Bounds on the order of biregular graphs with even girth at least 8

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Abstract
Let 2 ≤ r < m and g ≥ 4 even be three positive integers. A graph with a degree set \{r, m\}, girth g and minimum order is called a bi-regular cage or an \(\{r, m\}; g\)-cage, and its order is denoted by \(n(\{r, m\}; g)\). In this paper we obtain constructive upper bounds on \(n(\{r, m\}; g)\) for some values of \(r, m\) and even girth at least 8.

Keywords: bi-regular, cage, girth.

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1 Introduction

For terminology and definitions not included here see Biggs [4] and Chartrand and Lesniak [6]. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The girth is the length of the shortest cycle in $G$. The degree set $D$ of $G$ is the set of distinct degrees of the vertices of $G$. A $(D; g)$-graph is a graph with degree set $D$ and girth $g$. A $(D; g)$-cage is a $(D; g)$-graph with minimum order $n(D; g)$. These graphs have been studied in [1,2,7,8,11,12]. Thus Chartrand, Gould and Kapoor [7] proved the existence of $(D; g)$-cages. Clearly if $D = \{r\}$, then $(D; g)$-cages coincide with $(r; g)$-cages, which have been intensely studied since Erdős and Sachs [9] proved the existence of $(r; g)$-cages for all two positive integers $r \geq 2$ and $g \geq 3$. See the more recent survey by Exoo and Jajcay [10].

We focus our attention in $(D; g)$-graphs with degree set $D = \{r, m\}$, for $2 < r \leq m$. An $(\{r, m\}; g)$-graph with minimum order $n(\{r, m\}; g)$ is called $(\{r, m\}; g)$-cage or bi-regular cage. A first lower bound on $n(\{r, m\}; g)$ was given by Downs et al. [8], however for even girth $g \geq 8$ this lower bound has been improved in [1]. Here we consider $(\{r, m\}; g)$-graphs with $3 \leq r < m$ and even girth at least 8. Our first contribution is to give an upper bound on $n(\{r, k(r - 1) + \varepsilon\}; g)$ for $\varepsilon = 0, 1$, and any even girth $g \geq 8$. When $g = 8$ and $r - 1$ is a prime power we enhanced even more this upper bound for $m = kr + t$, $k \geq r - t$ and $t = 1, \ldots, r - 1$. And for $r = 3$ and $g = 8$, we also improve the lower bound given in [1]. Finally, we construct a $(\{3, m\}; 8)$-cage for $m = 4, 5, 7$ of order 39, 48 and 66 respectively. Using these cages, we also construct $(\{3, m\}; 8)$-graphs of order $9m + 3$ which yields an improvement of our above upper bound for $m = 8, 10, 13, 16$.

2 Small bi-regular graphs of even girth $g \geq 8$

Let $G$ be an $(r; g)$-graph with $r \geq 2$ and even girth $g \geq 8$. Let $d_G(u, v) = d(u, v)$ denote the distance in $G$ between any two vertices $u$ and $v$. Since the girth of $G$ is even, it follows that $G$ contains a tree of length $g/2 - 1$ rooted in an edge $xy$. Thus, for $0 \leq j \leq g/2 - 1$ the following mutually disjoint sets are defined:

\begin{align}
X_j &= \{u \in V(G): d(u, x) = j, \ d(u, y) = j + 1\}, \\
Y_j &= \{v \in V(G): d(v, y) = j, \ d(v, x) = j + 1\}.
\end{align}

Since $G$ is $r$-regular, we have $|X_j| = |Y_j| = (r - 1)^j$. These sets allow us to efficiently describe the constructive methods used here.
Construction 2.1 Let $G$ be an $(r; g)$-graph with $r \geq 2$ and even girth $g \geq 8$. Let $xy \in E(G)$ and consider the sets $X_j, Y_j$, $0 \leq j \leq g/2 - 1$, defined in (1). Let us take $k \geq 2$ vertex-disjoint copies $G^{(i)}$, $i = 1, 2, \ldots, k$, of $G$ and denote by $X_j^{(i)}, Y_j^{(i)}$ the copies of $X_j, Y_j$ in each $G^{(i)}$. For $t = 1, 2$ let $G_{t,k}$ be an $\{r, k(r-1)+t-1\}; g'$-graph with $g' \geq g$ obtained from $\bigcup_{i=1}^k G^{(i)}$ by performing the following operations:

(a) Remove from each $G^{(i)}$ the set of vertices $(\bigcup_{j=0}^{[g/4]-1} X_j^{(i)}) \cup (\bigcup_{j=0}^{[g/4]-2} Y_j^{(i)})$, for $i = t, \ldots, k$.

(b) Identify each vertex of $X_{[g/4]}^{(i)} \cup Y_{[g/4]-1}^{(i)}$ with its copy in $G^{(1)}$ for all $i = 2, \ldots, k$.

Construction 2.1 allows us to obtain the following theorem that shows the best upper bound on the order of a bicage with even girth at least 8.

Theorem 2.1 Let $r \geq 3$ and $k \geq 2$ be integers. If $g \geq 8$ is even, $m = k(r-1) + \varepsilon$ with $\varepsilon = 0, 1$, and whenever that (i) $C = m - r$ if $g \equiv 0 \pmod{4}$ or (ii) $C = 2(\varepsilon - 1)$ if $g \equiv 2 \pmod{4}$, then

$$n(\{r, m\}; g) \leq kn(r; g) - 2(k - \varepsilon) \sum_{i=0}^{[g/4]-1} (r-1)^i - C(r-1)^{[g/4]-1}.$$ 

3 Small $(\{r, m\}; 8)$-graphs with $r - 1$ a prime power

In what follows, we need the following result proved in [2].

Theorem 3.1 ([2]) If $k \geq 2$ and $r - 1$ is a prime power, then an $(\{r, rk\}; 8)$-graph is constructed by identifying the $(r - 1)^2 + 1$ vertices of an ovoid in $k$ copies of an $(r; 8)$-cage. Thus

$$n(\{r, kr\}; 8) \leq 2k \sum_{i=0}^{3} (r - 1)^i - (k - 1)((r - 1)^2 + 1).$$

Construction 3.1 Let $\Gamma$ be an $(r; 8)$-cage with $r - 1$ a prime power, $x \in V(G)$ and $y \in N(x)$. If $N = \{x\} \cup (N(x) - \{y\})$ then $G - N$ is a graph with $(r-1)^2 + 1$ vertices of degree $r - 1$. An $(\{r, k(r-1)\}; 8)$-graph with $(r - 1)^2$ vertices of degree $k(r-1)$ and order $2k \sum_{i=0}^{3} (r-1)^i - kr - (k-1)((r-1)^2 + 1)$ is obtained by identifying $k$ copies of $G - N$ through the vertices of degree $r - 1$.

Let $G_1$ be a graph obtained by Theorem 3.1 and $G_2$ a graph obtained by Construction 3.1. We obtained for $r - 1$ a prime power a constructive upper bound of an $(\{r, m\}; 8)$-graph, by identifying an adequate number of copies of $G_1$ and $G_2$ through its vertices of degree $m$. 
Theorem 3.2 Let $r - 1$ be a prime power, $k \geq r - t$ and $t = 1, \ldots, r - 1$. Then $n({r, kr + t}; 8) \leq (k + 1)(2r - 1)((r - 1)^2 + 1) + (t - 2)r + 2$.

4 Small $({3, m}; 8)$-graphs

We need to prove the following lemma for obtaining Theorem 4.2.

Lemma 4.1 Let $m \geq 4$, $m \not\equiv 0 \pmod{3}$. Let $G$ be a $({3, m}; 8)$-graph having at least two vertices of degree $m$ at distance at most three. Then $|V(G)| \geq 9m + 3$.

Using Lemma 4.1 and analysing the number of vertices of degree $m$ and the distance that separates them, we obtain the following result.

Theorem 4.2 Let $m \geq 7$, $m \not\equiv 3k$. Then $n({3, m}; 8) \geq \lceil 25m/3 \rceil + 7$.

As a consequence of Theorem 4.2, it turns out that the graphs constructed in Theorem 3.2 are close to being cages. We have the following result.

Corollary 4.3 Assume $m = 3k + t$ with $k \geq 2$ and $t = 1, 2$. Then

$$25k + 8t + 8 \leq n({3, 3k + t}; 8) \leq 25k + 3t + 21.$$  

The following construction allows us to obtain a family of $({3, m}; 8)$-cages for $m = 4, 5, 7$.

Construction 4.1 Let $T$ be a tree of length three rooted in a vertex $x$ such that $N(x) = \{x_0, \ldots, x_{m-1}\}$ and such that every vertex in $T - x$ within distance at most two from $x$ has degree three. Let $D_2(x_i) = \{v \in V(T) : d_T(v, x_i) = 2\} \setminus N_T(x_i), i = 0, \ldots, m - 1$. Let label the four vertices of $D_2(x_i)$ as $p_i, q_i, (i, 1), (i, 2)$ such that $d_T(p_i, q_i) = 2$, $d_T((i, 1), (i, 2)) = 2$ and $d_T((i, j), w) = 4$ for $j = 1, 2$, $w = p_i, q_i, i = 0, \ldots, m - 1$. Let $T_1$ be the graph obtained from $T$ by adding two new vertices $p$ and $q$ such that $N_{T_1}(p) = \{p_0, \ldots, p_{m-1}\}$ and $N_{T_1}(q) = \{q_0, \ldots, q_{m-1}\}$. And let $T^*$ be the graph obtained from $T_1$ by adding $2m$ new vertices labeled as $p_{i1}, q_{i1}$ adjacent to $p_i$ and $q_i$ respectively. Let $G$ be the graph obtained from $T^*$ by adding the following edges:

$$\{p_{i1}, (i + 1, 1)\}, \{q_{i1}, (i + 2, 2)\}, \{q_{i1}, (i + 1, 2)\}, \{q_{i1}, (i - 1, 1)\},$$

where the sum is taken modulo $m$. Then $G$ is a $({3, m}; 8)$-cage for $m = 4, 5, 7$ with $|V(G)| = 9m + 3$.

Using the same idea used by Construction 2.1 we obtain a family of $({3, m}; 8)$-graphs with $9m + 3$ vertices by identifying an adequate number of vertices of $({3, m}; 8)$-cages for $m = 4, 5, 7$. 

**Theorem 4.4** Let \( m \geq 8 \), \( m \not\equiv 0 \pmod{3} \). There exists a \((\{3, m\}; 8)\)-graph on \( 9m + 3 \) vertices having three vertices of degree \( m \) mutually at distance at least four.

**References**


