A Two-stage Image Segmentation Method Using a Convex Variant of the Mumford-Shah Model and Thresholding

Raymond H. Chan
Department of Mathematics
Chinese University of Hong Kong

Joint work with
Xiaohao Cai (Kaiserslautern Technical University)
Hongfei Yang (University of Nottingham)
Tieyong Zeng (Hong Kong Baptist University)

Bob’s Birthday Workshop
Nov 17-18, 2013

Supported by HKRGC
Aim: One Method to Segment Different Images

anti-mass

noisy

tubular

3-phase

4-phase blurry

non-Gaussian
Multiphase Segmentation for Blurry Image

Noisy & blurry

Li et al. (10)

Yuan et al. (10)

Sandberg et al. (10)

Steidl et al. (12)

Our result
1. Mumford-Shah Model

2. Our Two-stage Image Segmentation Method

3. Experimental Results

4. Extensions to Other Noise Models

5. Conclusions
Problem Setting and Notation

Given a noisy and blurry image $f$,

$$f = \begin{array}{c}
\Omega \\
\end{array} = \Gamma$$

want a $K$-phase segmentation:

$$g = \begin{array}{c}
\Omega_1 \\
\Omega_2 \\
\Omega_3 \\
\Omega_4
\end{array}$$

$$K = 4$$

$$\Omega \setminus \Gamma = \bigcup_i \Omega_i, \quad g = c_i \text{ in } \Omega_i, \quad i = 1, 2, 3, 4$$
Mumford-Shah Model (1989)

\[ \text{minimize} \quad E_{MS}(g, \Gamma) \]

\[ \frac{\lambda}{2} \int_\Omega (f - g)^2 \, dx \]

\[ + \quad \frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 \, dx \]

\[ + \quad \text{Length}(\Gamma) \]

Data fidelity: control \( g \) not far away from \( f \)

Regularization: impose smoothness of \( g \) on \( \Omega \setminus \Gamma \)

Regularization: require boundary \( \Gamma \) be short
Mumford-Shah Model (1989)

\[
E_{MS}(g, \Gamma) = \frac{\lambda}{2} \int_{\Omega} (f - g)^2 dx + \frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 dx + \text{Length}(\Gamma)
\]

Nonconvex: due to the edge term \(\Gamma\)
Finding Good Approximation of M-S Model

\[ \frac{\lambda}{2} \int_{\Omega} (f - g)^2 dx + \frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 dx + \text{Length}(\Gamma) \]

Simplifying Mumford-Shah Model

\[ E_{MS}(g, \Gamma) = \frac{\lambda}{2} \int_{\Omega} (f - g)^2 \, dx + \frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 \, dx + \text{Length}(\Gamma) \]

Simplify it:
\[ \nabla g \equiv 0 \text{ on } \Omega \setminus \Gamma \]

Multiphase Chan-Vese Model (02)
(minimizer \( \hat{g} \) is piecewise constant):
\[ E_{MS}(\{c_i\}, \Gamma) = \frac{\lambda}{2} \sum_{i=1}^{K} \int_{\Omega_i} (f - c_i)^2 + \text{Length}(\Gamma) \]

2-phase
\[ \Omega \setminus \Gamma = \Omega_1 \cup \Omega_2 \]
\[ \hat{g} = c_i \text{ in } \Omega_i \]

ACWE (01):
1. Mumford-Shah Model

2. Our Two-stage Image Segmentation Method

3. Experimental Results

4. Extensions to Other Noise Models

5. Conclusions
Our Motivation

(a): True binary image

(d): Difference of (a) and (c) (nonzero pixel values only at the boundary)
Aim in Segmentation

Chan-Vese: Get a piecewise constant approximation $\hat{g}$ of $f$

Our idea:

- $f$ (smooth approx. $\hat{g}$ of $f$)
- threshold $\hat{g}$ (stage 2)
- segmentation

Stage One: Convex Variant of the M-S Model

\[
\frac{\lambda}{2} \int_{\Omega} (f - g)^2 \, dx + \frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 \, dx + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 \, dx + \text{Length}(\Gamma) + \int_{\Omega} |\nabla g| \, dx
\]

\[
E_{\text{MS}}(g, \Gamma)
\]

Restrict: \( g \in W^{1,2}(\Omega) \)

Convex M-S Energy \( E(g) \)
**Stage One: Lemma 1**

**Lemma 1**

*If* $g \in W^{1,2}(\Omega)$ *and* $\Gamma$ *is a closed curve with Lebesgue measure* $m(\Gamma) = 0$, *then*

$$\int_{\Gamma} |\nabla g|^2 dx = 0.$$

**Proof:** Since $g \in W^{1,2}(\Omega)$, we have $\nabla g \in L^2(\Omega)$. Because of $m(\Gamma) = 0$, we get $\int_{\Gamma} |\nabla g|^2 dx = 0$ immediately.
Stage One: Convex Variant of the M-S Model

\[ E_{\text{MS}}(g, \Gamma) \]

\[ g \in W^{1,2}(\Omega) \]

\[ \frac{\lambda}{2} \int_{\Omega} (f - g)^2 dx \]

\[ \frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 dx \]

\[ \text{Length}(\Gamma) \]

\[ \frac{\lambda}{2} \int_{\Omega} (f - g)^2 dx \]

\[ \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 dx \]

\[ \int_{\Omega} |\nabla g| dx \]

Convex M-S Energy

\[ E(g) \]

Lemma 1

Approx. Thm. 2
Stage One: Theorem 2 (for 2-phase)

For $K = 2$, let $\Sigma = \text{Inside}(\Gamma)$, $g_1 \in W^{1,2}(\Sigma \setminus \Gamma)$, and $g_2 \in W^{1,2}(\Omega \setminus \Sigma)$.

Rewrite the Mumford-Shah model as:

$$E_{MS}(\Sigma, g_1, g_2) = \frac{\lambda}{2} \int_{\Sigma \setminus \Gamma} (f - g_1)^2 \, dx + \frac{\mu}{2} \int_{\Sigma \setminus \Gamma} |\nabla g_1|^2 \, dx$$

$$+ \frac{\lambda}{2} \int_{\Omega \setminus \Sigma} (f - g_2)^2 \, dx + \frac{\mu}{2} \int_{\Omega \setminus \Sigma} |\nabla g_2|^2 \, dx$$

$$+ \text{Length}(\Gamma).$$

Even if $g_1$ and $g_2$ are given, finding $\Sigma$ is still non-convex.
**Stage One: Theorem 2 (for 2-phase)**

**Theorem 2** (cf. T. Chan, Esedoglu, and Nikolova (06))

Given $g_1$ and $g_2 \in W^{1,2}(\Omega)$, a global minimizer $\Sigma$ for MS model $E_{MS}(\Sigma; g_1, g_2)$ can be found by solving the convex minimization:

$$
\min_{0 \leq g \leq 1} \left\{ \int_\Omega \left[ \frac{\lambda}{2} (f - g_1)^2 + \frac{\mu}{2} |\nabla g_1|^2 - \frac{\lambda}{2} (f - g_2)^2 - \frac{\mu}{2} |\nabla g_2|^2 \right] g(x) + \int_\Omega |\nabla g| \right\},
$$

for $\tilde{g}$ and setting $\Sigma = \{x : \tilde{g}(x) \geq \rho\}$ for a.e. $\rho \in [0, 1]$.

- $\Sigma$ is determined by thresholding $\tilde{g}$
- $\text{Length}(\Gamma) \approx \int_\Omega |\nabla g|$
- equal if $\tilde{g}$ is binary and piecewise constant
Mumford-Shah Model for SBV

When restricted to special functions of bounded variations ([Ambrosio-Giorgi, 88]), Mumford-Shah model becomes

$$\min_{g \in SBV} \left\{ \frac{\lambda}{2} \int_{\Omega} |f - g|^2 + \frac{\mu}{2} \int_{\Omega \setminus J_g} |\nabla g|^2 + \mathcal{H}^1(J_g) \right\},$$

where $J_g$ is the jump set of $g$ and $\mathcal{H}^1$ is the Hausdorff measure of dimension 1.

- See [Cagnetti & Scardia, 08] and [Strekalovskiy et al., 12]
- If $g$ is binary and piecewise-constant, then $J_g = \Gamma$ and

$$\mathcal{H}^1(J_g) = \text{Length}(\Gamma) = \int_{\Omega} |\nabla g|$$
Mumford-Shah Model for Binary Disk

Consider segmenting a binary disk \( f = a\chi_{B(0,1)} \).

The M-S model has two solutions:

(i) \( \Gamma = \partial B(0, 1) \) and \( g \) minimizes

\[
\min_g \left\{ \frac{\lambda}{2} \int_{\Omega} |f - g|^2 + \int_{\Omega} |\nabla g| \right\},
\]

(ii) \( \Gamma = \emptyset \) and \( g \) minimizes

\[
\min_g \left\{ \frac{\lambda}{2} \int_{\Omega} |f - g|^2 + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 \right\}.
\]

Both solutions can be reproduced by our model:

\[
E(g) = \frac{\lambda}{2} \int_{\Omega} (f - g)^2 dx + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 dx + \int_{\Omega} |\nabla g| dx
\]
Stage One: Convex Variant of the M-S Model

Mumford-Shah Energy

$$E_{MS}(g, \Gamma)$$

- $\frac{\lambda}{2} \int_\Omega (f - g)^2 \, dx$
- $\frac{\mu}{2} \int_\Omega |\nabla g|^2 \, dx$
- Length($\Gamma$)

- $\frac{\lambda}{2} \int_\Omega (f - g)^2 \, dx$
- $\frac{\mu}{2} \int_\Omega |\nabla g|^2 \, dx$
- $\int_\Omega |\nabla g| \, dx$

Convex M-S Energy

$$E(g)$$
Stage One: Extension to Blurred Problems

\[
\frac{\lambda}{2} \int_{\Omega} (f - g)^2 dx + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 dx + \int_{\Omega} |\nabla g| dx
\]

extendable to blurred image with blur \( \mathcal{A} \)

\[
\frac{\lambda}{2} \int_{\Omega} (f - \mathcal{A}g)^2 dx + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 dx + \int_{\Omega} |\nabla g| dx
\]
Stage One: Unique Minimizer

Our convex variant of the Mumford-Shah model is:

$$E(g) = \frac{\lambda}{2} \int_{\Omega} (f - Ag)^2 dx + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 dx + \int_{\Omega} |\nabla g| dx$$

Its discrete version is:

$$\frac{\lambda}{2} \|f - Ag\|_2^2 + \frac{\mu}{2} \|\nabla g\|_2^2 + \|\nabla g\|_1$$

Theorem 3

Let $\Omega$ be a bounded connected open subset of $\mathbb{R}^2$ with a Lipschitz boundary. Let $\text{Ker}(A) \cap \text{Ker}(\nabla) = \{0\}$ and $f \in L^2(\Omega)$, where $A$ is a bounded linear operator from $L^2(\Omega)$ to itself. Then $E(g)$ has a unique minimizer $g \in W^{1,2}(\Omega)$. 
Stage Two: Thresholding

\[ \frac{\lambda}{2} \int_{\Omega} (f - Ag)^2 dx + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 dx + \int_{\Omega} |\nabla g| dx \]

\[ \hat{g} \]

Stage 2

Threshold \( \hat{g} \) to piecewise constant
Our Two-stage Segmentation Algorithm

Given $f$

\[ \text{Stage 1: solve } \hat{g} \]
\[ \min_{g} \left\{ \frac{\lambda}{2} \| f - Ag \|_2^2 + \frac{\mu}{2} \| \nabla g \|_2^2 + \| \nabla g \|_1 \right\} \]

No iterations between Stages 1 and Stages 2

\[ \text{Stage 2: determine thresholds from } \hat{g} \]
\[ \rho_1 \leq \cdots \leq \rho_{K-1} \]

$K$ phases

$\rho = 0.19$
Numerical Aspects: Stage One

Given $f$

Stage 1: solve $\hat{g}$

$$\min_g \left\{ \frac{\lambda}{2} \| f - Ag \|_2^2 + \frac{\mu}{2} \| \nabla g \|_2^2 + \| \nabla g \|_1 \right\}$$

Stage 2: determine thresholds from $\hat{g}$

$$\rho_1 \leq \cdots \leq \rho_{K-1}$$

$K$ phases

Split-Bregman (ADMM) (Goldstein and Osher, 09);
Augmented Lagrangian (Tai, et. al., 09);
Chambolle-Pock method (Chambolle and Pock, 10)
Etc.
Numerical Aspects: Stage One

Split-Bregman method for solving our model

\[
\min_g \left\{ \frac{\lambda}{2} \| f - \mathcal{A}g \|_2^2 + \frac{\mu}{2} \| \nabla g \|_2^2 + \| \nabla g \|_1 \right\}.
\]

Note that \( F(g) \) is quadratic in \( g \).

Idea is to separate the \( g \)'s in \( F(g) \) and \( \| \nabla g \|_1 \). Set

\[
\begin{align*}
  d_x &= \nabla_x g, \\
  d_y &= \nabla_y g.
\end{align*}
\]
Numerical Aspects: Stage One

Solve:

\[
\min_{g,d_x,d_y} \left\{ F(g) + \| (d_x, d_y) \|_1 \right\}
\]

s.t. \[ d_x = \nabla_x g, \quad d_y = \nabla_y g \]

Split-Bregman iteration:

\[
(g^{k+1}, d_x^{k+1}, d_y^{k+1}) = \arg\min_{g,d_x,d_y} \left\{ F(g) + \| (d_x, d_y) \|_1 \right. \\
+ \frac{\sigma}{2} \| d_x - \nabla_x g - b_x^k \|_2^2 + \frac{\sigma}{2} \| d_y - \nabla_y g - b_y^k \|_2^2 \right\}
\]

\[
b_x^{k+1} = b_x^k + (\nabla_x g^{k+1} - d_x^{k+1}), \quad b_y^{k+1} = b_y^k + (\nabla_y g^{k+1} - d_y^{k+1}).
\]

Numerical Aspects: Stage One

Stage 1: solve \( \hat{g} \)

\[
\min_g \left\{ \frac{\lambda}{2} \|f - Ag\|_2^2 + \frac{\mu}{2} \|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}
\]

Stage 2: determine thresholds from \( \hat{g} \)

\[\rho_1 \leq \cdots \leq \rho_{K-1}\]

\( K \) phases

Solved!
Numerical Aspects: Stage Two

**Stage 1:** solve \( \hat{g} \)

\[
\min_g \left\{ \frac{\lambda}{2} \| f - Ag \|_2^2 + \frac{\mu}{2} \| \nabla g \|_2^2 + \| \nabla g \|_1 \right\}
\]

**Stage 2:** determine thresholds from \( \hat{g} \)

\[
\rho_1 \leq \cdots \leq \rho_{K-1}
\]

**Automatic way to determine thresholds by K-means**

1. Segment the histogram of \( \hat{g} \) into \( K \) clusters, and compute the mean value of each cluster:

\[
m_1 \leq m_2 \leq \cdots \leq m_K.
\]

2. Define \((K - 1)\) thresholds:

\[
\rho_i = \frac{m_i + m_{i+1}}{2}, \quad i = 1, \ldots, K - 1.
\]
Numerical Aspects: Stage Two

Stage 1: solve \( \hat{g} \)
\[
\min_g \left\{ \frac{\lambda}{2} \| f - Ag \|_2^2 + \frac{\mu}{2} \| \nabla g \|_2^2 + \| \nabla g \|_1 \right\}
\]

Stage 2: determine thresholds from \( \hat{g} \)
\[ \rho_1 \leq \cdots \leq \rho_{K-1} \]

Other Ways

1. Choose by the user: \( \rho^U \).
2. Two-phase: \( \rho^M = \text{mean}(\hat{g}) \).
Advantages of the 2-Stage Method

Given $f$

**Stage 1:** solve $\hat{g}$

$$\min_g \left\{ \frac{\lambda}{2} \| f - Ag \|_2^2 + \frac{\mu}{2} \| \nabla g \|_2^2 + \| \nabla g \|_1 \right\}$$

**Stage 2:** determine thresholds from $\hat{g}$

$$\rho_1 \leq \cdots \leq \rho_{K-1}$$

**Advantages**

- Stage 1 model for finding $\hat{g}$ is convex
- Stage 2 uses the same $\hat{g}$ when thresholds $\rho_i$ or $K$ change (No need to recompute $g$)
- No need to fix $K$ at the very beginning
- Easily adapted to blurry and noisy images

$K$ phases
1. Mumford-Shah Model

2. Our Two-stage Image Segmentation Method

3. Experimental Results
   a. Two-Phase Segmentation
   b. Multi-Phase Segmentation

4. Extensions to Other Noise Models

5. Conclusions
Anti-mass Image: Stage 1 Solution

Given image

Our solution \( \hat{g} \)
Anti-mass Image: Results Comparison

- **Given image**
- **Chan-Vese (01)**
- **Our:** $\rho^M = 0.1898$
- **Dong et al. (10)**
- **Yuan et al. (10)**
- **Our:** $\rho_1 = 0.2669$
Anti-mass Image: Our Results

\[ \rho^M = 0.1898 \quad \rho^U = 0.2 \quad \rho_1 = 0.2669 \]

Different thresholds give different meaningful segmentation results. No need to solve the convex model again when thresholds changed.
Tubular Image: Stage 1 Solution

Given image

Our solution $\hat{g}$
Tubular Image: Results Comparison

Chan-Vese (01)  

Yuan et al. (10)  

\( \rho_1 = 0.4019 \)

Dong et al. (10)  

Cai et al. (13)  

\( \rho^M = 0.1760 \)
Motion Blurred and Noisy Image

Clean image

Given blurred image

Chan-Vese (01)

Dong et al. (10)

Yuan et al. (10)

$\rho_1 = 0.5048$
Motion Blurred and Noisy Image

\[ \rho^M = 0.7661 \quad \rho^U = 0.6 \quad \rho_1 = 0.5048 \]

Robust with respect to the thresholds chosen
\( \Gamma \) changes as \( \rho \) changes. But no need to solve for \( \hat{g} \) again.
Convergence History

Log of difference in iterates versus CPU time

Anti-mass image

Motion blurred and noisy image

Our method is very stable.
**CPU Time**

**Two-phase:** iteration numbers and CPU time in second

<table>
<thead>
<tr>
<th>Example</th>
<th>C-V (01)</th>
<th>Dong (10)</th>
<th>Yuan (10)</th>
<th>Our method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>iter.</td>
<td>time</td>
<td>iter.</td>
<td>time</td>
</tr>
<tr>
<td>Anti-mass</td>
<td>1000</td>
<td>263.73</td>
<td>300</td>
<td>83.82</td>
</tr>
<tr>
<td>Tubular</td>
<td>1000</td>
<td>76.62</td>
<td>300</td>
<td>32.17</td>
</tr>
<tr>
<td>Motion</td>
<td>1300</td>
<td>28.19</td>
<td>300</td>
<td>10.18</td>
</tr>
</tbody>
</table>

Our method is faster than others except Yuan’s, but our segmentation results are better.
1. Mumford-Shah Model
2. Our Two-stage Image Segmentation Method
3. Experimental Results
   a. Two-Phase Segmentation
   b. Multi-Phase Segmentation
4. Extensions to Other Noise Models
5. Conclusions
Three-phase Segmentation

- Given image
- Li et al. (10)
- Sandberg et al. (10)
- Yuan et al. (10)
- Our solution \( \hat{g} \)
- Our 3 phases from \( \hat{g} \) using K-means \( \rho_i \)
Four-phase Segmentation: Noisy image

Given noisy image

Yuan et al. (10)

Li et al. (10)

Sandberg et al. (10)

Steidl et al. (12)

Our 4 phases from \( \hat{g} \) using K-means \( \rho_i \)
Four-phase Segmentation: Noisy and blurry image

Noisy & blurry

Yuan et al. (10)

Li et al. (10)

Sandberg et al. (10)

Steidl et al. (12)

Our 4 phases from $\hat{g}$ using K-means $\rho_i$
Convergence History

Log of difference in iterates versus CPU time

Three-phase image

Four-phase noisy image

Our method is very stable.
**CPU Time**

**Multiphase:** iteration numbers and CPU time in second

<table>
<thead>
<tr>
<th>Example</th>
<th>Li (10)</th>
<th>Sandberg (10)</th>
<th>Yuan (10)</th>
<th>Our method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>iter.</td>
<td>iter.</td>
<td>iter.</td>
<td>iter.</td>
</tr>
<tr>
<td>3-phase</td>
<td>100</td>
<td>2</td>
<td>32</td>
<td>62</td>
</tr>
<tr>
<td>4-phase</td>
<td>100</td>
<td>12</td>
<td>134</td>
<td>112</td>
</tr>
<tr>
<td>4-phase-blur</td>
<td>100</td>
<td>13</td>
<td>57</td>
<td>78</td>
</tr>
</tbody>
</table>

Our method is the best and fastest for multiphase segmentation.

Real MRI Brain Image with CPU Timing

MRI brain image

Yuan et. al. (10)

Li et. al. (10)

Sandberg et. al. (10)

Steidl et. al. (12)

Our using $\rho^K_i$
1. Mumford-Shah Model
2. Our Two-stage Image Segmentation Method
3. Experimental Results
4. Extensions to Other Noise Models
5. Conclusions
Poisson and Multiplicative Gamma Noises

- Poisson noise: observed image $f(x)$ follows

$$p_{f(x)}(n; g(x)) = \frac{(g(x))^n e^{-g(x)}}{n!}$$

with mean $g(x)$.

- Multiplicative Gamma noise: $f = g \cdot \eta$ where $\eta(x)$ follows:

$$p_{\eta(x)}(y; \theta, L) = \frac{1}{\theta^L \Gamma(L)} y^{L-1} e^{-\frac{y}{\theta}} \text{ for } y \geq 0.$$  

with mean 1 and variance of $\frac{1}{L}$. 

Two-stage Method:

First stage: given $f$, solve

$$
\min_g \left\{ \lambda \int_{\Omega} (Ag - f \log Ag) dx + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 dx + \int_{\Omega} |\nabla g| dx \right\}.
$$

- data fitting term good for Poisson noise from MAP analysis
- also suitable for Gamma noise (Steidl and Teuber (10))
- objective functional is convex (solved by Chambolle-Pock)
- admits unique solution if $\text{Ker}(A) \cap \text{Ker}(\nabla) = \{0\}$.

Second stage: threshold the solution $\hat{g}$ to get the phases.
3-object Image with Poisson Noise and Motion Blur

Original image

Noisy & blurred

Yuan et al. (10)

Dong et al. (10)

Sawatzky et al. (13)

Our method
Tree with Gamma Noise with Gaussian Blur

Original image  Noisy & blurred  Yuan et al. (10)

Dong et al. (10)  Sawatzky et al. (13)  Our method
Boat with Poisson Noise

Noisy image

Yuan et al. (10)

Dong et al. (10)

Sawatzky et al. (13)

With $\mu = 0.05$, $\rho^K = 142$

With $\mu = 0$, $\rho^K = 104$
Segmentation Changes with Threshold

Real cell image from an automated cell tracking system.
### CPU Time

2-phase: iteration numbers and CPU time in second

<table>
<thead>
<tr>
<th>Test</th>
<th>Yuan* iter.</th>
<th>Yuan* time</th>
<th>Dong* iter.</th>
<th>Dong* time</th>
<th>Sawatzky iter.</th>
<th>Sawatzky time</th>
<th>Our method iter.</th>
<th>Our method time</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-Object</td>
<td>22</td>
<td>0.17</td>
<td>500</td>
<td>40.7</td>
<td>13</td>
<td>37.2</td>
<td>325</td>
<td>4.1</td>
</tr>
<tr>
<td>Tree</td>
<td>39</td>
<td>4.1</td>
<td>500</td>
<td>190.6</td>
<td>14</td>
<td>660.1</td>
<td>263</td>
<td>18.9</td>
</tr>
<tr>
<td>Boat</td>
<td>54</td>
<td>2.1</td>
<td>500</td>
<td>189.6</td>
<td>13</td>
<td>324.5</td>
<td>61</td>
<td>1.5</td>
</tr>
<tr>
<td>Anti-mass</td>
<td>51</td>
<td>5.1</td>
<td>500</td>
<td>138.0</td>
<td>9</td>
<td>137.8</td>
<td>80</td>
<td>3.2</td>
</tr>
<tr>
<td>Cells</td>
<td>46</td>
<td>6.4</td>
<td>500</td>
<td>255.5</td>
<td>17</td>
<td>1546.2</td>
<td>101</td>
<td>6.3</td>
</tr>
<tr>
<td>Bacteria</td>
<td>51</td>
<td>5.1</td>
<td>500</td>
<td>189.6</td>
<td>12</td>
<td>548.7</td>
<td>74</td>
<td>3.9</td>
</tr>
</tbody>
</table>

*Yuan’s and Dong’s algorithms were applied on images after Anscombe transformation.*
Airplane with Multiplicative Gamma Noise

Original image

Noisy image

Yuan et al. (10)

Li et al. (10)

Our method with $\rho^K_i$ from K-means
4-phase under Gamma Noise with Gaussian Blur

Original image

Noisy & blurred

Yuan et al. (10)

Li et al. (10)

Our method with $\rho_i^K$
4-phase with Close Intensity under Poisson Noise

Original image

Poisson noise

intensities of segmented images enlarged to reflect the details.

Yuan et al. (10)

Li et al. (10)

Our method with $\rho^K_i$
Image with Close and Varying Intensities
Real MRI Image

Original image

Yuan et al. (10)

Li et al. (10)

Our method with $\rho^K_i$
# CPU Time

**Multi-phase:** iteration numbers and CPU time in second

<table>
<thead>
<tr>
<th>Test</th>
<th>Yuan iter.</th>
<th>Yuan time</th>
<th>Li iter.</th>
<th>Li time</th>
<th>Our Method iter.</th>
<th>Our Method time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airplane</td>
<td>127</td>
<td>1.0</td>
<td>95</td>
<td>1.0</td>
<td>86</td>
<td>0.2</td>
</tr>
<tr>
<td>4-phase</td>
<td>57</td>
<td>2.2</td>
<td>49</td>
<td>1.6</td>
<td>184</td>
<td>2.3</td>
</tr>
<tr>
<td>Close-intensity</td>
<td>34</td>
<td>1.8</td>
<td>110</td>
<td>4.0</td>
<td>84</td>
<td>0.5</td>
</tr>
<tr>
<td>Varying-intensity</td>
<td>114</td>
<td>4.4</td>
<td>332</td>
<td>9.9</td>
<td>444</td>
<td>3.0</td>
</tr>
<tr>
<td>MRI</td>
<td>76</td>
<td>25.7</td>
<td>114</td>
<td>26.4</td>
<td>19</td>
<td>0.6</td>
</tr>
</tbody>
</table>

C., Yang, and Zeng, *A two-stage image segmentation method for blurry images with Poisson or multiplicative Gamma noise,* Accepted by SIAM J. Imag. Sci.
1. Mumford-Shah Model
2. Our Two-stage Image Segmentation Method
3. Experimental Results
4. Extensions to Other Noise Models
5. Conclusions
Relationship with Image Restoration

Convex M-S Energy

\[ E(g) \]

\[ \frac{\lambda}{2} \int_{\Omega} (f - Ag)^2 \, dx \]  
\[ + \quad \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 \, dx \]  
\[ + \quad \int_{\Omega} |\nabla g| \, dx \]

reduce by introducing higher-order derivative:
T. Chan (00), Lysaker (03), Steidl (08), Bredies (10), Hintermüller (10), etc.

ROF Model (1992)
Rudin, Osher and Fatemi

staircase
Relationship with Image Restoration

\[ \frac{\lambda}{2} \int_{\Omega} (f - Ag)^2 dx + \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 dx + \int_{\Omega} |\nabla g| dx \]

Hintermüller (2010): image restoration model
ROF + Thresholding = Chan-Vese

Solve ROF model in the 1st stage:

\[
\min_g \left\{ \frac{\lambda}{2} \int_{\Omega} (f - g)^2 + \int_{\Omega} |\nabla g| \right\}
\]

for \( \tilde{g} \). Then threshold \( \tilde{g} \) properly in the 2nd stage.

One can get a 2-phase segmentation \( (\Sigma, c_1, c_2) \) which satisfies the Chan-Vese model:

\[
\min_{\Sigma, c_1, c_2} \left\{ \frac{\mu}{2} \int_{\Sigma} (f - c_1)^2 + \frac{\mu}{2} \int_{\Omega \setminus \Sigma} (f - c_2)^2 + \text{Length}(\partial \Sigma) \right\}
\]

- See [Cai & Steidl, EMMCVPR, 2013]
- \( \text{Length}(\Gamma) \approx \int_{\Omega} |\nabla g| \)
Conclusions

- Look for **smooth** solutions of Mumford-Shah model

- **Convex** segmentation model with **unique** solution — can be solved easily and fast

- Model solved only once — no need to solve the model again when threshold or number of phases changes

- Easily **extendable** to blurry images and non-Gaussian noise

- Link **image segmentation** and **image restoration**
References


URL: www.math.cuhk.edu.hk/~rchan
Happy 75th Birthday Bob!
Welcome to Hong Kong
and Chinese University of Hong Kong

Thank you!