Chance-constrained games: A mathematical programming approach

Vikas Vikram Singh\textsuperscript{a}, Oualid Jouini\textsuperscript{b}, Abdel Lisser\textsuperscript{a}

\textsuperscript{a}Laboratoire de Recherche en Informatique, Université Paris Sud XI, Bât 650, 91405, Orsay, France.
\textsuperscript{b}Laboratoire Génie Industriel, Ecole Centrale Paris, Grande Voie des Vignes, 92290, Châtenay-Malabry, France.

Abstract

We consider a two player bimatrix game where the entries of the payoff matrices are random variables. We formulate this problem as a chance-constrained game by considering that the payoff of each player is defined using a chance constraint. We consider the case where the entries of the payoff matrices are independent normal/Cauchy random variables. We show a one-to-one correspondence between a Nash equilibrium of a chance-constrained game corresponding to normal distribution and a global maximum of a certain mathematical program. Further as a special case where the payoffs are independent and identically distributed normal random variables, we show that a strategy pair, where each player’s strategy is a uniform distribution over his action set, is a Nash equilibrium. We show a one-to-one correspondence between a Nash equilibrium of a chance-constrained game corresponding to Cauchy distribution and a global maximum of a certain quadratic program.

Keywords: Chance-constrained game, Nash equilibrium, Normal distribution, Cauchy distribution, Mathematical program, Quadratic program.

1. Introduction

It is well known that there exists a mixed strategy saddle point equilibrium for a two player zero sum matrix game (see Neumann (1928)).

\textit{Email addresses: vikas.singh@lri.fr (Vikas Vikram Singh), oualid.jouini@ecp.fr (Oualid Jouini), abdel.lisser@lri.fr (Abdel Lisser)}

Preprint submitted to Discrete Applied Mathematics December 30, 2015
Nash (1950) showed the existence of a mixed strategy equilibrium for the
games where there are finite number of players and each player has finite
number of actions. Later such equilibrium was called Nash equilibrium. For
two player case the game can be described using two payoff matrices (one
for each player), and it is called a bimatrix game. Mangasarian and Stone
(1964) showed a one-to-one correspondence between a Nash equilibrium of a
bimatrix game and a global maximum of a certain quadratic program, i.e., a
Nash equilibrium of a bimatrix game can be computed by solving a quadratic
program. A bimatrix game problem can also be equivalently written as a
linear complementarity problem. Lemke and Howson (1964) proposed an
algorithm, based on solving an equivalent linear complementarity problem,
to compute a Nash equilibrium of a bimatrix game.

Both Nash (1950), and Neumann (1928) considered the games where
the payoffs of the players are exact real values. In some cases the payoffs
of the players may be within certain ranges. Using fuzzy theory, Collins
and Hu (2008) modeled such a situation as an interval valued matrix game.
The computational algorithms to solve an interval valued matrix game are
available in the literature (see Deng-Feng Li and Zhang (2012), Li (2011),
Mitchell et al. (2014)). However, in many situations the payoffs are random
variables due to uncertainty which arises from various external factors. The
wholesale electricity markets are the good examples (see Mazadi et al. (2013),
Couchman et al. (2005), Valenzuela and Mazumdar (2007), Wolf and Smeers
(1997)). One way to handle this type of game is by taking the expectation
of random payoffs and consider the corresponding deterministic game (see
Valenzuela and Mazumdar (2007), Wolf and Smeers (1997)). Some recent
papers on the games with random payoffs using expected payoff criterion
include Ravat and Shanbhag (2011), Xu and Zhang (2013), Jadamba and

The expected payoff criterion captures the situation where the deci-
sion makers are risk neutral. The risk averse situation can be handled by
chance constraint programming where the payoff of a player is defined using
a chance constraint. For more details about chance constraint program-
ming (see Charnes and Cooper (1963), Cheng and Liss (2012), Cheng and
Lisser (2013), Prékopa (1995)). There are few papers on zero sum chance-
constrained games available in the literature (see Blau (1974), Cassidy et al.
(1972), Charnes et al. (1968), Song (1992)). Recently, a chance-constrained
game with finite number of players is considered in Singh et al. (2015a), Singh
et al. (2015b), where authors showed the existence of a mixed strategy Nash
equilibrium. Singh et al. (2015a) considered the case where the components of the payoff vector of each player are independent normal/Cauchy random variables, and they also consider the case where the payoff vector of each player follow a multivariate elliptically symmetric distribution. Singh et al. (2015b) considered the case where the distribution of the payoff vector of each player is not known completely. They consider a distributionally robust approach to handle these games (see Cheng et al. (2014)). Singh et al. (2016) formulated a chance-constrained game corresponding to normal/Cauchy distribution as an equivalent complementarity problem. In application regimes there are few papers that consider chance-constrained game models to analyze the situation, e.g., see Mazadi et al. (2013), Couchman et al. (2005). In Mazadi et al. (2013), the randomness in payoffs is due to the installation of wind generators in the electricity market. They consider the case of independent normal random variables. Later, for better representation and ease in computation the authors considered, in detail, the case where only one wind generator is installed in the electricity market. In Couchman et al. (2005), the payoffs are random due to consumers’ random demand which is assumed to be normally distributed.

In this paper, we consider a two player bimatrix game where the entries of the payoff matrices are independent normal/Cauchy random variables. For the case of normal distribution, we show a one-to-one correspondence between a Nash equilibrium of a chance-constrained game and a global maximum of a certain mathematical program. Further we consider a special case where the entries of the payoff matrices are also identically distributed. We show that a strategy pair, where each player’s strategy is a uniform distribution over his action set, is a Nash equilibrium. For the case of Cauchy distribution, we show a one-to-one correspondence between a Nash equilibrium of a chance-constrained game and a global maximum of a certain quadratic program. Further if the entries of the payoff matrices are also identically distributed, all the strategy pairs are Nash equilibrium.

Now, we describe the structure of rest of the paper. Section 2 contains the definition of a chance-constrained game. Section 3 presents a mathematical programming formulation for chance-constrained game. Section 4 shows numerical results of few instances of random bimatrix game. We conclude the paper in Section 5.
2. The Model

We consider a two player bimatrix game where the entries of the payoff matrix of each player are random variables following certain distributions. Let $I = \{1, 2, \cdots, m\}$, and $J = \{1, 2, \cdots, n\}$ be the sets of actions of player 1 and player 2 respectively. Let $A^w = [a^w_{ij}]$ and $B^w = [b^w_{ij}]$ be the $m \times n$ random payoff matrices of player 1 and player 2 respectively; $w$ denotes some uncertainty parameter. That is, if player 1 chooses an action $i$ and player 2 chooses an action $j$, the payoffs of player 1 and player 2 are given by random variables $a^w_{ij}$ and $b^w_{ij}$ respectively. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then, for each $i \in I$, $j \in J$, $a^w_{ij} : \Omega \rightarrow \mathbb{R}$, and $b^w_{ij} : \Omega \rightarrow \mathbb{R}$. The sets $I$ and $J$ are also called the sets of pure strategies of player 1 and player 2 respectively. A mixed strategy of a player is a probability distribution over his actions. Let $X = \{x \in \mathbb{R}^m | \sum_{i \in I} x_i = 1, x_i \geq 0, \forall i \in I\}$ and $Y = \{y \in \mathbb{R}^n | \sum_{j \in J} y_j = 1, y_j \geq 0, \forall j \in J\}$ be the sets of mixed strategies of player 1 and player 2 respectively. For each $(x, y) \in X \times Y$, the payoffs $x^T A^w y$ and $x^T B^w y$ of player 1 and player 2 respectively are random variables. At strategy pair $(x, y)$, each player is interested in the highest level of his payoff that he can attain with at least a specified level of confidence. The confidence level of each player is given a priori. We assume that the confidence level of one player is known to another player. Let $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, 1]$ be the confidence levels of player 1 and player 2 respectively; $\alpha_1$ and $\alpha_2$ also represent the risk levels of player 1 and player 2 respectively. Let $\alpha = (\alpha_1, \alpha_2)$ be a confidence (risk) level vector. For a given strategy pair $(x, y)$ and a given $\alpha$, the payoff of player 1 is given by

$$u_1^{\alpha_1}(x, y) = \sup \{ u | P(x^T A^w y \geq u) \geq \alpha_1 \},$$

(1)

and the payoff of player 2 is given by

$$u_2^{\alpha_2}(x, y) = \sup \{ v | P(x^T B^w y \geq v) \geq \alpha_2 \}.$$  

(2)

We call this game a chance-constrained game because the payoff of each player is defined using a chance constraint. We assume that the probability distributions of the entries of the payoff matrix of one player are known to another player. Then, for a given $\alpha \in [0, 1]^2$ the payoffs of one player defined above are known to another player. That is, a chance-constrained game is a non-cooperative game with complete information. For a given $\alpha$, the set of
best response strategies of player 1 against the fixed strategy $y$ of player 2 is given by

$$BR^{\alpha_1}(y) = \{ \bar{x} \in X | u^{\alpha_1}_1(\bar{x}, y) \geq u^{\alpha_1}_1(x, y), \forall x \in X \},$$

and the set of best response strategies of player 2 against the fixed strategy $x$ of player 1 is given by

$$BR^{\alpha_2}(x) = \{ \bar{y} \in Y | u^{\alpha_2}_2(x, \bar{y}) \geq u^{\alpha_2}_2(x, y), \forall y \in Y \}.$$

**Definition 2.1** (Nash equilibrium). For a given $\alpha \in [0,1]^2$, a strategy pair $(x^*, y^*)$ is said to be a Nash equilibrium of a chance-constrained game if the following inequalities hold:

$$u^{\alpha_1}_1(x^*, y^*) \geq u^{\alpha_1}_1(x, y^*), \forall x \in X,$$

$$u^{\alpha_2}_2(x^*, y^*) \geq u^{\alpha_2}_2(x^*, y), \forall y \in Y.$$

That is, a strategy pair $(x^*, y^*)$ is a Nash equilibrium if and only if $x^* \in BR^{\alpha_1}(y^*)$ and $y^* \in BR^{\alpha_2}(x^*)$.

**3. Mathematical programming formulation**

We consider the case where the entries of the payoff matrices $A^w$ and $B^w$ are independent random variables following certain distributions. Then, for a given strategy pair $(x, y)$, the payoff $x^T A^w y$ (resp. $x^T B^w y$) of player 1 (resp. player 2) can be viewed as a linear combination of the independent random variables. We consider the probability distributions that are closed under a linear combination of the independent random variables. That is, if $Y_1, Y_2, \ldots, Y_k$ are independent random variables following the same distribution (possibly with different parameters), for any $b \in \mathbb{R}^k$, the distribution of $\sum_{i=1}^k b_i Y_i$ is the same as $Y_i$ up to parameters. The normal and Cauchy distributions satisfy this property (see Johnson et al. [1994]). We show that a Nash equilibrium of a chance-constrained game corresponding to normal distribution has a one-to-one correspondence with a global maximum of a certain mathematical program, and for the case of Cauchy distribution we show that a Nash equilibrium of a chance-constrained game has a one-to-one correspondence with a global maximum of a certain quadratic program.
3.1. Payoffs following normal distribution

We assume that all the entries of payoff matrix \( A^w \) are independent normal random variables, and all the entries of payoff matrix \( B^w \) are independent normal random variables. For each \( i \in I, j \in J \), let \( a^w_{ij} \) follows a normal distribution with mean \( \mu_{1,ij} \) and variance \( \sigma_{1,ij}^2 \), and \( b^w_{ij} \) follows a normal distribution with mean \( \mu_{2,ij} \) and variance \( \sigma_{2,ij}^2 \). A linear combination of independent normal random variables is a normal random variable. Then, for a given strategy pair \((x, y)\), \( x^T A^w y \) follows a normal distribution with mean \( \mu_1(x, y) = \sum_{i \in I, j \in J} \mu_{1,ij} x_i y_j \) and variance \( \sigma_1^2(x, y) = \sum_{i \in I, j \in J} x_i^2 y_j^2 \sigma_{1,ij}^2 \), and \( x^T B^w y \) follows a normal distribution with mean \( \mu_2(x, y) = \sum_{i \in I, j \in J} \mu_{2,ij} x_i y_j \) and variance \( \sigma_2^2(x, y) = \sum_{i \in I, j \in J} x_i^2 y_j^2 \sigma_{2,ij}^2 \). Then, \( Z_1^N = \frac{x^T A^w y - \mu_1(x, y)}{\sigma_1(x, y)} \) and \( Z_2^N = \frac{x^T B^w y - \mu_2(x, y)}{\sigma_2(x, y)} \) follow a standard normal distribution. Let \( F_{Z_1^N}^{-1}(\cdot) \) and \( F_{Z_2^N}^{-1}(\cdot) \) be the quantile functions of a standard normal distribution. From (1), for a given strategy pair \((x, y)\) and a given confidence level \( \alpha_1 \), the payoff of player 1 is given by

\[
u_1^{\alpha_1}(x, y) = \sup \{ u | P(x^T A^w y \geq u) \geq \alpha_1 \} = \sup \left\{ u | P \left( \frac{x^T A^w y - \mu_1(x, y)}{\sigma_1(x, y)} \leq \frac{u - \mu_1(x, y)}{\sigma_1(x, y)} \right) \leq 1 - \alpha_1 \right\} = \sup \left\{ u | u \leq \mu_1(x, y) + \sigma_1(x, y) F_{Z_1^N}^{-1}(1 - \alpha_1) \right\}.
\]

That is,

\[
u_1^{\alpha_1}(x, y) = \sum_{i \in I, j \in J} \mu_{1,ij} x_i y_j + \left( \sum_{i \in I, j \in J} x_i^2 y_j^2 \sigma_{1,ij}^2 \right)^{1/2} F_{Z_1^N}^{-1}(1 - \alpha_1).
\] (3)

Similarly, from (2), for a given strategy pair \((x, y)\) and a given confidence level \( \alpha_2 \), the payoff of player 2 is given by

\[
u_2^{\alpha_2}(x, y) = \sum_{i \in I, j \in J} \mu_{2,ij} x_i y_j + \left( \sum_{i \in I, j \in J} x_i^2 y_j^2 \sigma_{2,ij}^2 \right)^{1/2} F_{Z_2^N}^{-1}(1 - \alpha_2).
\] (4)

Singh et al. (2015a) showed that there always exists a mixed strategy Nash equilibrium for all \( \alpha \in [0.5, 1]^2 \). At \( \alpha = (1, 1) \), \( F_{Z_1^N}^{-1}(1 - \alpha_1) = F_{Z_2^N}^{-1}(1 - \alpha_2) = -\infty \). Therefore, at \( \alpha = (1, 1) \), the value of the payoff functions defined by (3)
and (4) is $-\infty$ for all $(x, y) \in X \times Y$. That is, at confidence level 1, a player cannot improve his payoff by changing his strategies. Hence, all the strategies of a player would be part of a Nash equilibrium if his confidence level is 1, and all strategy pairs $(x, y)$ would be a Nash equilibrium at $\alpha = (1, 1)$. Therefore, we consider the case where $\alpha \in [0.5, 1]^2$, so that the payoff functions defined by (3) and (4) have finite values.

Let $\mu_1(y) = (\mu_{1,i}(y))_{i \in I}$ be an $m \times 1$ vector, where $\mu_{1,i}(y) = \sum_{j \in J} \mu_{1ij} y_j$, and let $\Sigma_1(y) = \text{diag}(\sum_{j \in J} \sigma_{1,1j}^2 y_j^2, \sum_{j \in J} \sigma_{1,2j}^2 y_j^2, \ldots, \sum_{j \in J} \sigma_{1,mj}^2 y_j^2)$ be an $m \times m$ diagonal matrix. Let $\Sigma_2(x) = \text{diag}(\sum_{i \in I} \sigma_{2,1i}^2 x_i^2, \sum_{i \in I} \sigma_{2,2i}^2 x_i^2, \ldots, \sum_{i \in I} \sigma_{2,mi}^2 x_i^2)$ be an $n \times n$ diagonal matrix. Using these expressions, (3) and (4) can be written as

$$u_1^{\alpha_1}(x, y) = x^T \mu_1(y) + ||\Sigma_1^{1/2}(y)x|| F_{Z_1}^{-1}(1 - \alpha_1), \quad (5)$$

and

$$u_2^{\alpha_2}(x, y) = y^T \mu_2(x) + ||\Sigma_2^{1/2}(y)y|| F_{Z_2}^{-1}(1 - \alpha_2), \quad (6)$$

where $|| \cdot ||$ is the Euclidean norm. For $\alpha_1 \in [0.5, 1)$ and $\alpha_2 \in [0.5, 1)$, $F_{Z_1}^{-1}(1 - \alpha_1) \leq 0$ and $F_{Z_2}^{-1}(1 - \alpha_2) \leq 0$. Therefore, for fixed $y \in Y$ and $\alpha_1 \in [0.5, 1)$, $u_1^{\alpha_1}(\cdot, y)$ defined by (5) is a concave function of $x$, and for fixed $x \in X$ and $\alpha_2 \in [0.5, 1)$, $u_2^{\alpha_2}(x, \cdot)$ defined by (6) is a concave function of $y$.

3.1.1. Best response convex programs

Let $X^+ = \{ x \in \mathbb{R}^m | x_i \geq 0, \forall i \in I \}$. For a fixed strategy $y$ of player 2 and $\alpha_1 \in [0.5, 1)$, the best response strategy of player 1 can be obtained by solving the following convex quadratic program:

$$[P1] \min_x -x^T \mu_1(y) - ||\Sigma_1^{1/2}(y)x|| F_{Z_1}^{-1}(1 - \alpha_1)$$

s.t.

$$(i) \sum_{i \in I} x_i = 1,$$

$$(ii) x_i \geq 0, \ i \in I.$$

The Lagrangian dual problem of $[P1]$ is

$$\max_{\lambda_1 \in \mathbb{R}} \min_{x \in X^+} \left\{ -x^T \mu_1(y) - ||\Sigma_1^{1/2}(y)x|| F_{Z_1}^{-1}(1 - \alpha_1) + \lambda_1 \left( 1 - \sum_{i \in I} x_i \right) \right\}.$$
We have,

\[
\begin{align*}
\min_{x \in X^+} & -x^T \mu_1(y) - \|\Sigma_1^{1/2}(y)x\|_{F^{-1/2}Z_N^N}^2 (1 - \alpha_1) + \lambda_1 \left(1 - \sum_{i \in I} x_i\right) \\
= & \min_{x \in X^+} \max_{\|u_1\| \leq 1} \left(-x^T \mu_1(y) - \left(\Sigma_1^{1/2}(y)x\right)^T u_1 F_{Z_N^N}^{-1} (1 - \alpha_1) + \lambda_1 - \lambda_1 \sum_{i \in I} x_i\right) \\
= & \max_{\|u_1\| \leq 1} \min_{x \in X^+} x^T \left(-\mu_1(y) - \Sigma_1^{1/2}(y)u_1 F_{Z_N^N}^{-1} (1 - \alpha_1) - \lambda_1 1_m\right) + \lambda_1,
\end{align*}
\]

where $1_m$ is an $m \times 1$ vector of all 1’s. The first equality follows from Cauchy-Schwartz inequality, and the last equality follows from [Rockafellar (1970), Corollary 37.3.2]. The minimum in the last equality is $-\infty$, unless

\[\lambda_1 1_m \leq -\mu_1(y) - \Sigma_1^{1/2}(y)u_1 F_{Z_N^N}^{-1} (1 - \alpha_1).\]

Then, the Lagrangian dual problem of $[P1]$ is

\[
\begin{align*}
[D1] \quad & \max_{\lambda_1, u_1} \lambda_1 \\
\text{s.t.} & \\
(i) & \lambda_1 1_m \leq -\mu_1(y) - \Sigma_1^{1/2}(y)u_1 F_{Z_N^N}^{-1} (1 - \alpha_1), \\
(ii) & \|u_1\| \leq 1.
\end{align*}
\]

Similarly, for fixed $x \in X$ and $\alpha_2 \in [0.5, 1)$, a best response strategy of player 2 can be obtained by solving the following convex quadratic program:

\[
\begin{align*}
[P2] \quad & \min_y -y^T \mu_2(x) - \|\Sigma_2^{1/2}(x)y\|_{F^{-1/2}Z_N^N}^2 (1 - \alpha_2) \\
\text{s.t.} & \\
(i) & \sum_{j \in J} y_j = 1, \\
(ii) & y_j \geq 0, \ j \in J.
\end{align*}
\]

From the similar arguments used above, the dual of $[P2]$ is
\[\text{[D2]} \quad \max_{\lambda_2, u_2} \lambda_2 \]
\[
\text{s.t.}
\begin{align*}
(i) \quad \lambda_2 1_m & \leq -\mu_2(x) - \Sigma_2^{1/2}(x) u_2 F_{Z_2}^{-1}(1 - \alpha_2), \\
(ii) \quad ||u_2|| & \leq 1.
\end{align*}
\]

3.1.2. Mathematical program

We denote the decision variables and the objective function of the mathematical program [MP] by \(\zeta = (\lambda_1, \lambda_2, u_1, u_2, x, y)\) and \(\psi(\cdot)\) respectively. By using the best response convex programs [P1], [D1], [P2], [D2] we have the following result.

**Theorem 3.1.** Consider a bimatrix game \((A^w, B^w)\), where all the entries of matrix \(A^w\) be independent normal random variables, and all the entries of matrix \(B^w\) also be independent normal random variables. For all \(i \in I, j \in J\), the mean and variance of \(a^w_{ij}\) are \(\mu_{1,ij}\) and \(\sigma_{1,ij}^2\) respectively, and the mean and variance of \(b^w_{ij}\) are \(\mu_{2,ij}\) and \(\sigma_{2,ij}^2\) respectively. Consider a point \(\zeta^* = (\lambda_1^*, \lambda_2^*, u_1^*, u_2^*, x^*, y^*)\). Then, for a given \(\alpha \in [0.5, 1)^2\), a strategy part \((x^*, y^*)\) of \(\zeta^*\) is a Nash equilibrium of a chance-constrained game if and only if it is a global maximum of the mathematical program [MP] given below

\[
\text{[MP]} \quad \max_\zeta \left[ \left( \lambda_1 + x^T \mu_1(y) + ||\Sigma_1^{1/2}(y)x||F_{Z_1}^{-1}(1 - \alpha_1) \right) \\
+ \left( \lambda_2 + y^T \mu_2(x) + ||\Sigma_2^{1/2}(x)y||F_{Z_2}^{-1}(1 - \alpha_2) \right) \right]
\]
\[
\text{s.t.}
\begin{align*}
(i) \quad \lambda_1 1_m & \leq -\mu_1(y) - \Sigma_1^{1/2}(y) u_1 F_{Z_1}^{-1}(1 - \alpha_1), \\
(ii) \quad \lambda_2 1_n & \leq -\mu_2(x) - \Sigma_2^{1/2}(x) u_2 F_{Z_2}^{-1}(1 - \alpha_2), \\
(iii) \quad ||u_1|| & \leq 1, \\
(iv) \quad ||u_2|| & \leq 1, \\
(v) \quad \sum_{i \in I} x_i = 1, \\
(vi) \quad \sum_{j \in J} y_j = 1,
\end{align*}
\]
(vii) $x_i \geq 0$, $\forall i \in I$,
(viii) $y_j \geq 0$, $\forall j \in J$,

with objective function value $\psi(\zeta^*) = 0$.

Proof. Fix $\alpha \in [0.5, 1)^2$. Let $(x^*, y^*)$ be a Nash equilibrium of a chance-constrained game. The constraints (v)-(viii) are satisfied by $(x^*, y^*)$ because $x^*$ and $y^*$ are probability distributions. For fixed $y^*$, $x^*$ is a best response of player 1, i.e., it is an optimal solution of [P1]. The convex quadratic program [P1] satisfies all the conditions of strong duality Theorem 6.2.4 of Bazaraa et al. (2006). Hence, there exists an optimal solution $(u_1^*, \lambda_1^*)$ of dual program [D1] such that the constraints (i) and (iii) are satisfied at $(u_1^*, \lambda_1^*, y^*)$ and the objective values of both [P1] and [D1] are same, i.e.,

$$
\lambda_1^* = -x^T \mu_1(y^*) - ||\Sigma_1^{1/2}(y^*)x^*|| F_{Z_1}^{-1}(1 - \alpha_1).
$$

Similarly, for fixed $x^*$, $y^*$ is an optimal solution of [P2]. Then, there exists an optimal solution $(u_2^*, \lambda_2^*)$ of [D2] such that the constraints (ii) and (iv) are satisfied at $(u_2^*, \lambda_2^*, x^*)$, and

$$
\lambda_2^* = -y^T \mu_2(x^*) - ||\Sigma_2^{1/2}(x^*)y^*|| F_{Z_2}^{-1}(1 - \alpha_2).
$$

That is, $\zeta^*$ is a feasible point of [MP] and from the construction of the objective function we have $\psi(\zeta^*) = 0$. Next, we show that $\zeta^*$ is a global maximum of [MP]. Let $\zeta$ be a feasible point of [MP]. Multiply constraint (i) by $x$ and by using Cauchy-Schwartz inequality and constraints (iii), (v), (viii), we have

$$
\lambda_1 \leq -x^T \mu_1(y) - ||\Sigma_1^{1/2}(y)x|| F_{Z_1}^{-1}(1 - \alpha_1). \tag{7}
$$

Similarly,

$$
\lambda_2 \leq -y^T \mu_2(x) - ||\Sigma_2^{1/2}(x)y|| F_{Z_2}^{-1}(1 - \alpha_2). \tag{8}
$$

From (7) and (8), $\psi(\zeta) \leq 0$ for all feasible point of $\zeta$ of [MP]. Therefore, $\zeta^*$ is a global maximum of [MP].

Let $\zeta^*$ be a global maximum of [MP] with $\psi(\zeta^*) = 0$. Since, $\zeta^*$ is a feasible point of [MP], then $\zeta^*$ will satisfy (7) and (8). That is, each term of the objective function is non-positive at $\zeta^*$. This implies that, (7) and (8) are equalities at $\zeta^*$. Fix $\zeta^*$, and then multiply (i) by $x$ and by using Cauchy-Schwartz inequality, we have

10
\[ \lambda^*_i \leq -x^T \mu_1(y^*) - ||\Sigma_1^{1/2} (y^*) x|| F_{Z_1}^{-1} (1 - \alpha_1), \; \forall \; x \in X. \]

Using the fact that (7) is an equality at \( \zeta^* \), we have

\[ u_1^{\alpha_1}(x^*, y^*) \geq u_1^{\alpha_1}(x, y^*), \; \forall \; x \in X. \]  

(9)

Similarly,

\[ u_2^{\alpha_2}(x^*, y^*) \geq u_2^{\alpha_2}(x^*, y), \; \forall \; y \in Y. \]  

(10)

From (9) and (10), \((x^*, y^*)\) is a Nash equilibrium of a chance-constrained game at \( \alpha \in [0.5, 1)^2 \).

### 3.1.3. Special case

We consider the case where the entries of payoff matrix \( A^w \) are independent and identically distributed (i.i.d.) normal random variables with mean \( \mu_1 \) and variance \( \sigma_1^2 \), and the entries of payoff matrix \( B^w \) are i.i.d. normal random variables with mean \( \mu_2 \) and variance \( \sigma_2^2 \).

**Theorem 3.2.** Consider a bimatrix game \((A^w, B^w)\), where all the entries of matrix \( A^w \) are i.i.d. normal random variables with mean \( \mu_1 \) and variance \( \sigma_1^2 \), and all the entries of matrix \( B^w \) are i.i.d. normal random variables with mean \( \mu_2 \) and variance \( \sigma_2^2 \). The strategy pair \((x^*, y^*)\), where,

\[ x_i^* = \frac{1}{m}, \; \forall \; i \in I, \; \text{ and } \; y_j^* = \frac{1}{n}, \; \forall \; j \in J, \]  

(11)

is a Nash equilibrium of a chance-constrained game for all \( \alpha \in [0.5, 1)^2 \).

**Proof.** Fix \( \alpha \in [0.5, 1)^2 \). To show that \((x^*, y^*)\) defined by (11) is a Nash equilibrium, it is sufficient, from Theorem 3.1, to show that there exists a vector \((\lambda_1^*, \lambda_2^*, u_1^*, u_2^*)\) that together with \((x^*, y^*)\) is a feasible point of [MP] with objective function value zero. By using i.i.d. property, for all \((x, y)\) we have \( \mu_1(y) = \mu_1 1_m, \mu_2(x) = \mu_2 1_n, \Sigma_1^{1/2}(y) = \sigma_1 ||y|| I_{m \times m}, \) and...
\[\Sigma_2^{1/2}(x) = \sigma_2 \|x\|_{I_{n \times n}}; \quad I_{k \times k} \text{ denotes a } k \times k \text{ identity matrix.} \]

\[
\begin{align*}
\lambda_1^* &= -\mu_1 - \frac{\sigma_1 F_{Z_1}^{-1}(1 - \alpha_1)}{\sqrt{mn}} \\
\lambda_2^* &= -\mu_2 - \frac{\sigma_2 F_{Z_2}^{-1}(1 - \alpha_2)}{\sqrt{mn}} \\
u_1^* &= \frac{1}{\sqrt{m}} \mathbf{1}_m \\
u_2^* &= \frac{1}{\sqrt{n}} \mathbf{1}_n.
\end{align*}
\]

It is easy to check that \(\zeta^* = (\lambda_1^*, \lambda_2^*, u_1^*, u_2^*, x^*, y^*)\) is a feasible point of [MP] and \(\psi(\zeta^*) = 0\). Hence, \((x^*, y^*)\) defined by (11) is a Nash equilibrium of the chance-constrained game.

3.2. Payoffs following Cauchy distribution

We assume that all the entries of payoff matrix \(A^w\) are independent Cauchy random variables, and all the entries of payoff matrix \(B^w\) are independent Cauchy random variables. For each \(i \in I, j \in J\), let \(a_{ij}^w\) follows a Cauchy distribution with location and scale parameters \(\mu_{1,ij}\) and \(\sigma_{1,ij}\) respectively, and \(b_{ij}^w\) follows a Cauchy distribution with location and scale parameters \(\mu_{2,ij}\) and \(\sigma_{2,ij}\) respectively. Therefore, for a given strategy pair \((x, y)\), \(x^TA^wy\) follows a Cauchy distribution with location parameter \(\mu_1(x, y) = \sum_{i \in I, j \in J} x_iy_j \mu_{1,ij}\) and scale parameter \(\sigma_1(x, y) = \sum_{i \in I, j \in J} x_iy_j \sigma_{1,ij}\), and \(x^TB^wy\) follows a Cauchy distribution with location parameter \(\mu_2(x, y) = \sum_{i \in I, j \in J} x_iy_j \mu_{2,ij}\) and scale parameter \(\sigma_2(x, y) = \sum_{i \in I, j \in J} x_iy_j \sigma_{2,ij}\). Then, \(Z_1^C = \frac{x^TA^w y - \mu_1(x, y)}{\sigma_1(x, y)}\) and \(Z_2^C = \frac{x^TB^w y - \mu_2(x, y)}{\sigma_2(x, y)}\) follow a standard Cauchy distribution. Let \(F_{Z_1}^{-1}(\cdot)\) and \(F_{Z_2}^{-1}(\cdot)\) be the quantile functions of a standard Cauchy distribution. Similar to the previous case, for a given strategy pair \((x, y)\) and a given \(\alpha\) the payoff of player 1 is given by

\[
u_{1,1}^\alpha(x, y) = \sup \{ u | P(x^TA^w y \geq u) \geq \alpha \}
\]

\[
= \sup \left\{ u | P\left( \frac{x^TA^w y - \mu_1(x, y)}{\sigma_1(x, y)} \leq \frac{u - \mu_1(x, y)}{\sigma_1(x, y)} \right) \leq 1 - \alpha_1 \right\}
\]

\[
= \sup \left\{ u | u \leq \mu_1(x, y) + \sigma_1(x, y) F_{Z_1}^{-1}(1 - \alpha_1) \right\}.
\]
That is,

\[ u_{1}^{\alpha_1}(x, y) = \sum_{i \in I, j \in J} x_i y_j \left( \mu_{1,ij} + \sigma_{1,ij} F_{Z_1}^{-1}(1 - \alpha_1) \right), \quad (12) \]

The payoff of player 2 is given by

\[ u_{2}^{\alpha_2}(x, y) = \sum_{i \in I, j \in J} x_i y_j \left( \mu_{2,ij} + \sigma_{2,ij} F_{Z_2}^{-1}(1 - \alpha_2) \right). \quad (13) \]

The quantile function of a standard Cauchy distribution is not finite at 0 and 1. Therefore, we consider the case of \( \alpha \in (0, 1)^2 \), so that the payoff functions defined by (12) and (13) have finite values.

### 3.2.1. Equivalent bimatrix game

Define, a matrix \( \tilde{A}(\alpha_1) = [\tilde{a}_{ij}(\alpha_1)] \), where

\[ \tilde{a}_{ij}(\alpha_1) = \mu_{1,ij} + \sigma_{1,ij} F_{Z_1}^{-1}(1 - \alpha_1), \]

and a matrix \( \tilde{B}(\alpha_2) = [\tilde{b}_{ij}(\alpha_2)] \), where

\[ \tilde{b}_{ij}(\alpha_2) = \mu_{2,ij} + \sigma_{2,ij} F_{Z_2}^{-1}(1 - \alpha_2). \]

Then, we can write (12) as

\[ u_{1}^{\alpha_1}(x, y) = x^T \tilde{A}(\alpha_1) y, \]

and we can write (13) as

\[ u_{2}^{\alpha_2}(x, y) = x^T \tilde{B}(\alpha_2) y. \]

Then, for a given \( \alpha \in (0, 1)^2 \), a chance-constrained game is equivalent to the deterministic bimatrix game \( (\tilde{A}(\alpha_1), \tilde{B}(\alpha_2)) \). Hence, the existence of a Nash equilibrium in this case follows from [Nash 1950].

**Remark 3.3.** For case of i.i.d. Cauchy random variables each strategy pair \((x, y)\) is a Nash equilibrium because from (12) and (13), the payoff functions of both the players are constant.
3.2.2. Quadratic program

Mangasarian and Stone (1964) showed a one-to-one correspondence between a Nash equilibrium of a bimatrix game and a global maximum of a quadratic program. Hence, we have the following characterization for a chance-constrained game corresponding to Cauchy distribution.

**Theorem 3.4.** Consider a bimatrix game \((A^w, B^w)\), where all the entries of matrix \(A^w\) are independent Cauchy random variables, and all the entries of matrix \(B^w\) are also independent Cauchy random variables. For all \(i \in I, j \in J\), the location and scale parameters of \(a^w_{ij}\) are \(\mu_{1,ij}\) and \(\sigma_{1,ij}\) respectively, and the location and scale parameters of \(b^w_{ij}\) are \(\mu_{2,ij}\) and \(\sigma_{2,ij}\) respectively. Consider a point \(\zeta^* = (\lambda_1^*, \lambda_2^*, x^*, y^*)\). Then, for a given \(\alpha \in (0, 1)^2\), a strategy part \((x^*, y^*)\) of \(\zeta^*\) is a Nash equilibrium of a chance-constrained game if and only if it is a global maximum of the quadratic program \([QP]\) given below

\[
[QP] \quad \max_{\zeta} \left[ (x^T \tilde{A}(\alpha_1)y - \lambda_1) + (x^T \tilde{B}(\alpha_2)y - \lambda_2) \right]
\]

s.t.

(i) \(\tilde{A}(\alpha_1)y \leq \lambda_1 1_m\),
(ii) \(\tilde{B}^T(\alpha_2)x \leq \lambda_2 1_n\),
(iii) \(\sum_{i \in I} x_i = 1\),
(iv) \(\sum_{j \in J} y_j = 1\),
(v) \(x_i \geq 0, \forall i \in I\),
(vi) \(y_j \geq 0, \forall j \in J\),

with objective function value \(\psi(\zeta^*) = 0\).

**Proof.** For a given \(\alpha\), a chance-constrained game corresponding to Cauchy distribution is equivalent to the bimatrix game \(\left(\tilde{A}(\alpha_1), \tilde{B}(\alpha_2)\right)\) defined in Section 3.2.1 Then, the proof follows from Mangasarian and Stone (1964).

4. Numerical results

We illustrate our theoretical results given in Section 3 by considering some examples of random bimatrix game. We consider different sizes of ran-
dom bimatrix game corresponding to normal as well as Cauchy distributions. We compute the Nash equilibria of chance-constrained game by solving the corresponding mathematical program [MP] or quadratic program [QP]. Our numerical experiments were carried out on an Intel(R) 32-bit core(TM) i3-3110M CPU @ 2.40GHz × 4 and 3.8 GiB of RAM machine. We use fmincon in MATLAB optimization toolbox to solve the corresponding optimization problem.

4.1. Bimatrix game with normally distributed random payoffs

We consider few instances of bimatrix game of different sizes where the entries of the payoff matrices are independent normal random variables. We compute the Nash equilibria of the chance-constrained game by solving the corresponding mathematical program [MP]. The running time of each instance is within 1 minute. We compute a global maximum $\zeta^*$ of the mathematical program [MP] corresponding to each data set. The strategy part $(x^*, y^*)$ of $\zeta^*$ is a Nash equilibrium.

4.1.1. 5 × 5 random bimatrix game

We consider two instances of random bimatrix game of size 5 × 5. The mean $\mu_1$, $\mu_2$, and variance $\sigma_1^2$, $\sigma_2^2$ of independent normal random variables that characterize chance-constrained game are as follows:

1. $\mu_1 = \begin{pmatrix} 3 & 2 & 3 & 1 & 2 \\ 2 & 3 & 3 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 2 & 3 \end{pmatrix}$, $\sigma_1^2 = \begin{pmatrix} 2 & 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 2 & 3 \\ 2 & 3 & 1 & 3 & 2 \\ 1 & 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 & 1 \end{pmatrix}$,

2. $\mu_2 = \begin{pmatrix} 5 & 4 & 3 & 2 & 2 \\ 2 & 5 & 4 & 1 & 1 \\ 3 & 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 1 & 2 & 2 & 3 & 2 \end{pmatrix}$, $\sigma_2^2 = \begin{pmatrix} 2 & 2 & 3 & 1 & 1 \\ 2 & 3 & 1 & 3 & 4 \\ 1 & 1 & 2 & 5 & 4 \\ 1 & 2 & 3 & 2 & 1 \\ 2 & 3 & 4 & 1 & 3 \end{pmatrix}$,
\[ \mu_2 = \begin{pmatrix} 5 & 4 & 1 & 3 & 2 \\ 2 & 4 & 4 & 2 & 1 \\ 3 & 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 4 & 2 & 1 & 3 & 2 \end{pmatrix}, \quad \sigma_2^2 = \begin{pmatrix} 2 & 1 & 3 & 2 & 1 \\ 2 & 3 & 1 & 2 & 4 \\ 1 & 3 & 2 & 4 & 4 \\ 1 & 3 & 3 & 1 & 1 \\ 2 & 1 & 4 & 1 & 3 \end{pmatrix}. \]

The entries of \( \mu_1, \sigma_1^2, \mu_2, \sigma_2^2 \) defined above are the mean and variance of the corresponding normal random variables. For example, in dataset 1, random payoff \( a_{11} \) is a normal random variable with mean 3 and variance 2. Table 1 summarizes the Nash equilibria of the chance-constrained game corresponding to the datasets given for two 5 \( \times \) 5 instances of normally distributed random bimatrix game.

<table>
<thead>
<tr>
<th>No.</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( x^* )</th>
<th>( y^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>( (1,0,0,0,0) )</td>
<td>( (1,0,0,0,0) )</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6</td>
<td>( \left( \frac{7614}{10000}, \frac{2386}{10000}, 0, 0, 0 \right) )</td>
<td>( \left( \frac{5672}{10000}, \frac{4328}{10000}, 0, 0, 0 \right) )</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.7</td>
<td>( \left( \frac{8362}{10000}, \frac{1638}{10000}, 0, 0, 0 \right) )</td>
<td>( \left( \frac{7125}{10000}, \frac{2875}{10000}, 0, 0, 0 \right) )</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.5</td>
<td>( (0,0,0,0,1) )</td>
<td>( (1,0,0,0,0) )</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6</td>
<td>( \left( 0, \frac{2373}{10000}, \frac{4743}{10000}, \frac{2884}{10000} \right) )</td>
<td>( \left( \frac{965}{10000}, 0, \frac{35}{10000}, 0, 0 \right) )</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.7</td>
<td>( \left( 0, \frac{2488}{10000}, \frac{4760}{10000}, \frac{2752}{10000} \right) )</td>
<td>( \left( \frac{7343}{10000}, 0, \frac{1809}{10000}, \frac{848}{10000}, 0 \right) )</td>
</tr>
</tbody>
</table>

4.1.2. 7 \( \times \) 7 random bimatrix game

We consider two instances of random bimatrix game of size 7 \( \times \) 7. The mean \( \mu_1, \mu_2 \), and variance \( \sigma_1^2, \sigma_2^2 \) of independent normal random variables are as follows:

1. \( \mu_1 = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 & 4 & 2 \\ 1 & 1 & 2 & 2 & 3 & 2 & 1 \\ 2 & 3 & 2 & 1 & 1 & 2 & 2 \\ 4 & 2 & 2 & 1 & 3 & 1 & 2 \\ 3 & 4 & 1 & 2 & 4 & 3 & 3 \\ 2 & 1 & 3 & 4 & 2 & 2 & 3 \\ 1 & 1 & 2 & 3 & 2 & 1 & 3 \end{pmatrix}, \quad \sigma_1^2 = \begin{pmatrix} 2 & 2 & 1 & 3 & 2 & 1 & 1 \\ 3 & 2 & 2 & 2 & 1 & 1 & 4 \\ 2 & 1 & 2 & 2 & 1 & 3 & 2 \\ 2 & 3 & 4 & 1 & 2 & 4 & 3 \\ 2 & 3 & 4 & 2 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 1 & 3 & 2 & 4 & 5 & 1 & 2 \end{pmatrix}. \)
\[\mu_2 = \begin{pmatrix} 2 & 3 & 1 & 5 & 1 & 3 & 2 \\ 3 & 4 & 1 & 2 & 4 & 2 & 3 \\ 2 & 4 & 5 & 3 & 1 & 2 & 1 \\ 3 & 1 & 2 & 1 & 3 & 1 & 2 \\ 2 & 5 & 1 & 2 & 3 & 3 & 2 \\ 1 & 2 & 3 & 4 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 & 2 & 1 & 1 \end{pmatrix}, \quad \sigma_2^2 = \begin{pmatrix} 2 & 2 & 1 & 3 & 1 & 2 & 3 \\ 4 & 2 & 1 & 3 & 2 & 2 & 4 \\ 2 & 1 & 2 & 3 & 2 & 4 & 1 \\ 2 & 3 & 4 & 1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 4 & 1 & 3 \\ 1 & 3 & 4 & 2 & 2 & 1 & 2 \\ 1 & 2 & 3 & 2 & 1 & 3 & 4 \end{pmatrix}.\]

\[\mu_1 = \begin{pmatrix} 1 & 2 & 2 & 3 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 & 1 & 2 & 1 \\ 2 & 3 & 2 & 4 & 5 & 2 & 1 \\ 5 & 2 & 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 1 & 3 & 1 & 3 & 2 \\ 1 & 1 & 2 & 5 & 1 & 2 & 3 \\ 2 & 1 & 3 & 3 & 4 & 1 & 1 \end{pmatrix}, \quad \sigma_1^2 = \begin{pmatrix} 1 & 3 & 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 3 & 4 \\ 2 & 2 & 2 & 4 & 1 & 2 & 2 \\ 2 & 1 & 4 & 2 & 2 & 1 & 2 \\ 2 & 3 & 4 & 2 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 3 & 1 & 2 & 2 & 2 & 1 & 1 \end{pmatrix}.\]

2. \[\mu_2 = \begin{pmatrix} 5 & 3 & 1 & 3 & 1 & 3 & 1 \\ 3 & 2 & 4 & 2 & 1 & 3 & 3 \\ 2 & 4 & 5 & 3 & 1 & 2 & 1 \\ 3 & 1 & 2 & 2 & 1 & 1 & 2 \\ 2 & 5 & 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 2 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1 & 4 & 3 & 2 \end{pmatrix}, \quad \sigma_2^2 = \begin{pmatrix} 1 & 3 & 1 & 3 & 1 & 2 & 3 \\ 4 & 2 & 4 & 2 & 2 & 2 & 4 \\ 2 & 1 & 2 & 2 & 2 & 4 & 1 \\ 2 & 3 & 2 & 1 & 3 & 2 & 1 \\ 1 & 4 & 2 & 3 & 4 & 1 & 3 \\ 1 & 3 & 1 & 2 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 & 1 & 3 & 1 \end{pmatrix}.\]

Table 2 summarizes the Nash equilibria of the chance-constrained game corresponding to the datasets given for two 7×7 instances of normally distributed random bimatrix game.

4.2. Bimatrix game with Cauchy distributed random payoffs

We consider few instances of bimatrix game of different sizes where the entries of the payoff matrices are independent Cauchy random variables. We compute the Nash equilibria of chance-constrained game by solving the corresponding quadratic program [QP]. The running time of each instance is within 30 seconds. We compute the global maximum \(\zeta^*\) of the quadratic program [QP] corresponding to each data set. The strategy part \((x^*, y^*)\) of \(\zeta^*\) is a Nash equilibrium.
Table 2: Nash equilibria for various values of $\alpha$

<table>
<thead>
<tr>
<th>No.</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$x^*$</th>
<th>$y^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>$(0,0,0,1,0,0,0)$</td>
<td>$(0,0,0,0,4979,0,0)$</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6</td>
<td>$(0,0,\frac{8323}{10000},0,\frac{1677}{10000},0,0)$</td>
<td>$(0,\frac{4396}{10000},\frac{5604}{10000},0,0,0,0)$</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.7</td>
<td>$(0,0,\frac{4684}{10000},0,\frac{1444}{10000},\frac{3872}{10000},0)$</td>
<td>$(0,\frac{3606}{10000},\frac{5506}{10000},\frac{888}{10000},0,0,0)$</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.5</td>
<td>$(0,0,0,0,1,0,0)$</td>
<td>$(0,1,0,0,0,0,0)$</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6</td>
<td>$(0,\frac{3652}{10000},\frac{3698}{10000},0,\frac{2650}{10000},0,0)$</td>
<td>$(0,\frac{5188}{10000},\frac{4812}{10000},0,0,0,0)$</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.7</td>
<td>$(0,0,\frac{2577}{10000},\frac{5111}{10000},\frac{2312}{10000},0,0)$</td>
<td>$(\frac{2989}{10000},\frac{3812}{10000},\frac{3199}{10000},0,0,0,0)$</td>
</tr>
</tbody>
</table>

4.2.1. $5 \times 5$ random bimatrix game

We consider two instances of random bimatrix game of size $5 \times 5$. The location parameters $\mu_1$, $\mu_2$, and scale parameters $\sigma_1$, $\sigma_2$ of independent Cauchy random variables are as follows:

1. $\mu_1 = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & 3 & 1 & 1 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 3 & 4 & 2 \\ 1 & 2 & 4 & 5 & 2 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 2 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 3 & 1 \\ 2 & 1 & 3 & 4 & 2 \\ 3 & 1 & 2 & 5 & 2 \end{pmatrix}$

2. $\mu_2 = \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 2 & 2 & 1 & 3 \\ 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 4 & 2 & 1 \\ 1 & 1 & 2 & 1 & 3 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 5 & 2 & 3 & 2 & 3 \\ 2 & 4 & 3 & 2 & 1 \\ 1 & 3 & 4 & 2 & 3 \\ 2 & 1 & 3 & 5 & 1 \\ 2 & 1 & 2 & 3 & 4 \end{pmatrix}$

2. $\mu_1 = \begin{pmatrix} 1 & 2 & 4 & 2 & 1 \\ 2 & 1 & 3 & 2 & 2 \\ 1 & 2 & 4 & 2 & 1 \\ 2 & 2 & 3 & 4 & 1 \\ 1 & 2 & 4 & 5 & 2 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 2 & 3 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 2 \\ 3 & 1 & 3 & 3 & 1 \\ 2 & 2 & 5 & 4 & 2 \\ 3 & 1 & 3 & 5 & 2 \end{pmatrix}$
\[
\mu_2 = \begin{pmatrix}
1 & 2 & 3 & 2 & 1 \\
3 & 1 & 2 & 1 & 4 \\
2 & 1 & 3 & 4 & 2 \\
3 & 2 & 4 & 2 & 1 \\
2 & 4 & 2 & 1 & 3
\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}
5 & 2 & 4 & 2 & 1 \\
2 & 4 & 3 & 2 & 1 \\
4 & 3 & 3 & 2 & 3 \\
2 & 1 & 3 & 5 & 3 \\
1 & 3 & 4 & 3 & 4
\end{pmatrix}.
\]

The entries of \(\mu_1, \sigma_1, \mu_2, \sigma_2\) defined above are the location and scale parameters of the corresponding Cauchy random variables. For example, in dataset 1, random payoff \(a_{11}\) is a Cauchy random variable with location parameter 1 and scale parameter 2. Table 3 summarizes the Nash equilibria of the chance-constrained game corresponding to the datasets given for two 5 \times 5 instances of Cauchy distributed random bimatrix game.

<table>
<thead>
<tr>
<th>No.</th>
<th>(\alpha_1)</th>
<th>(\alpha_2)</th>
<th>(x^*)</th>
<th>(y^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>((0, \frac{2}{5}, \frac{1}{5}, \frac{1}{6}, 0))</td>
<td>((\frac{1}{2}, 0, 0, 0, \frac{1}{2}))</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6</td>
<td>((0, \frac{5271}{10000}, \frac{1443}{10000}, \frac{3286}{10000}, 0))</td>
<td>((\frac{5699}{10000}, 0, \frac{1398}{10000}, 0, \frac{2903}{10000}))</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.7</td>
<td>((0, \frac{3841}{10000}, \frac{6159}{10000}, 0, 0))</td>
<td>((\frac{1}{2}, 0, 0, 0, \frac{1}{2}))</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.5</td>
<td>((0, 0, \frac{1}{5}, 0, \frac{1}{2}))</td>
<td>((0, 0, 1, 0, 0))</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6</td>
<td>((0, 0, \frac{4743}{10000}, 0, \frac{5257}{10000}))</td>
<td>((0, 1, 0, 0, 0))</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.7</td>
<td>((\frac{1304}{10000}, \frac{2226}{10000}, 0, 0, \frac{6470}{10000}))</td>
<td>((\frac{1962}{10000}, \frac{4813}{10000}, 0, 0, \frac{3225}{10000}))</td>
</tr>
</tbody>
</table>

4.2.2. 7 \times 7 random bimatrix game

We consider two instances of random bimatrix game of size 7 \times 7. The location parameters \(\mu_1, \mu_2\), and scale parameters \(\sigma_1, \sigma_2\) of independent Cauchy random variables are as follows:

\[
\mu_1 = \begin{pmatrix}
1 & 2 & 2 & 4 & 3 & 2 & 1 \\
1 & 1 & 2 & 1 & 3 & 2 & 2 \\
3 & 2 & 1 & 2 & 4 & 2 & 1 \\
2 & 4 & 2 & 2 & 3 & 4 & 1 \\
1 & 2 & 4 & 5 & 2 & 2 & 3 \\
1 & 3 & 4 & 3 & 2 & 2 & 3 \\
2 & 1 & 4 & 2 & 3 & 2 & 1
\end{pmatrix}, \quad \sigma_1 = \begin{pmatrix}
2 & 3 & 1 & 2 & 1 & 1 & 2 \\
1 & 1 & 3 & 1 & 2 & 2 & 4 \\
2 & 1 & 3 & 1 & 3 & 3 & 1 \\
2 & 2 & 5 & 4 & 2 & 1 & 3 \\
2 & 1 & 3 & 1 & 3 & 5 & 2 \\
1 & 2 & 3 & 1 & 2 & 3 & 2 \\
2 & 1 & 4 & 2 & 3 & 1 & 2
\end{pmatrix},
\]
2. $\mu_2 = \begin{pmatrix} 1 & 2 & 3 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 & 2 & 1 & 4 \\ 2 & 1 & 2 & 1 & 3 & 4 & 2 \\ 1 & 2 & 3 & 2 & 4 & 2 & 1 \\ 2 & 3 & 1 & 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 2 & 1 & 3 & 4 \\ 2 & 3 & 1 & 2 & 3 & 4 & 2 \end{pmatrix}$, \[ \sigma_2 = \begin{pmatrix} 5 & 2 & 4 & 2 & 1 & 2 & 3 \\ 1 & 2 & 2 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 3 & 3 & 2 & 3 \\ 2 & 3 & 2 & 1 & 3 & 5 & 3 \\ 2 & 1 & 2 & 3 & 4 & 3 & 4 \\ 1 & 2 & 2 & 3 & 1 & 3 & 1 \end{pmatrix}. \]

\[ \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 1 & 3 & 4 & 1 \\ 2 & 1 & 2 & 1 & 2 & 4 & 2 \\ 1 & 2 & 1 & 5 & 3 & 2 & 1 \\ 1 & 3 & 2 & 2 & 3 & 2 & 1 \\ 2 & 3 & 4 & 5 & 2 & 1 & 3 \\ 1 & 3 & 2 & 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 & 1 & 2 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 3 & 1 & 2 & 1 & 2 & 2 \\ 2 & 1 & 3 & 1 & 2 & 2 & 4 \\ 2 & 1 & 3 & 2 & 3 & 4 & 1 \\ 2 & 2 & 3 & 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 & 3 & 2 & 2 \\ 1 & 2 & 3 & 2 & 2 & 4 & 2 \\ 2 & 3 & 4 & 1 & 3 & 1 & 2 \end{pmatrix}. \]

Table 4 summarizes the Nash equilibria of the chance-constrained game corresponding to the datasets given for two $7 \times 7$ instances of Cauchy distributed random bimatrix game.

5. Conclusions

We consider two player random bimatrix game where the entries of the payoff matrices are independent normal/Cauchy random variables. We formulate this problem as a chance-constrained game. For both the cases we show that a Nash equilibrium of the corresponding chance-constrained game can be computed by solving an equivalent optimization problem. For illustrating our theoretical results, we consider few instances of random bimatrix game and perform numerical experiments using MATLAB. We solve the equivalent optimization problem corresponding to each dataset using \texttt{fmincon} in MATLAB optimization toolbox. Our approaches can be used
Table 4: Nash equilibria for various values of $\alpha$

<table>
<thead>
<tr>
<th>No.</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$x^*$</th>
<th>$y^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>$(\frac{2}{3}, 0, 0, 0, \frac{1}{3}, 0, 0)$</td>
<td>$(0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0)$</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6</td>
<td>$(4308_{10000}^0, 0, 0, 373_{10000}^{\frac{3}{10}}, 2202_{10000}^{\frac{2}{10}}, 3117_{10000}^{\frac{1}{10}}, 0)$</td>
<td>$(0, 3211_{10000}^{\frac{1}{10}}, 1362_{10000}^{\frac{1}{10}}, 1691_{10000}^{\frac{1}{10}}, 3736_{10000}^{\frac{1}{10}}, 0, 0)$</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.7</td>
<td>$(1014_{10000}^{\frac{1}{10}}, 0, 3991_{10000}^{\frac{3}{10}}, 1187_{10000}^{\frac{1}{10}}, 3808_{10000}^{\frac{1}{10}}, 0, 0)$</td>
<td>$(0, 3687_{10000}^{\frac{1}{10}}, 2474_{10000}^{\frac{1}{10}}, 3370_{10000}^{\frac{1}{10}}, 469_{10000}^{\frac{1}{10}}, 0)$</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.5</td>
<td>$(0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0)$</td>
<td>$(0, 0, 0, 5126_{10000}^{\frac{1}{10}}, 3249_{10000}^{\frac{1}{10}}, 0, 1625_{10000}^{\frac{1}{10}})$</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6</td>
<td>$(1527_{10000}^{\frac{1}{10}}, 0, 0, 0, 4156_{10000}^{\frac{1}{10}}, 4317_{10000}^{\frac{1}{10}}, 0)$</td>
<td>$(0, 0, 0, 432_{10000}^{\frac{1}{10}}, 5755_{10000}^{\frac{1}{10}}, 0, 3813_{10000}^{\frac{1}{10}})$</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.7</td>
<td>$(2885_{10000}^{\frac{1}{10}}, 0, 3764_{10000}^{\frac{1}{10}}, 0, 3351_{10000}^{\frac{1}{10}}, 0, 0)$</td>
<td>$(0, 0, 0, 3614_{10000}^{\frac{1}{10}}, 3798_{10000}^{\frac{1}{10}}, 2588_{10000}^{\frac{1}{10}}, 0)$</td>
</tr>
</tbody>
</table>

for solving large instances as shown by the low computational effort in our numerical examples. Recently, the electricity markets over the past few years have been transformed from nationalized monopolies into competitive markets with privately owned participants. The randomness in the electricity markets is present due to many external factors. These situations can be modeled as chance-constrained games and the approaches developed in this paper can be useful in computing the Nash equilibria of the chance-constrained game.

References


