{(1)} $ on the internal vertices of the $ 4n $ trees $ F_i $ attached to $ v $. Let $ K^{(2)} := w_0 + w_4 + \cdots + w_{8(n-1)} $. In every coloring the error is at least $ K^{(2)} $ on the internal vertices of the $ n $ variable setting gadgets. Let $ K^{(3)} := m/2 \cdot a_{m-1} $. In every coloring the error is at least $ K^{(3)} $ on the internal vertices of the $ m/2 $ copies of $ A_{8n-1} $. Let $ K^{(4)} := \sum_{j=1}^{m} s_{c_{j,1},c_{j,2},c_{j,3}} $ where $ c_{j,k} $ is the color corresponding to the $ k $-th literal in clause $ C_j $. In every coloring of $ G $, the error on the internal vertices of the $ m $ satisfaction testing gadgets is at least $ K^{(4)} $. Finally, set $ K := K^{(1)} + K^{(2)} + K^{(3)} + K^{(4)} $. It is clear that every coloring of $ G $ has error at least $ K $. We claim that $ G $ has a coloring with error exactly $ K $ if and only if $ \varphi $ is satisfiable.

Assume first that coloring $ \psi $ has error $ K $. This is possible only if $ \psi $ has zero error on $ v $, and the error is exactly $ K $ on the internal vertices of the attached gadgets. By Lemmas 5.1, 5.3, 5.4, and 5.5, this implies that

- $ \psi(u_{i}v) = i $ for $ i = 2, 4, \ldots, 8n $,
- for every $ i = 0, 1, \ldots, n-1 $, either
\[ \psi(w_i v) = 8i + 1 \text{ and } \psi(w_i v) = 8i + 3, \text{ or} \]
\[ \psi(w_i v) = 8i + 5 \text{ and } \psi(w_i v) = 8i + 7, \]
\bullet \psi(s_i v) \in \{c_{j,1}, c_{j,2}, c_{j,3}\} \text{ for every } j = 1, \ldots, m, \text{ and} \]
\bullet \psi(t_i v) \leq 8n - 1 \text{ and odd for every } i = 1, 2, \ldots, m/2.

Consider the following variable assignment: set variable \( x_i \) to true if \( \psi(w_{i,1} v) = 8i + 5 \) and \( \psi(w_{i,2} v) = 8i + 7 \), and set \( x_i \) to false if \( \psi(w_{i,1} v) = 8i + 1 \) and \( \psi(w_{i,2} v) = 8i + 3 \). We show that this is a satisfying assignment of \( \varphi \), i.e., every clause \( C_j \) is satisfied. Assume that \( \psi(s_j v) = c_{j,k} \) for some \( k = 1, 2, 3 \), and let the \( k \)-th literal in clause \( C_j \) be an occurrence of the variable \( x_i \). In this case, the \( k \)-th literal of clause \( C_j \) is true in the constructed variable assignment: otherwise color \( c_{j,w} \) would appear also on edge \( w_{i,1} v \) or \( w_{i,2} v \). Therefore, every clause contains at least one true literal, and the formula is satisfied by the variable assignment.

Now assume that \( \varphi \) has a satisfying variable assignment. Consider the following (partial) coloring \( \psi \):
\bullet \psi(u_i v) = i \text{ for } i = 2, 4, \ldots, 8n,
\bullet \text{ for every } i = 0, 1, \ldots, n - 1,
\quad \text{if variable } x_i \text{ is true, then } \psi(w_{i,1} v) = 8i + 5 \text{ and } \psi(w_{i,2} v) = 8i + 7,
\quad \text{if variable } x_i \text{ false, then } \psi(w_{i,1} v) = 8i + 1 \text{ and } \psi(w_{i,2} v) = 8i + 3,

It is clear from the construction that for every \( j = 1, 2, \ldots, m \), one of the colors \( c_{j,1}, c_{j,2}, c_{j,3} \) is not already assigned: otherwise this would imply that clause \( C_j \) contains only false literals in the satisfying variable assignment, a contradiction. Therefore, we can set \( \psi(s_j v) \) to one of these three colors. So far coloring \( \psi \) assigns \( 4n \) even and \( 2n + m \) odd colors to the edges incident to \( v \), thus there remains exactly \( m/2 \) odd colors not greater than \( 8n \). Assign these colors to the edges \( t_1 v, t_2 v, \ldots, t_m v \) in some order. Now every color not greater than \( 8n \) is used exactly once at \( v \), hence there is zero error on vertex \( v \) in \( \psi \). It is straightforward to verify that this coloring can be extended to the whole graph \( G \) such that the resulting coloring has error exactly \( K \): in every gadget, the edges incident to \( v \) are colored in such a way that makes this extension possible.

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