An Optimal Sequential Optimization Approach in Application to Dynamic Weapon Allocation in Naval Warfare

Dany Dionne*, Edward Pogossian†, Arthur Grigoryan†, Jean Couture* and Elisa Shahbazian*

*R&D Department, Lockheed Martin Canada, Montreal, Canada
†Institute for Problems of Informatics and Automation, Academy of Sciences of Armenia, Yerevan, Armenia

Abstract—A sequential optimization technique is presented in application to the dynamic weapon-target allocation problem. This problem is in general NP-Complete. The purpose of the optimization technique is to decompose the original search space into a sequence of smaller subspaces. Optimization over these subspaces (still NP-Complete) is of reduced dimensionality. The sequential decomposition technique is proven to preserve optimality. This sequential optimization technique is applied to an illustrative weapon-target allocation example relevant to naval warfare.

Keywords: Resource management, optimization

I. INTRODUCTION

Combat platforms in modern naval warfare deal with many and varied threats in a diversity of complex and dynamic operational scenarios and environments. The core of these naval combat platforms is a command and control system (CCS) used to support the information gathering and decision making processes. The CCS objective is to ensure survival and mission success.

A successful CCS in modern warfare requires increasingly rapid and effective completion of its cycles. This mandate stems from two key observations: (a) the speed, complexity, and number of events is ever increasing, and (b) a distinguishing factor of success is for friendly combat platforms to complete their operation cycles faster and more effectively than those of the adversary [1].

A fundamental cycle of the CCS is the Resource Management (RM) support that plans for the allocation and scheduling of resources. A RM cycle involves two steps: (a) the construction of a decision tree describing the admissible countermeasures against the possible threat situations, and (b) the selection of a defense plan (i.e., a strategy) by optimization of paths in the decision tree.

In general, the RM plan needs to be repetitively updated on-line due to: (a) the continuously evolving environment, (b) the incomplete and uncertain incoming information, and (c) the uncertainty about the outcome of the decisions made. These repetitive on-line optimizations are constrained by the time allowed to compute a plan.

Several RM approaches have been proposed, they can be broadly classified into two categories: reactive planning, and deliberative planning. Reactive planning approaches use low-level reasoning techniques for a simple response to a situation. Their two major benefits are the simplicity of the approach, and the guarantee that a plan will be available within a short period of time. It is overwhelmingly the usual approach to planning in naval RM. Examples of such systems are the Rapid Anti-ship missile Integrated Defense System (RAIDS) [2], and the Reactive Resource Allocation at Single Ship Level (RRASSL) system [3]. RAIDS is a reactive rule-based planning system that exploit a set of naval doctrines under the real-time stress of anti-ship missiles. RRASSL is a prototype combat decision aid that employs an expert-system reactive planner for the ship’s hardkill and softkill weapons against an attack by anti-ship missiles.

Deliberative planning approaches involve high-level reasoning techniques over a planning time horizon. There main benefit is to extensively optimize the resources to foster survival and mission completion. Deliberative planning in naval RM is subdivided into two categories: static weapon-target allocation (WTA), and dynamic WTA. In static WTA, all weapons are assigned and fired simultaneously; it means that the planning of the future weapons’ allocation is independent of the outcome of the previous allocations. In dynamic WTA, the planner explicitly accounts for future allocations of weapons that are conditional on the outcomes of the previous allocations [4]; those outcomes are typically assessed through a repetitive “shoot-look-shoot” approach where threats are acquired, then weapons are engaged, and the assessment of the outcomes follows. In both static and dynamic WTA, the optimization problem is NP-Complete [5].

Two examples of deliberative planners for dynamic WTA are provided in Refs [6], [7]. In Ref. [6], a Tabu search algorithm is employed for planning, while in Ref. [7], a genetic algorithm is exploited. Both were demonstrated to better optimize plans than a reactive planner. However, these deliberative planners required an exceedingly large amount of time when the number of threats is large (typically, larger than five threats).

In this paper, a novel RM deliberative planning algorithm for dynamic WTA is proposed. The algorithm decomposes the search space into a sequence of subspaces of smaller dimension. The objective of the decomposition is to increase numerical efficiency, while preserving optimality. The two main contributions are the decomposition technique and the theoretical assessment of the RM solution optimality. The application of the sequential optimization algorithm is illustrated in an example.
highly scenario dependent (e.g., number of weapons, weapon
is guaranteed for all
plans in the sense that satisfaction of the engagement rules
and (b) the outcome at
tgagement rules. Let
S
if: (a) the outcome at
trealization is the lowest branch of the tree which will happe n
realizations of the plan based on those outcomes. An example
outcomes per weapon is depicted in Fig. 1. Since the plan
weapon allocations with specified outcomes.

A defense plan, D, prescribes the allocation of the defense
weapons over a time interval. All the possible outcomes for
the allocated weapons are accounted for by the plan, making
the plan in the form of a tree with \( N_{\text{leaf}} \) branches. This tree is
a solution of the dynamic WTA. Each branch represents one
attack on the ship. The ship engages the

Let \( p_{\text{surv}} : D \rightarrow [0,1] \) be the probability of survival of the
ship using plan \( D \in S_D \). The output of the RM algorithm
is a defense plan \( D^* \in S_D \) that maximizes \( p_{\text{surv}} \), i.e.,

\[
D^* = \arg \max_{D \in S_D} p_{\text{surv}}(D)
\]

The optimum \( D^* \) is in general non-unique.

The following assumptions are made:

(i) There is a finite number of defense weapons, \( N_{\text{weapon}} \),
and of threats, \( N_{\text{threat}} \).

(ii) The outcome of the threats’ action and of the planned
weapons is unknown at the current time instant. How-
ever, each outcome is either a success or a miss.

(iii) Each defense weapon has a known, mutually indepen-
dent, probability of success, \( p_{\text{success}} \).

(iv) Prior to a strike on the ship, a threat must survive
to all the defense weapons. Upon a strike, the given
probability of success (i.e., the ship being destroyed) is
\( p_{\text{success}} \); this probability is independent of the defense
actions and of the other threats.

(v) A single defense action can affect only a single threat.

III. RESOURCE MANAGEMENT

The proposed resource management approach is described
in the four following subsections. In § III-A, the optimization’s
cost function is derived with respect to the assumptions (i)-(v)
enumerated in § II. In § III-B, theoretical optimization
results are derived. The steps of the optimization algorithm
are then presented in § III-C. An assessment of the benefits
and drawbacks of the approach follows in § III-D.

A. Probability of survival

The optimization of the defense plan requires a metric for
performance evaluation. The adopted metric is the survival
probability of the ship. In general, such probability is function
of the number of defense weapons and of threats, on their
probability of success, and on the engagement rules. The
survival probability is here calculated by Prop. 3.1. 

Proposition 3.1: Let the assumptions in § II be satisfied. Let
\( x_i, i \in \{1, \ldots, N_{\text{leaf}}\} \), be a realization of a plan \( D \). Let
\( N_{\text{action}} \) be the total number of defense actions in realization
\( x_i \).

Then, the expected probability of survival of the ship using
plan \( D \) is

\[
P_{\text{surv}}(D) = \sum_{i=1}^{N_{\text{leaf}}} p(x_i) \alpha_i
\]

with

\[
p(x_i) = \prod_{k=1}^{N_{\text{action}}} p_{\text{outcome}}(k|x_i)
\]

\[
\alpha_i = \prod_{j=1}^{N_{\text{threat}}} (1 - p_{\text{surv}}(j|x_i) p_{\text{threat}}(j))
\]
and
\[
P_{\text{outcome}}(k|x_i) = \begin{cases} 
P_{\text{success}}(k), & \text{if } x_i \text{ specifies success of action } k \\ 
1 - P_{\text{success}}(k), & \text{otherwise} 
\end{cases}
\]
(2d)
where \( p(x_i) \) is the probability of realization \( x_i \); \( \alpha_t \) is the probability of the ship surviving to the strike of all the threats that reach it under realization \( x_i \); \( P_{\text{outcome}}(k|x_i) \) is the probability of the outcome specified by \( x_i \) for weapon \( k \); \( P_{\text{surv}}(j|x_i) \) indicates whether \( x_i \) survives under \( x_i \); and \( P_{\text{success}} \) and \( P_{\text{threat}} \) are given.

**Proof:**
Consider two random variables, \( X \) and \( Y \).

Let \( X : D \rightarrow x_i \) be the mapping of a defense plan \( D \) into a realization \( x_i \), \( i \in \{1, \ldots, N_{\text{leaf}}\} \). By virtue of assumption (ii), such mapping is random due to the unknown outcomes of the defense weapons. By virtue of assumption (i), \( X \) takes value in a finite discrete sample space. Let \( p(x_i) \triangleq P(X = x_i) \) be the probability mass function (p.m.f.) of \( X \).

Let \( Y : x_i \rightarrow y, y \in \{0, 1\} \) be a binary random variable that represents the state of the ship after the strike of all the threats that survived \( x_i \). A value \( y = 1 \) indicates the ship survived to all the strikes, while \( y = 0 \) otherwise. By virtue of assumptions (ii) and (iv), \( Y \) is random because: a) the list of threats that reach (strike) the ship is determined by the random variable \( X \), and b) the outcome of each threat striking the ship is itself random. Let \( p(y, x_i) \triangleq P(Y = y, X = x_i) \) be the p.m.f. of \( Y \). By virtue of \( X \) being independent of \( Y \), the p.m.f. of \( Y \) can be rewritten in the form of a conditional p.m.f., i.e., \( p(y, x_i) = p(y|x_i) \triangleq P(Y = y|X = x_i) \).

The probability of survival of the ship can be expressed as \( P_{\text{surv}}(D) = P(Y = 1) \). By virtue of assumption (iii), the total probability theorem can be applied to deliver
\[
P_{\text{surv}}(D) = P(Y = 1) = \sum_{i=1}^{N_{\text{leaf}}} p(x_i)P(Y = 1|x_i)
\]
(3)

**Calculation of the p.m.f. \( p(x_i) \)**
A realization \( x_i \) defines a sequence of \( N_{\text{action}} \) specific outcomes. By virtue of assumption (iii), the probability of each of these outcomes, \( p_{\text{outcome}}(k|x_i), k \in \{1, \ldots, N_{\text{action}}\} \), is mutually independent and known. Hence, using the chain rule, the value of \( p(x_i) \) is
\[
p(x_i) = \prod_{k=1}^{N_{\text{action}}} p_{\text{outcome}}(k|x_i)
\]
(4)

**Calculation of \( P(Y = 1|x_i) \)**
By definition, the survival of the ship, \( Y = 1 \), requires that the ship survives to all the threats, i.e.,
\[
\{Y = 1\} = \bigcap_{j=1}^{N_{\text{threat}}} \{B_j = 1\}
\]
(5)
where \( B_j = 1 \) is defined as the ship surviving to threat \( j \) (otherwise, \( B_j = 0 \)). Hence,
\[
P(Y = 1|x_i) = P \left( \bigcap_{j=1}^{N_{\text{threat}}} B_j = 1 \mid x_i \right)
\]
(6)

Since the threat outcomes are mutually independent by virtue of assumption (iv), the chain rule is applied to deliver
\[
P(Y = 1|x_i) = \prod_{j=1}^{N_{\text{threat}}} P(B_j = 1 \mid x_i)
\]
(7)

In general, only a subset of threats reach the ship (because some are intercepted by the defense weapons). When a threat \( j \) reaches the ship, the ship survives to it with a probability calculated according to assumption (iv). Otherwise, when a threat fails to reach the ship, the ship survives to it with probability one, i.e.,
\[
P(B_j = 1) = \begin{cases} 1 - P_{\text{threat}}(j), & \text{if threat } j \text{ reaches the ship} \\ 1, & \text{otherwise} \end{cases}
\]
(8)

The outcomes of the defense weapons being specified under realization \( x_i \), the subset of threats reaching the ship is also known under \( x_i \). Then, the Eqs. (7) and (8) can be combined, delivering Eqs. (2c) and (2e).

**B. Theoretical optimization results**

Theoretical results are presented for the optimization problem defined in Eq. (1) when the cost function is of the form of Eq. (2a). The solution to this optimization problem is here forth referred to as the global optimum.

The first theoretical result is Prop. 3.2 where: (a) the original optimization problem is replaced by another one within a subspace of smaller dimension, and (b) the optimal solution within this subspace is guaranteed to be a part of the global optimum. In reference to Fig. 1, the global optimum is the whole plan \( D^* \), and the Prop. 3.2 delivers only one of the branches of this plan.

The second theoretical result is Prop. 3.3 where: (a) the original optimization problem is also replaced by another one within a subspace of smaller dimension, (b) the selected subspace is orthogonal to the previous subspace(s), and (c) the optimal solution within this orthogonal subspace is also guaranteed to be a part of the global optimum. In reference to Fig. 1, a Prop. 3.3 solution delivers a subset of the branches (in general, more than one).

In order to state the Prop. 3.2, let the sets \( S_x \) and \( S_x^{\text{WC}} \) be defined as follow. Let the set \( S_x \) contain all the feasible realizations of defense actions. A realization is feasible provided that it satisfies the engagement rules. Notice that the difference
between $S_x$ and $S_D$ (the set of feasible plans) is that: (a) a plan is a collection of feasible realizations, and (b) there are additional feasibility constraints imposed by this collection since a plan must prevent allocation conflicts between two realizations. For example, consider a realization allocating weapon $A$ to a threat, while another allocates weapon $B$ to this threat; a plan cannot contain both realizations simultaneously since they conflict in terms of the allocation to that threat.

Let $S_x^{WC}$ be a subset of $S_x$ containing only the worst-case realizations, $S_x^{WC} \subseteq S_x$; a worst-case realization is identified when a realization specifies that the outcome of all the defense actions is a failure to intercept the threats.

**Proposition 3.2:** Let the global optimization problem be given by Eq. (1) and let the global cost function be of the form of Eq. (2). Let the plan $D^*$ be the global optimum. Consider $S_x^{WC}$, the set of all the feasible worst-case realizations.

Let $x_1^* \in S_x^{WC}$ be the optimal worst-case realization in the sense that it minimizes the probability of its occurrence, i.e.,

$$x_1^* = \arg \min_{x_1 \in S_x^{WC}} p(x_1) \quad (9)$$

Then,

$$x_1^* \in D^* \quad (10)$$

That is, the optimal defense plan necessarily contains $x_1^*$.

**Proof:** From Eq. (1), the cost function $P_{\text{ship}}^{\text{surv}}$ is

$$P_{\text{surv}}^{\text{ship}}(D) = \sum_{i=1}^{N^{\text{leaf}}} p(x_i) \alpha_i, \quad \alpha_i \triangleq P(Y = 1|x_i) \quad (11)$$

where the value of $\alpha_i$ is conditioned on the outcomes of the defense actions in $x_i \in D$.

Without loss of generality, let $\alpha_1$ denote

$$\alpha_1 = \min \{\alpha_i, \quad i \in \{1, \ldots, N^{\text{leaf}}\} \quad (12)$$

The minimum $\alpha_1$ is necessarily unique because: (a) the minimum of $\alpha_i$ necessarily occurs when the outcomes of all the defense actions fail to intercept the threats (i.e., all the threats strike the ship), and (b) any defense plan $D$ necessarily contains a single realization $x_i$ with such outcomes (i.e., the worst-case realization). Then,

$$P_{\text{surv}}^{\text{ship}}(D) = p(x_1) \alpha_1 + \sum_{i=2}^{N^{\text{leaf}}} p(x_i) \alpha_i \quad (13a)$$

$$= p(x_1) \alpha_1 + (\alpha_1 + \delta) \sum_{i=2}^{N^{\text{leaf}}} p(x_i) \quad (13b)$$

$$= p(x_1) \alpha_1 + (\alpha_1 + \delta) (1 - p(x_1)) \quad (13c)$$

where $\delta > 0$ and $x_1 \in S_x^{WC}$. Notice that the value of $\alpha_1$ is necessary the same for all $x_1 \in S_x^{WC}$ because in all those realizations, all the threats always reach the ship.

Consider

$$\max_{D \in S_D} P_{\text{surv}}^{\text{ship}}(D) = \max_{D \in S_D} \left( p(x_1) \alpha_1 + (\alpha_1 + \delta) (1 - p(x_1)) \right) \quad (14)$$

Since $(\alpha_1 + \delta) > \alpha_1$, it necessarily follows that

$$\arg \max_{p(x_1) \in S^{\text{WC}}} P_{\text{surv}}^{\text{ship}}(D) = \min_{x_1 \in S_x^{\text{WC}}} p(x_1) \quad (15)$$

Then, from the definition (9),

$$\arg \max_{p(x_1) \in S^{\text{WC}}} P_{\text{surv}}^{\text{ship}}(D) = p(x_1^*) \quad (16)$$

Since $P_{\text{surv}}^{\text{ship}}(D^*) = \max_{D \in S_D} P_{\text{surv}}^{\text{ship}}(D)$, it follows that

$$\arg \max_{p(x_1)} P_{\text{surv}}^{\text{ship}}(D^*) = p(x_1^*) \quad (17)$$

Thus, it is necessary that $x_1^* \in D^*$.

The following proposition generalizes the Prop. 3.2 when some branches of the optimal plan are already known. In order to state the Prop. 3.3, let define a partial plan, the set $\Omega$, and the notion of $\omega$-completeness. A partial plan contains a collection of $m$ realizations, but this collection is not complete; completeness requires the collection to contain a realization for all the possible allocation outcomes. Hence, $m < N^{\text{leaf}}$. With respect to Fig. 1, a partial plan is missing some of the branches.

The set $\Omega$ contains all the different values that can be adopted by $\alpha_i$, $i \in \{1, \ldots, N^{\text{leaf}}\}$, as calculated by Eq. (2c). Let $s$ be the number of different values, i.e., $\Omega = \{\omega_1, \ldots, \omega_s\}$. Let these elements be enumerated in increasing value order so that the worst case value, $\alpha_1$, is given by $\alpha_1 = \omega_1$. In general, $s \leq N^{\text{leaf}}$ since several $\alpha_i$ may share the same value.

The notion of $\omega$-completeness applies to partial plans. A partial plan that is said to be $\omega$-complete up to some value $\omega_p \in \Omega$ means that the partial plan cannot contain anymore realizations with $\alpha_i \leq \omega_p$. For example, a partial plan containing only the worst-case realization is necessarily $\omega$-complete up to $\omega_1$ since any plan contains only one worst-case realization.

**Proposition 3.3:** Let $\Omega \subset \Omega$ be a subset containing the first $r$ values in $\Omega$, i.e., $\Omega = \{\omega_1, \ldots, \omega_r\}$, $r < s$.

Let $D^*$ be a partial plan containing $m$ realizations: $x_i$, $i \in \{1, \ldots, m\}$. Let these realizations be also parts of the optimal plan $D^*$ such that $D^* \subset D^*$. Let $D^*$ be $\omega$-complete up to $\omega_r$. Let $X$ be a subset of realizations such that $D^* \oplus X$ is $\omega$-complete up to $\omega_{r+1}$. Let $S_X^D \subset S_X$ contains only the feasible $X$ subsets that do not conflict with the partial plan $D^*$.

Let $X^*$ be the optimal realizations’ subset in the sense that

$$X^* = \arg \min_{X \in S_X^D} \sum_{j=1}^{q} p(x_j) \quad (18)$$

where $q$ is the number of realizations in $X$ (two different subsets $X$ can have different values of $q$). Then,

$$X^* \in D^* \quad (19)$$

That is, the optimal defense plan necessarily contains the set of realizations $X^*$. 

619
Thus, it is necessary that
\[ X = D^* \oplus X \oplus Z \] (20)
where the subset \( Z \) contains any feasible set of realizations that permit to complete the plan.

The cost function in Eq. (1) can be written
\[
P_{\text{surv}}^{\text{ship}}(D) = \sum_{i=1}^{m} p(x_i^1) \alpha_i + \sum_{j=m+1}^{m+q} p(x_j) \alpha_j + \sum_{l=m+q+1}^{N_{\text{lead}}} p(x_l) \alpha_l
\]
where \( x_i^1 \in D^*, x_j \in X, \) and \( x_l \in Z. \)

By definition of \( X, \) all realizations \( x_j \in X \) share the same value \( \alpha_j = \omega_{r+1}. \) Moreover, since the partial plan \( (D^* \oplus X) \) is \( \omega \)-complete up to \( \omega_{r+1}, \) any realization \( x_l \in Z \) necessarily has \( \alpha_l > \omega_{r+1}. \) Hence, the cost function can be re-written
\[
P_{\text{surv}}^{\text{ship}}(D) = \sum_{i=1}^{m} p(x_i^1) \alpha_i + \omega_{r+1} \sum_{j=m+1}^{m+q} p(x_j) + (\omega_{r+1} + \delta) \sum_{l=m+q+1}^{N_{\text{lead}}} p(x_l)
\]
(22a)
\[
= \sum_{i=1}^{m} p(x_i^1) \alpha_i + \omega_{r+1} \sum_{j=m+1}^{m+q} p(x_j) + (\omega_{r+1} + \delta) \left( 1 - \sum_{i=1}^{m} p(x_i^*) \right) \sum_{j=m+1}^{m+q} p(x_j)
\]
(22b)
where \( \delta > 0. \)

Since \( D^* \subset D^* \), it is necessary that
\[
\max_{D \in S_D} p_{\text{surv}}^{\text{ship}}(D) = \sum_{i=1}^{m} p(x_i^1) \alpha_i + \max_{D \in S_D} \left[ \omega_{r+1} \sum_{j=m+1}^{m+q} p(x_j) + (\omega_{r+1} + \delta) \left( 1 - \sum_{i=1}^{m} p(x_i^*) \right) \sum_{j=m+1}^{m+q} p(x_j) \right]
\]
(23)

Since \( \sum_{i=1}^{m} p(x_i^1) \) is constant and \( (\omega_{r+1} + \delta) > \omega_{r+1}, \) it follows that
\[
\arg \max_{X^* \in S_D} p_{\text{surv}}^{\text{ship}}(D) = \min_{x \in S^*_X} \sum_{j=m+1}^{m+q} p(x_j) = p(X^*)
\] (24)

Thus, it is necessary that \( X^* \in D^*. \)

C. Global optimization algorithm

The global optimization algorithm partitions the search space into a sequence of (orthogonal) lower dimensional subspaces. Within each subspace, some local optimization algorithm is selected to deliver a local optimum. The algorithm ends when there is no subspace left; the global optimum is obtained from the combination of the local optimizers.

The optimization algorithm involves the following steps

i. Construct the set \( \Omega = \{ \omega_1, \cdots, \omega_s \}. \) It requires the value of \( p_{\text{threat}}^{\text{success}}(T) \) for each of the threats. The number \( s \) of elements in \( \Omega \) is the number of subspaces. The subspace being optimized is designated by the index \( k. \) Define an initial set \( \Omega(1) = \Omega \) and set \( k = 1. \)

ii. Select the minimum value element in \( \Omega(k). \) If this minimum is \( \omega_1, \) apply Prop. 3.2, the output is then \( x^*_1. \) Otherwise, apply Prop. 3.3 and the output is \( X^*. \)

iii. Build a partial optimal plan, \( D^*(k) \), from the output of step (ii), i.e.,
\[
D^*(k) = \begin{cases} 
  x^*_1, & \text{if output is } x^*_1 \\
  D^*(k-1) \oplus X^*, & \text{otherwise}
\end{cases}
\]
(25)

The optimization over each subspace is defined by the Props. 3.2 and 3.3 in step (ii). Any algorithms can be employed for this subspace optimization. However, to guarantee optimality of the global solution, each subspace optimization must deliver the local optimum within its subspace.

D. Benefits and drawbacks of the optimization algorithm

The four main benefits of this algorithm are:

- The search space is significantly reduced by partitioning a plan \( D \) into subsets and by optimizing only over these subsets rather than the whole plan.
- Global optimality is preserved.
- Graceful degradation: interrupting the algorithm before completing \( D^* \) still provides parts of the optimal \( D^*. \)
- The worst-case subsets are optimized first (it enables a more graceful degradation).

Drawbacks of this algorithms are:

- The optimization problem within each lower dimensional subspace is still NP-complete.
- In order to guarantee global optimality, the local solution found within each lower dimensional subspace must be optimal within that subspace (in compliance with the Bellman’s principle of optimality).
- The dimension of the subspaces is not known in advance; each subspace is constructed on-line as it depends on the solutions found with the previous subspaces. Nonetheless, the number of subspaces is known, it is the number of elements in the set \( \Omega. \)

IV. ILLUSTRATIVE EXAMPLE

Consider a scenario with two threats denoted \( T_1 \) and \( T_2. \) To ease notation, let \( P_1 \triangleq p_{\text{threat}}^{\text{success}}(T_1) \) and \( P_2 \triangleq p_{\text{threat}}^{\text{success}}(T_2) \)
Let \( P_1 > P_2, \) which means that \( T_1 \) is more dangerous than \( T_2. \) To ease explanation, let the optimal plan be illustrated in Fig. 1. Let the planning time instant be \( t_0. \)

The initial step to construct the optimal plan \( D^* \) is to determine the possible strike scenarios and construct the set \( \Omega. \)
(enumerating the different survival probabilities) from it. The possible strike scenarios are: (i) \( T_1 \) and \( T_2 \) strike the ship, (ii) only \( T_1 \) strikes, (iii) only \( T_2 \) strikes, and (iv) there is no strike. Hence, \( \Omega = \{ P_1P_2, P_1, P_2, 1 \} \) since \( P_1 \) and \( P_2 \) have different values. The four elements in \( \Omega \) means that the optimal plan is constructed through a sequence of four subspace optimizations.

The first subspace optimization delivers \( D^*(1) \) and is stated:

1.1 The worst-case value in \( \Omega(1) \) is \( P_1P_2 \) (both \( T_1 \) and \( T_2 \) strike the ship).

1.2 By optimization, find a realization \( x_1^* \) where the outcome of all the defense actions is a failure to intercept the threats. Let \( x_1^* \) be the branch A in Fig. 1.

1.3 By virtue of Prop. 3.2, the realization \( x_1^* \) is a branch of the optimal plan. Hence, set \( D^*(1) = x_1^* \).

The second subspace optimization is

2.1 The worst-case value in \( \Omega(2) \) is \( P_1 \) (only \( T_1 \) strikes the ship).

2.2 By optimization, find a subset \( X^* \) where the outcome of all the defense actions is a failure to intercept \( T_1 \), but a success with \( T_2 \).

To initialize optimization with \( D^*(1) \), find all the occurrences of \( T_2 \) in \( D^*(1) \). Only one occurrence is found at instant \( t_2 \) of branch A. So, the second optimization’s output will be branches issue from that node in A.

Let \( X^* \) be the branch C in Fig. 1.

2.3 By virtue of Prop. 3.3, the realization \( X^* \) is a branch of the optimal plan. Set \( D^*(2) = D^*(1) \oplus X^* \).

The third subspace optimization repeats the previous process:

3.1 The worst-case value in \( \Omega(3) \) is \( P_2 \) (only \( T_2 \) strikes the ship).

3.2 By optimization, find a subset \( X^* \) where the outcome of all the defense actions is a success to intercept \( T_1 \), but a failure with \( T_2 \).

To initialize optimization with \( D^*(2) \), find all the occurrences of \( T_1 \) in \( D^*(2) \). Two occurrences are found: at instant \( t_1 \) and \( t_3 \), in branch A. So, the output of the third optimization will be two branches issue from these nodes in A.

Let \( X^* \) be the branches B and E in Fig. 1.

3.3 By virtue of Prop. 3.3, the realization \( X^* \) is a branch of the optimal plan. Set \( D^*(3) = D^*(2) \oplus X^* \).

The last subspace optimization delivers the optimal plan:

4.1 The worst-case value in \( \Omega(4) \) is 1 (no strike on the ship).

4.2 By optimization, find a subset \( X^* \) where the outcome of all the defense actions is a success to intercept \( T_1 \) and \( T_2 \).

To initialize optimization with \( D^*(3) \), find all the occurrences of \( T_1 \) and \( T_2 \) in \( D^*(3) \). Two occurrences are found: at instant \( t_3 \) in branches C and at instant \( t_2 \) in branch E. So, the output of the third optimization will be two branches issue from the respective node in C and E.

Let \( X^* \) be the branches D and F in Fig. 1.

4.3 By virtue of Prop. 3.3, the realization \( X^* \) is a branch of the optimal plan. Set \( D^* = D^*(3) \oplus X^* \).

To summarize, the branch A was first independently optimized. Then, a second optimization delivered branch C. The third optimization delivered both branches B and E. Finally, the last optimization delivered branches D and F.

Notice that if \( P_{\text{success}}(T_1) < P_{\text{success}}(T_2) \) the order of optimization of the branches in \( D^* \) is different, see Table I.

Moreover, if \( P_{\text{success}}(T_1) = P_{\text{success}}(T_2) \) the number of subspaces is reduced to three.

### V. Conclusion

This work presented a sequential decomposition technique for the optimization of a class of dynamic WTA problems. The main benefits of such decomposition were to reduce the search space and to inherently provide graceful degradation. The proposed sequential decomposition technique was proven to preserve global optimality provided that the optimal solution is obtained in each of the subspaces.

In a future work, numerical results from this sequential decomposition technique will be presented in a naval warfare scenario and compare with those of other deliberative and reactive planners.

### Acknowledgement

This work was financially supported by NATO.

### References


