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LOCAL GEOMETRY OF DEFORMABLE TEMPLATES*

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Abstract. In this paper, we discuss a geometrical model of a space of deformable images or shapes, in which infinitesimal variations are combinations of elastic deformations (warping) and of photometric variations. Geodesics in this space are related to velocity-based image warping methods, which have proved to yield efficient and robust estimations of diffeomorphisms in the case of large deformation. Here, we provide a rigorous and general construction of this infinite dimensional "shape manifold" on which we place a Riemannian metric. We then obtain the geodesic equations, for which we show the existence and uniqueness of solutions for all times. We finally use this to provide a geometrically founded linear approximation of the deformations of shapes in the neighborhood of a given template.

Key words. infinite dimensional Riemannian manifolds, deformable templates, shape representation and recognition, warping

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1. Introduction.

The theoretical developments which are addressed in this paper are motivated by the theory of deformable templates, as it emerged from the work of Grenander and his collaborators in the 1980's [19, 21, 22, 20], to handle image processing problems. This theory has an abstract formulation, in which the purpose is to represent the variability within an object class by the variations in shape, or color, etc., of a single object, submitted to the action of "deformations." For instance, a model designed to describe a picture of a human face should be able to explain inter-individual variations but also variations caused by the change of expression of a given individual, and by the changing of imaging conditions, such as lighting, occultations, etc. The interesting feature in Grenander's construction is that it assigns a large part, sometimes all, of the variations to a fixed structure, describing the deformation, which is independent of the particular instance of the observed image. This structure most of the time belongs to a group, the group of deformations, which is acting on the set of objects. The specific choice of the group depends on the application and on the type of visual features which are modeled, like pixelized images [18] and discretized shapes [20, 29, 23]. In such discrete settings, the group action is used to generate variations of the constituting generators of the object (pixels for an image, segments for polygons) and therefore are modeled as finite dimensional groups, generally products of linear or affine groups. In the simple example of labeled collections of points (landmarks), the deformation may simply correspond to independent translations of each point, but when the question is raised of the similarity of two collections of landmarks, one would like to figure out the amount of deformation which is required to transform one of them into the other. When evaluating this deformation, it is clear that the lengths of the induced translations should have some impact, but that this is not the

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only factor and often not even the main factor. One would also like to draw conclusions on the smoothness of the deformation, based on the fact that, in the context of large deformations of shapes, a lower similarity must be associated to a collection of translations which point to erratic directions, compared to a more homogeneous displacement. We see, in this case, that a global point of view on the displacements is needed. Spline-based landmark matching [9, 26] specifically addresses this issue by seeking the smoothest function which interpolates the considered displacements.

When dealing with image deformation, the need to pass to the continuum is even more obvious. In this case, deformations, which should provide nonambiguous point displacements, must be diffeomorphisms on the image support. This nonambiguity constraint, however, has been relaxed in most of the early attempts to deal with this issue, working preferably with linear spaces of deformations [6, 7, 8, 2, 1, 14], which can be seen as first order approximations. Dealing explicitly with true deformations, i.e., diffeomorphisms acting on the support of images, was rigorously formalized by Riemannian metric arguments on the groups of diffeomorphisms in [32] for one-dimensional problems, and in [31] in full generality (see also [30]). Stemming from the simple representation of right invariant metrics on groups of diffeomorphisms along a path in this space, i.e., time-dependent deformations, in terms of the Eulerian velocity, this last reference built diffeomorphisms as flows associated to ODEs (a construction which was already present in [3]) and transferred the modeling effort to the linear space of velocities, i.e., of vector fields defined on the image support. Under suitable Banach space structures on these linear spaces, the extension of the ODE solutions for infinite time and the existence of minimizers to general variational problems in this space can be ensured, providing rigorous sufficient conditions for the well-posedness of many practical problems in template matching. This analysis rejoined the line of work of Miller and his collaborators on the estimation of large deformation diffeomorphisms [13, 26], in which velocity-based models have been introduced, and variational properties studied in [16]. In [27], the interest in considering a lifted group action, on the cross product of the group itself and of the image space, was demonstrated in a wide variety of applications. The final metric on the image space was obtained by projecting a right-invariant Riemannian distance designed on the product space.

The approach we follow in this paper addresses the same kind of construction as in [27], which focused on the metric aspects, but from a different point of view. Our purpose is to start from the infinitesimal analysis of small deformations of images in order to model and measure image variations and define differentiable and geodesic curves in the image space. We shall accept conditions which ensure enough smoothness on the diffeomorphisms but try whenever possible to avoid placing such smoothness assumptions on the images themselves. Such a choice, which is very important given the discontinuous nature of images, is made at the cost of increasing technicalities and notation, as will be seen in section 3, in which the basic geometry of the model is presented. Here, we define the tangent space at a given square integrable image i as an equivalent class for all possible variations resulting from an infinitesimal combination of a deformation (geometry) and of the addition of a square integrable function (photometry), yielding what can be called a morphometrical variation. We then equip it with an inner product and define from it lengths and energies of curves. This metric is based on the best tradeoff between geometrical and photometrical variations. Still, in this general setting, we show the existence of minimizing geodesics (curves of minimal energy) between any two images.

The rest of the paper is devoted to the study of geodesics and their generation from initial conditions. The motivation in this study is the possibilities it offers for prototype-based image representation and the generation of image variations and deformations from initial conditions belonging to a vector space. In this context, the geodesic equations are derived under the assumption that the deformed prototype is smooth (H^1), but with no restriction on the other endpoint. This is done in section 5.2.2. The obtained evolution equations are then generalized to a form which does not require the smoothness of the initial position and that we conjecture to represent a comprehensive class of image evolutions. The equations, under this form, are studied in section 7, where we prove that they have a unique solution over arbitrary finite time intervals. Our last result shows the local nonambiguity of this representation, at least in the smooth case: from a smooth prototype, the solutions of the geodesic equations in small time cannot coincide if they have been generated from distinct smooth initial conditions. This is done in section 9. The last section, 10, presents numerical experiments, which illustrate the feasibility of retrieving a target from the initial conditions associated to the minimizing geodesic starting from the template.

2. Notation. For further reference, we present in a single definition some of the main functional spaces we use throughout the paper.

DEFINITION 1. *Let $k, p \in \mathbb{N}_*$, $l \in \mathbb{N}$, and Ω be a bounded domain of $\mathbb{R}^k$ with C^1 boundary.*

(1) *We denote $C_c^\infty(\Omega, \mathbb{R}^p)$ the space of smooth compactly supported $\mathbb{R}^p$-valued functions on Ω.*
(2) *We denote $C^l(\overline{\Omega}, \mathbb{R}^p)$ the set of the restrictions to Ω of the l times continuously differentiable $\mathbb{R}^p$-valued functions on $\mathbb{R}^k$.*
Let $f \in C^l(\overline{\Omega}, \mathbb{R}^p)$. We define the norm $|f|_{l,\infty}$ by

$$|f|_{l,\infty} \triangleq \sum_{\alpha,\ 0 \le |\alpha| \le l} \sup_{x \in \Omega} \left| \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \right|,$$

where for any $\alpha \triangleq (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_*^d$ we denote $|\alpha| \triangleq \sum \alpha_i$.
(3) *We denote $C_0^l(\Omega, \mathbb{R}^p)$ the completion of $C_c^\infty(\Omega, \mathbb{R}^p)$ for the norm $|\ |_{l,\infty}$.*
(4) *We denote $L^2(\Omega, \mathbb{R}^p)$ the Hilbert space of square integrable functions in $\mathbb{R}^p$ with dot product defined for $f, g \in L^2(\Omega, \mathbb{R}^p)$ by*

$$\langle f, g \rangle_2 \triangleq \int_\Omega \langle f(x), g(x) \rangle_{\mathbb{R}^p} dx.$$

(5) *We denote $H^1(\Omega, \mathbb{R}^p)$ the Hilbert space of square integrable $\mathbb{R}^p$-valued functions with square integrable first partial (generalized) derivatives. The dot product is defined for any $f, g \in H^1(\Omega, \mathbb{R}^p)$ by*

$$\langle f, g \rangle_{H^1} \triangleq \langle f, g \rangle_2 + \sum_{i=1}^k \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i} \right\rangle_2.$$

3. Measuring distances on the image space.

3.1. Infinitesimal transformations. Let us consider a space $\mathcal{J}_W$ of functions defined on $\bar{\Omega}$, and taking values on $\mathbb{R}^d$, which will be explicitly defined later. To somewhat fix the ideas, we shall speak of elements of $\mathcal{J}_W$ as "images" and use the corresponding photometric vocabulary, although our constructions apply to generic graphs of vector-valued functions.

We want to build a distance, denoted hereafter $d_{\mathcal{J}_W}$, on $\mathcal{J}_W$ through a Riemannian analysis. Let $j \in \mathcal{J}_W$ and $h \in \mathbb{R}$, and consider a small perturbation j_h of j such that

$$j_h(x) = j(x - hv(x)) + h\sigma^2 z(x) + o(h),$$

where v is a displacement field and z is an $\mathbb{R}^d$-valued function on Ω. Here and in the following, σ^2 is a fixed positive parameter. The transformation from j to j_h is therefore divided in two complementary processes. The first, which we call the "geometric transformation," is a pure deformation of the support for which a point located at x in the first image is pushed to location $x + hv(x)$. The second process, called the "photometric transformation," is the residual, obtained by the addition of $\sigma^2 hz$. Both transformations are the main ingredients of any morphing process between two images. When j is smooth, we have

$$\frac{\partial j}{\partial h}_{|h=0} \triangleq \lim_{h\to 0} \frac{j_h - j}{h} = \sigma^2 z - dj(v). \tag{1}$$

The usual geometric interpretation is that $\gamma \triangleq \frac{\partial j}{\partial h}_{|h=0}$ is an element of the tangent space $T_j\mathcal{J}_W$, and, given our representation, it is sensible to let the length $|\gamma|_j$ depend on $w \triangleq (z, v)$ and to let w vary in some chosen vector space W. The solution cannot merely be to set $|\gamma|_j = |w|_W$, where $|\ |_W$ is a norm on W, because the representation $(z, v) \mapsto \gamma$ is not one-to-one: if $w' = (v', z')$ is such that

$$\sigma^2(z' - z) - dj(v' - v) = 0, \tag{2}$$

then the transformations along w and w' of j are infinitesimally equivalent. Hence, looking for the best tradeoff between geometric and photometric transformations, we can choose for the metric on the tangent space $T_j\mathcal{J}_W$

$$|\gamma|_j = \inf\left\{ |w|_W \mid w = (v, z),\ \gamma = \sigma^2 z - dj(v)\ \right\}. \tag{3}$$

Now, we can define formally

$$d_{\mathcal{J}_W}(j_0, j_1) \triangleq \inf\left\{ \int_0^1 \left|\frac{\partial \mathrm{j}}{\partial t}\right|_{j_t},\ \mathrm{j} \text{ path from } j_0 \text{ to } j_1 \right\}. \tag{4}$$

3.2. Differentiable structure. The previous construction is now made rigorous for $\mathcal{J}_W \triangleq L^2(\Omega, \mathbb{R}^d)$.

Remark 1. Since $L^2(\Omega, \mathbb{R}^d)$ is a Hilbert space, it has a natural structure of smooth infinite dimensional manifold. However, the differential structure we need to consider here is different from the standard L^2 structure. To see this, consider the following example: $\Omega =]0, 1[^k$, and $j_h(x) \triangleq j_0(x - hv(x))$, where

- $j_0(x) \triangleq \mathbf{1}_{x_1 \geq 1/2}$,
- $v \in C_c^\infty(\Omega, \mathbb{R}^k)$ is such that the first coordinate, v_1, of v is strictly positive at the center $c \triangleq (1/2, \dots, 1/2)$ of Ω.

Then, $|j_h - j_0|_2/h \to +\infty$ so that j_h is not differentiable at $h = 0$ for the usual L^2 differentiable structure, whereas, by the construction above, it will be so for the differential structure on $\mathcal{J}_W$ (this is a justification for keeping the nonstandard notation $\mathcal{J}_W$ for the image space).

Our construction starts with the definition of C^1 paths on $\mathcal{J}_W$. We first need to specify the allowed geometric as well as grey-level infinitesimal transformations.

3.2.1. Infinitesimal transformation spaces.

Geometric transformation. We denote $\mathcal{B}$ the space of the displacement fields underlying the infinitesimal geometric transformation. We assume that $\mathcal{B}$ is a Hilbert space with dot product denoted by $\langle\, ,\, \rangle_{\mathcal{B}}$ and norm denoted by $|\ |_{\mathcal{B}}$. We assume throughout this paper that $\mathcal{B}$ is continuously embedded in $C_0^p(\Omega, \mathbb{R}^k)$, where $p = 1$ at least but may be larger if specified. As a reminder, we recall that $\mathcal{B}$ is continuously embedded in some Banach space $\mathcal{B}'$ (with norm $|\ |_{\mathcal{B}'}$) of functions if and only if each element v of $\mathcal{B}$ can be considered as an element of $\mathcal{B}'$ and there exists a constant C such that, for all $v \in \mathcal{B}$,

$$|v|_{\mathcal{B}'} \leq C\, |v|_{\mathcal{B}}\, .$$

Moreover, $\mathcal{B}$ is compactly embedded in $\mathcal{B}'$ if it is continuously embedded and any bounded set for the norm on $\mathcal{B}$ is relatively compact in the $\mathcal{B}'$-topology.

We shall also assume that $C_c^\infty(\Omega, \mathbb{R}^k)$ is dense in $\mathcal{B}$.

Photometric transformation. Grey-level transformations are assumed to belong to the space $L^2(\Omega, \mathbb{R}^d)$.

Finally, we denote $W \triangleq \mathcal{B} \times L^2(\Omega, \mathbb{R}^d)$ on which we place the dot product defined for $w = (v, z)$ and $w' = (v', z')$ by

$$\langle w, w' \rangle_W \triangleq \langle v, v' \rangle_{\mathcal{B}} + \sigma^2 \langle z, z' \rangle_2.$$

3.2.2. Differentiable curves and tangent space. For any smooth image j, we have, for any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$ and any $w = (v, z) \in W$,

$$\langle \sigma^2 z - dj(v), u \rangle_2 = \sigma^2 \langle z, u \rangle_2 + \langle j, \operatorname{div}(u \otimes v) \rangle_2, \tag{5}$$

where $\operatorname{div}(u \otimes v) \in C_0(\Omega, \mathbb{R}^d)$ is defined by $\operatorname{div}(u \otimes v)_i = \operatorname{div}(u_i v)$. The right-hand side of the equality is well defined for arbitrary $j \in \mathcal{J}_W$, which leads us to the following definition.

Definition 2 (C^1 curves in $\mathcal{J}_W$). *Let I be an interval in $\mathbb{R}$. We say that* $\mathrm{j} : I \to \mathcal{J}_W$ *is a continuously differentiable curve if there exists* $\mathrm{w} \triangleq (\mathrm{v}, \mathrm{z}) \in C(I, W)$ *such that*

1. $\mathrm{j} \in C(I, L^2(\Omega, \mathbb{R}^d))$ *for the usual L^2-topology,*
2. *for any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$, $t \to \langle \mathrm{j}_t, u \rangle_2$ is a continuously differentiable real-valued function and* $\frac{\partial}{\partial t} \langle \mathrm{j}_t, u \rangle_2 = \sigma^2 \langle \mathrm{z}_t, u \rangle_2 + \langle \mathrm{j}_t, \operatorname{div}(u \otimes \mathrm{v}_t) \rangle_2$.

If we define as usual tangent vectors via classes of first order equivalent curves, we can identify the tangent bundle of $\mathcal{J}_W$ from the definition of C^1 path on $\mathcal{J}_W$ as follows.

Definition 3.

1. *For any $j \in \mathcal{J}_W$ and any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$, we denote $l_{j,u}$ the continuous linear form on W (the continuity stems from the continuous embedding of $\mathcal{B}$ in $C_0^1(\Omega, \mathbb{R}^k)$) defined for any $w = (v, z) \in W$ by*

$$l_{j,u}(w) \triangleq \sigma^2 \langle z, u \rangle_2 + \langle j, \operatorname{div}(u \otimes v) \rangle_2. \tag{6}$$

2. *We define*

$$E_j \triangleq \{\ w \in W \mid l_{j,u}(w) = 0,\ \forall u \in C_c^\infty(\Omega, \mathbb{R}^d)\ \}, \tag{7}$$

and

$$T_j \mathcal{J}_W \triangleq \{j\} \times W/E_j, \tag{8}$$

where W/E_j is the quotient space, the elements of which are denoted $\overline{w}$.

Remark 2. The use of a quotient space is a consequence of the nonuniqueness of the representation of the derivative by an element $w \in W$ as explained by (2).

We consider $T_j\mathfrak{J}_W$ as a vector space where for any $\gamma = (j, \overline{w})$ and $\gamma' = (j', \overline{w}') \in T_j\mathfrak{J}_W$, we have $\gamma + \lambda'\gamma' \triangleq (j, \overline{w} + \lambda\overline{w}')$. Now, if we define

$$T\mathfrak{J}_W \triangleq \bigcup_{j\in\mathfrak{J}_W} T_j\mathfrak{J}_W,$$

$T\mathfrak{J}_W$ plays the role of the tangent bundle of the manifold $\mathfrak{J}_W$.

DEFINITION 4.

(1) *We denote $\pi : T\mathfrak{J}_W \to \mathfrak{J}_W$ the canonical projection defined by $\pi(\gamma) = j$ for any $\gamma \triangleq (j, \overline{w}) \in T_j\mathfrak{J}_W$.*

(2) *Let $\gamma \triangleq (j, \overline{w}) \in T\mathfrak{J}_W$ and $w = (z, v) \in \overline{w}$. For any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$, we denote*

$$\langle \gamma\, ,\, u\rangle \triangleq \sigma^2\langle z, u\rangle_2 + \langle j, \mathrm{div}(u \otimes v)\rangle_2.$$

(Note that the right-hand side does not depend on the choice of $w \in \overline{w}$).

(3) *For any function $\gamma : I \to T\mathfrak{J}_W$ where I is a real interval, we say that γ is measurable if $\pi \circ \gamma$ is measurable from I to $\mathfrak{J}_W$ and for any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$, $\langle \gamma_t\, ,\, u\rangle$ is measurable from I to $\mathbb{R}$.*

Returning to Definition 2, we see that C^1 curves j admit a lifting $t \mapsto \gamma_t = (\mathrm{j}_t, \overline{w}_t)$ to $T\mathfrak{J}_W$ such that for all $u \in C_c^\infty(\Omega, \mathbb{R}^d)$

$$\frac{d}{dt}\langle \mathrm{j}_t\, ,\, u\rangle_2 = \langle \gamma_t\, ,\, u\rangle$$

so that it is natural to define $\frac{d\mathrm{j}_t}{dt} \triangleq \gamma_t \in T_{j_t}\mathfrak{J}_W$ leading to the formula

$$\frac{d}{dt}\langle \mathrm{j}_t\, ,\, u\rangle_2 = \left\langle \frac{d\mathrm{j}_t}{dt}\, ,\, u\right\rangle. \tag{9}$$

The next step, for our Riemannian construction, is to place a metric on $T_j\mathfrak{J}_W$ for all $j \in \mathfrak{J}_W$.

3.3. Riemannian structure.

DEFINITION 5. *For any $j \in \mathfrak{J}_W$, we define on $T_j\mathfrak{J}_W$ the norm*

$$|\gamma|_j \triangleq \inf\{\ |w|_W \mid (j, w) \in \gamma\ \}.$$

The infimum is attained at a unique point, as stated in the following proposition.

PROPOSITION 1. *For any $j \in \mathfrak{J}_W$ and any $\gamma = (j, \overline{w}) \in T_j\mathfrak{J}_W$, since $\overline{w}$ is a closed subspace of W, there exists a unique $w \in W$ denoted $\overline{p}(\gamma)$ such that*

$$\overline{p}(\gamma) \triangleq \underset{w\in\overline{w}}{\mathrm{Argmin}}\, |w|_W.$$

Hence, $|\gamma|_j \triangleq |\overline{p}(\gamma)|_W$. Moreover, $\overline{p}$ is linear from $T_j\mathfrak{J}_W$ to W.

Proof. Since $\overline{w}$ is a close subspace of W, it is sufficient to note that if p is the orthogonal projection from W to $E_j^\perp$, then $p(w) = 0$ for any $w \in E_j$ so that p can be factorized as a linear map $\overline{p}$ from W/E_j to $E_j^\perp$. Now, one easily checks that $\overline{p}(\gamma) \in \overline{w}$ and that $\overline{p}(\gamma)$ minimizes the norm. ◻

We can now define the geodesic distance between arbitrary points j_0, j_1 in $\mathcal{J}_W$ by

$$d_{\mathcal{J}_W}(j_0, j_1) \triangleq \inf \left\{ \int_0^1 \left|\frac{d\mathrm{j}}{dt}\right|_{\mathrm{j}_t} dt \mid \mathrm{j} \in C^1_{\mathrm{pw}}([0,1], \mathcal{J}_W),\ \mathrm{j}_0 = j_0,\ \mathrm{j}_1 = j_1 \right\}, \tag{10}$$

where $C^1_{\mathrm{pw}}([0,1], \mathcal{J}_W)$ is the set of piecewise C^1 curves in $\mathcal{J}_W$ which are straightforwardly defined from the definition of C^1 curves. This definition is the usual definition for finite dimensional Riemannian manifolds. There is, however, a measurability issue, since it is not obvious from our definition of a measurable path in $T\mathcal{J}_W$ that $t \mapsto |\gamma_t|_{\pi(\gamma_t)}$ is measurable. This issue is addressed in Proposition 2, the proof of which is provided in Appendix A.

PROPOSITION 2. *Let $\gamma : [0,1] \to T\mathcal{J}_W$ be a measurable path in $T\mathcal{J}_W$. Then, $\overline{p} \circ \gamma$ is a measurable path in W and $|\gamma|_{\pi\circ\gamma}$ is a measurable real-valued function.*

4. Groups of diffeomorphisms. Curves in W naturally generate diffeomorphisms on Ω by integration of their first component, which is a time-dependent vector field on Ω which vanishes at $\partial\Omega$. The relations between the Hilbert structure on $\mathcal{B}$ and the class of diffeomorphisms which can be generated in that way have been investigated, in particular, in [30] and [16], in which sufficient smoothness conditions on the vector field are derived to ensure existence, uniqueness, and smoothness of the flow for all time.

For $T > 0$, define the set $L^1([0,T], \mathcal{B})$ as the Banach space of measurable functions $\mathrm{v} : [0,T] \to \mathcal{B}$ such that

$$|\mathrm{v}|_{1,T} \triangleq \int_0^T |\mathrm{v}|_{\mathcal{B}}\, dt < \infty.$$

Similarly, $L^2([0,T], \mathcal{B})$ denotes the Hilbert space of square integrable functions defined on $[0,T]$ and taking values in $\mathcal{B}$, with the norm

$$|\mathrm{v}|_{2,T} \triangleq \left(\int_0^T |\mathrm{v}|^2_{\mathcal{B}}\, dt \right)^{1/2}.$$

For $\mathrm{v} \in L^1([0,T], \mathcal{B})$, consider the ODE

$$\frac{dy}{dt} = \mathrm{v}_t(y). \tag{11}$$

A global flow solution of this equation is a time-dependent family of functions $t \to \varphi_t$ such that, for all $x \in \Omega$, $\varphi_0(x) = x$ and

$$\varphi_t = \int_0^t \mathrm{v}_s \circ \varphi_s\, ds.$$

When the dependence of this flow on v must be emphasized, it is denoted by φ^{v}.

Results in [30, 16] essentially relate the existence and smoothness of such flows to embedding conditions of $\mathcal{B}$ into standard sets of continuous functions. We quote these results in the following theorem.

THEOREM 1 (Trouvé). *If $\mathcal{B}$ is continuously embedded in $C^1_0(\Omega, \mathbb{R}^k)$, then for all $T > 0$ and all $\mathrm{v} \in L^1([0,T], \mathcal{B})$, the ODE (11) can be integrated over $[0,T]$, and its associated flow φ^{v} is such that at all times $x \to \varphi^{\mathrm{v}}_t$ is a homeomorphism of Ω.*

NOTATION 1. *Assume that $\mathcal{B}$ is continuously embedded in $C_0^1(\Omega, \mathbb{R}^k)$, and introduce the map*

$$\begin{array}{rccc} \mathbf{A}_T: & L^1([0,T],\mathcal{B}) & \to & C(\overline{\Omega},\mathbb{R}^k), \\ & \mathrm{v} & \mapsto & \varphi_T^{\mathrm{v}}. \end{array}$$

Then, the set $\mathbf{A}_1(L^1([0,1],\mathcal{B}))$ will be denoted $G_{\mathcal{B}}$.

The fact that $G_{\mathcal{B}}$ is a group is proved in [30]. Further results on these groups and on A_T can be found in Appendix C.

The relation between algebraic and metric properties of groups of diffeomorphisms and some of the fundamental equations of fluid mechanics has been the subject of several studies, starting with [5], in which the Euler equation is related to the geodesic equations of groups of diffeomorphisms with an L^2 metric on its Lie algebra (see also [3, 4, 24]). Another important equation, the Camassa–Holm equation, which describes the motion of the waves in shallow water, can be interpreted along the same lines with an H^1_α metric on the Lie algebra [11, 17]. Here, since the energy derives from both geometric and photometric variations, the geodesic equations that we derive can be formally interpreted as conservation of momentum on a semidirect product of the group of diffeomorphisms and the space of images, as studied in [25]. However, our point of view of smooth deformations acting on nonsmooth images requires a specific approach. This is also related to developments in optimal design [28].

5. Geodesics on $\mathcal{J}_W$.

5.1. Minimizing geodesics. The space of C^1 curves is not well suited to deal with proofs of the existence of curves of minimal length between two images j_0 and j_1, i.e., minimizing geodesics. We introduce below the more tractable space of curves with square integrable speed.

We need first a preliminary proposition saying that square integrable paths in $T\mathcal{J}_W$ are uniquely identified by their trace on smooth space-time vector fields in $\mathbb{R}^d$. The proof of this proposition is postponed to Appendix A.

PROPOSITION 3. *Let $\gamma : [0,1] \to T\mathcal{J}_W$ be a measurable path in $T\mathcal{J}_W$. Then, if $\int_0^1 |\gamma_t|^2_{\pi(\gamma_t)} dt < +\infty$ and, for any $\mathrm{u} \in C_c^\infty(\Omega \times]0,1[, \mathbb{R}^d)$, we have $\int_0^1 \langle \gamma_t\,,\, \mathrm{u}_t\rangle dt = 0$, then $\gamma = 0$ a.e.*

We can now introduce the space $H^1([0,1], \mathcal{J}_W)$ of regular curves.

DEFINITION 6. *We say that a path $\mathrm{j} \in C([0,1], L^2(\Omega, \mathbb{R}^d))$ is regular if there exists a measurable path $\gamma : [0,1] \to T\mathcal{J}_W$ such that $\pi(\gamma) = \mathrm{j}$, $\int_0^1 |\gamma_t|^2 dt < \infty$, and, for any $\mathrm{u} \in C_c^\infty(]0,1[\times\Omega, \mathbb{R}^d)$, we have $-\int_0^1 \langle \mathrm{j}_t, \frac{\partial \mathrm{u}}{\partial t}\rangle_2 dt = \int_0^1 \langle \gamma_t\,,\, \mathrm{u}_t\rangle dt$. From Proposition 3, the path γ is uniquely defined; using the notation $\frac{\partial \mathrm{j}}{\partial t} \triangleq \gamma_t$, we get the integration by parts formula*

$$\int_0^1 \left\langle \mathrm{j}_t, \frac{\partial \mathrm{u}}{\partial t} \right\rangle_2 dt = -\int_0^1 \left\langle \frac{\partial \mathrm{j}}{\partial t}\,,\, \mathrm{u}_t \right\rangle dt. \tag{12}$$

We denote $H^1([0,1], \mathcal{J}_W)$ as the set of all the regular paths in $C([0,1], L^2(\Omega, \mathbb{R}^d))$.

PROPOSITION 4. *We have $C^1([0,1], \mathcal{J}_W) \subset H^1([0,1], \mathcal{J}_W)$ and both definitions of $\frac{\partial \mathrm{j}}{\partial t}$ coincide.*

Proof. Let $\mathrm{j} \in C^1([0,1], \mathcal{J}_W)$. There exists $\mathrm{w} = (\mathrm{v}, \mathrm{z}) \in C([0,1], W)$ such that for any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$, $t \to \langle \mathrm{j}_t, u\rangle_2$ is C^1 and

$$\frac{\partial}{\partial t}\langle \mathrm{j}_t, \mathrm{u}\rangle_2 = \sigma^2 \langle \mathrm{z}_t, \mathrm{u}\rangle_2 + \langle \mathrm{j}_t, \mathrm{div}(\mathrm{u} \otimes \mathrm{v}_t)\rangle_2.$$

Certainly, $\mathrm{w} \in L^2([0,1],W)$. Moreover, for any $\mathrm{u} \in C_c^\infty(\Omega,\mathbb{R}^d)$ and any $f \in C_c^\infty(]0,1[,\mathbb{R})$, we have by integration by parts (we denote $f'(t) \triangleq \frac{df}{dt}$)

$$\int_0^1 \langle j_t, f'(t)\mathrm{u}\rangle_2 = \int_0^1 f'(t)\langle \mathrm{j}_t, \mathrm{u}\rangle_2 dt = -\int_0^1 f(t)\frac{d}{dt}\langle \mathrm{j}_t, \mathrm{u}\rangle_2 dt$$

so that (12) is true for $u \otimes f \in C_c^\infty(]0,1[\times\Omega,\mathbb{R}^d)$. The complete proof follows by usual density arguments. □

We carry on with an important result which characterizes regular paths in $\mathcal{J}_W$. For a path v in $L^1([0,1],\mathcal{B})$, we define for any $s,t \in [0,1]$

$$\varphi^{\mathrm{v}}_{t,s} \triangleq \varphi^{\mathrm{v}}_s \circ (\varphi^{\mathrm{v}}_t)^{-1}.$$

THEOREM 2. *A path* $\mathrm{j} : [0,1] \to \mathcal{J}_W$ *is regular (resp., is in* $C^1([0,1],\mathcal{J}_W)$*) if and only if there exists* $\mathrm{w} = (\mathrm{v},\mathrm{z}) \in L^2([0,1],W)$ *(resp.,* $\in C([0,1],W)$*) such that*

$$\mathrm{j}_t = \mathrm{j}_0 \circ \varphi^{\mathrm{v}}_{t,0} + \sigma^2 \int_0^t \mathrm{z}_s \circ \varphi^{\mathrm{v}}_{t,s} ds.$$

Proof. The proof is postponed to Appendix B. □

THEOREM 3. *Let* j_0 *and* j_1 *be in* $\mathcal{J}_W$*. Then we have*

$$d_{\mathcal{J}_W}(j_0,j_1) = \inf\left\{ \int_0^1 \left|\frac{\partial \mathrm{j}}{\partial t}\right|_{j_t} dt \mid \mathrm{j} \in H^1([0,1],\mathcal{J}_W),\ \mathrm{j}_0 = j_0,\ \mathrm{j}_1 = j_1 \right\}. \tag{13}$$

Proof. Let $\mathrm{j} \in H^1([0,1],\mathcal{J}_W)$ be a regular path from j_0 to j_1 and let $\mathrm{w} \in L^2([0,1],W)$ such that $\mathrm{w}_t = \overline{p}_{\mathrm{j}_t}(\frac{\partial \mathrm{j}}{\partial t})$ for any t. There exists a sequence $(\mathrm{w}^n = (\mathrm{v}^n,\mathrm{z}^n) \in C([0,1],W),\ n \in \mathbb{N})$ such that $\int_0^1 |\mathrm{w}_t - \mathrm{w}^n_t|^2_W dt \to 0$. Define

$$\mathrm{j}^n_t = j_0 \circ \varphi^{\mathrm{v}^n}_{t,0} + \sigma^2 \int_0^t \mathrm{z}^n_s \circ \varphi^{\mathrm{v}^n}_{t,s} ds.$$

We get from Theorem 2 that $\mathrm{j}^n \in C^1([0,1],\mathcal{J}_W)$. Now, considering $\tilde{\mathrm{w}}^n \triangleq (\tilde{\mathrm{v}}^n,\tilde{\mathrm{z}}^n)$ with $\tilde{\mathrm{z}}^n_t \triangleq \mathrm{z}^n_t + (j_1 - \mathrm{j}^n_1)\circ\varphi^{\mathrm{v}^n}_{s,1}$ and $\tilde{\mathrm{v}}^n \triangleq \mathrm{v}^n$ we get from Theorem 9 (see Appendix C) that $\tilde{\mathrm{w}}^n \in C([0,1],W)$. Using Theorem 2, we deduce that if $\tilde{\mathrm{j}}^n$ is defined by

$$\tilde{\mathrm{j}}^n_t = j_0 \circ \varphi^{\mathrm{v}^n}_{t,0} + \sigma^2 \int_0^t \tilde{\mathrm{z}}^n_s \circ \varphi^{\mathrm{v}^n}_{t,s} ds,$$

then $\tilde{\mathrm{j}}^n \in C^1([0,1],\mathcal{J}_W)$ and $\tilde{\mathrm{j}}^n_1 = j_1$. However,

$$\int_0^1 \left|\frac{\partial \tilde{\mathrm{j}}^n}{\partial t}\right|_{\tilde{\mathrm{j}}^n_t} dt \le \int_0^1 |\tilde{\mathrm{w}}^n_t|_W dt \to \int_0^1 |\mathrm{w}_t|_W dt$$

when $n \to \infty$. Therefore, we deduce that $d_{\mathcal{J}_W}(j_0,j_1) \le \int_0^1 |\frac{\partial \mathrm{j}}{\partial t}|_{\mathrm{j}_t} dt$ for any regular path from j_0 to j_1. Finally, since $C^1([0,1],\mathcal{J}_W) \subset H^1([0,1],\mathcal{J}_W)$, we get the result. □

DEFINITION 7. *Let* $j_0, j_1 \in \mathcal{J}_W$*. We say that* $\mathrm{j} \in C([0,1],L^2(\Omega,\mathbb{R}^d))$ *is a minimizing geodesic path from* j_0 *to* j_1 *if* j *is regular and*

$$\left(\int_0^1 \left|\frac{\partial \mathrm{j}}{\partial t}\right|^2_{\mathrm{j}_t} dt\right)^{\frac{1}{2}} = d_{\mathcal{J}_W}(j_0,j_1).$$

We denote $G_{\mathcal{J}_W}(j_0,j_1)$ *as the set of the minimizing geodesic paths from* j_0 *to* j_1*.*

5.2. Characterization of geodesics.

5.2.1. Photometric optimality.

THEOREM 4. *Let* $j_0, j_1 \in \mathcal{J}_W$ *and* $j \in G_{\mathcal{J}_W}(j_0, j_1)$ *be a minimizing geodesic path from* j_0 *to* j_1. *Let* $\mathrm{w} = (\mathrm{v}, \mathrm{z}) \in L^2([0,1], W)$ *be defined by* $\mathrm{w}_t \triangleq \overline{p}(\frac{\partial j}{\partial t})$ *for any* $t \in [0,1]$. *Then* $\mathrm{z} \in C([0,1], L^2(\Omega, \mathbb{R}^d))$ *and for any* $t \in [0,1]$ *we have*

$$\mathrm{z}_t = \mathrm{z}_0 \circ \varphi^{\mathrm{v}}_{t,0} \left| d\varphi^{\mathrm{v}}_{t,0} \right|. \tag{14}$$

Proof. Let $\mathrm{j} \in H^1([0,1], \mathcal{J}_W)$ be a minimizing geodesic from j_0 to j_1, and let $\mathrm{w} = (\mathrm{v}, \mathrm{z}) \in L^2([0,1], W)$ such that for any $t \in [0,1]$, $\mathrm{w}_t = \overline{p}(\frac{d\mathrm{j}}{dt})$. For any $\mathrm{u} \in C_c^\infty(]0,1[\times\Omega, \mathbb{R}^d)$ and any $\varepsilon \in \mathbb{R}$, define

$$\tilde{\mathrm{j}}_t = j_0 \circ \varphi^{\mathrm{v}}_{t,0} + \sigma^2 \int_0^t \left(\mathrm{z}_s + \varepsilon \frac{\partial \mathrm{u}_s}{\partial s} \circ \varphi^{\mathrm{v}}_{s,1} \right) \circ \varphi_{t,s} ds.$$

Since $t \to (\mathrm{v}_t, \mathrm{z}_t + \varepsilon \frac{\partial \mathrm{u}_t}{\partial t} \circ \varphi^{\mathrm{v}}_{t,1}) \in L^2([0,1], W)$, we get from Theorem 2 that $\tilde{\mathrm{j}} \in H^1([0,1], \mathcal{J}_W)$. Moreover, $\tilde{\mathrm{j}}_0 = j_0$ and $\tilde{\mathrm{j}}_1 = j_1$ so that

$$\int_0^1 \left| \frac{d\mathrm{j}_t}{dt} \right|^2_{\mathrm{j}_t} dt = \int_0^1 \left(|\mathrm{v}_t|^2_{\mathcal{B}} + \sigma^2 |\mathrm{z}_t|^2_2 \right) dt \le \int_0^1 \left| \frac{d\tilde{\mathrm{j}}_t}{dt} \right|^2_{\tilde{\mathrm{j}}_t} dt$$
$$\le \int_0^1 \left(|\mathrm{v}_t|^2_{\mathcal{B}} + \sigma^2 \left| \mathrm{z}_t + \varepsilon \frac{\partial \mathrm{u}_t}{\partial t} \circ \varphi^{\mathrm{v}}_{t,1} \right|^2_2 \right) dt.$$

Since ε is arbitrary, we get

$$0 = \int_0^1 \left\langle \mathrm{z}_t, \frac{\partial \mathrm{u}_t}{\partial t} \circ \varphi^{\mathrm{v}}_{t,1} \right\rangle_2 dt = \int_0^1 \left\langle \mathrm{z}_t \circ \varphi^{\mathrm{v}}_{1,t} \left| d\varphi^{\mathrm{v}}_{1,t} \right|, \frac{\partial \mathrm{u}_t}{\partial t} \right\rangle_2 dt.$$

Choosing arbitrary $\mathrm{u} \in C_c^\infty(]0,1[\times\Omega, \mathbb{R}^d)$, we get that there exists $\tilde{z}_1 \in L^2(\Omega, \mathbb{R}^d)$ such that t-a.e. we have $\mathrm{z}_t \circ \varphi^{\mathrm{v}}_{1,t} \left| d\varphi^{\mathrm{v}}_{1,t} \right| = \tilde{z}_1$. Hence, if $\tilde{\mathrm{z}}_t = \tilde{z_1} \circ \varphi^{\mathrm{v}}_{t,1} \left| d\varphi^{\mathrm{v}}_{t,1} \right|$, we have $\tilde{\mathrm{z}} \in C([0,1), L^2([0,1], \mathbb{R}^d))$ and $\mathrm{z}_t = \tilde{\mathrm{z}}_t$ t-a.e. Note that $\tilde{\mathrm{z}}_0 \circ \varphi^{\mathrm{v}}_{1,0} \left| d\varphi^{\mathrm{v}}_{1,0} \right| = \tilde{z_1}$ so that

$$\tilde{\mathrm{z}}_t = \left(\tilde{\mathrm{z}}_0 \circ \varphi^{\mathrm{v}}_{1,0} \left| d\varphi^{\mathrm{v}}_{1,0} \right| \right) \circ \varphi^{\mathrm{v}}_{t,1} \left| d\varphi^{\mathrm{v}}_{t,1} \right| = \tilde{\mathrm{z}}_0 \circ \varphi^{\mathrm{v}}_{t,0} \left| d\varphi^{\mathrm{v}}_{t,0} \right|,$$

and the proof is ended. □

This leads to the following definition.

DEFINITION 8. *A regular path* $\mathrm{j} \in H^1([0,1], \mathcal{J}_W)$ *is called a pregeodesic path if and only if the following equations are satisfied almost everywhere in* t*:*

$$\begin{cases} \mathrm{j}_t = \mathrm{j}_0 \circ \varphi^{\mathrm{v}}_{t,0} + \sigma^2 \displaystyle\int_0^t \mathrm{z}_s \circ \varphi^{\mathrm{v}}_{t,s} ds, \\ \mathrm{z}_t = \mathrm{z}_0 \circ \varphi^{\mathrm{v}}_{t,0} \left| d\varphi^{\mathrm{v}}_{t,0} \right|, \\ (\mathrm{v}_t, \mathrm{z}_t) = \overline{p}\left(\dfrac{d\mathrm{j}}{dt} \right). \end{cases} \tag{15}$$

5.2.2. Study of the geodesic equation.

Directional derivatives in L^2. In this section, we try to clarify the last equation of system (15), at least in some situations of interest. The difficulty comes from the fact that, unless j_t is smooth enough, this equation does not, in general, specify a unique correspondence $\mathrm{z}_t \mapsto \mathrm{v}_t$.

To be more precise, let us analyze the condition that, for all t,

$$(\mathrm{v}_t, \mathrm{z}_t) = \overline{p}\left(\frac{d\mathrm{j}}{dt}\right).$$

For this purpose, we first introduce a weak version of the directional derivative $dj.v$ when $j \in \mathcal{J}_W$ and $v \in \mathcal{B}$.

DEFINITION 9. *Let $j \in \mathcal{J}_W$.*

(1) *We define the operator $Dj : \mathcal{D}_j \to L^2(\Omega, \mathbb{R}^d)$ by*

$$\mathcal{D}_j \triangleq \{\ v \in \mathcal{B} \mid \exists C,\ s.t.\ \forall u \in C_c^\infty(\Omega, \mathbb{R}^d),\ |\langle j\,,\, \mathrm{div}(u \otimes v)\rangle_2| \le C\,|u|_2\ \},$$

and for any $v \in \mathcal{D}_j$, $Dj.v$ is the unique element in $L^2(\Omega, \mathbb{R}^d)$ such that

$$\langle Dj.v\,,\, u\rangle_2 = -\langle j\,,\, \mathrm{div}(u \otimes v)\rangle_2$$

for any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$.

(2) *We define the adjoint operator $Dj^* : \mathcal{D}_j^* \to \mathcal{B}$, where*

$$\mathcal{D}_j^* \triangleq \{\ u \in L^2(\Omega, \mathbb{R}^d) \mid \exists C,\ s.t.\ \forall v \in \mathcal{D}_j\ \ |\langle Dj.v\,,\, u\rangle_2| \le C\,|v|_B\ \},$$

and, for any $u \in \mathcal{D}_j^*$, Dj^* is the unique element in $\overline{\mathcal{D}_j}$ (closure of $\mathcal{D}_j$) such that

$$\langle Dj^*.u\,,\, v\rangle_{\mathcal{B}} = \langle u\,,\, Dj.v\rangle_2 \tag{16}$$

for any $v \in \mathcal{D}_j$.

Remark 3. The existence of $Dj.v$ comes from the extension of the linear form $u \to \langle j\,,\, \mathrm{div}(u \otimes v)\rangle$ for smooth u into a continuous linear form on $L^2(\Omega, \mathbb{R}^d)$ for $v \in \mathcal{D}_j$. For the definition of the adjoint Dj^*, the adjoint is uniquely defined as an element of $\overline{\mathcal{D}_j}$ by (16) ($\mathcal{D}_j$ is not necessarily dense in $\mathcal{B}$).

Fix $j \in \mathcal{J}_W$. We may characterize elements $v \in \mathcal{D}_j$ as follows. (We denote hereafter $\varphi^v_{\cdot}$ as the flow associated with the constant speed $v_t \equiv v$ for any $t \in [0, 1]$.)

THEOREM 5. *The vector field $v \in \mathcal{B}$ belongs to $\mathcal{D}_j$ if and only if there exists a square integrable function $\xi : \Omega \to \mathbb{R}^d$ such that*

$$j \circ \varphi^v_{0,t}(x) = j(x)\left|d_x\varphi^v_{0,t}\right|^{-1} + \int_0^t \xi \circ \varphi^v_{0,s}(x)\left|d_{\varphi^v_{0,s}(x)}\varphi^v_{s,t}\right|^{-1} ds. \tag{17}$$

We have in such a case $Djv = \xi - j\mathrm{div}(v)$.

Proof. We first notice that, if $v \in \mathcal{B}$,

$$-\langle j\,,\, \mathrm{div}(u \otimes v)\rangle_2 = \frac{d}{d\varepsilon}\int_\Omega \left\langle j(x)\,,\, u \circ \varphi^v_{\varepsilon,0}(x)\right\rangle \left|d_x\varphi^v_{\varepsilon,0}\right| dx = \frac{d}{d\varepsilon}\int_\Omega \left\langle j \circ \varphi^v_{0,\varepsilon}(x)\,,\, u(x)\right\rangle dx.$$

Assuming that (17) holds, the last expression yields

$$\begin{aligned}\frac{d}{d\varepsilon}\int_\Omega \langle j(x)\,,\, u(x)\rangle \left|d_x\varphi^v_{0,\varepsilon}\right|^{-1} dx + \frac{d}{d\varepsilon}\int_\Omega\int_0^\varepsilon \left\langle \xi \circ \varphi^v_{0,s}\,,\, u(x)\right\rangle \left|d_{\varphi^v_{0,s}(x)}\varphi^v_{s,\varepsilon}\right|^{-1} dx \\ = -\int_\Omega \langle j(x)\,,\, u(x)\rangle \mathrm{div}(v)dx + \int_\Omega \langle \xi(x)\,,\, u(x)\rangle dx = \langle \xi - j\mathrm{div}(v)\,,\, u\rangle_2,\end{aligned}$$

which implies that $v \in \mathcal{D}_j$ and $Djv = \xi - j\mathrm{div}(v)$.

Conversely, let $v \in \mathcal{D}_j$ and $\xi = Djv + j\mathrm{div}(v)$. Fix $u \in C^1(\Omega, \mathbb{R}^d)$. Consider the function f, defined on $[0,1]$ by $f(t) = \left\langle j \circ \varphi^v_{0,t} \,,\, u \right\rangle_2$. Denote by $\tilde{\jmath}(t)$ the left-hand term of (17), and $g(t) = \langle \tilde{\jmath}_t \,,\, u \rangle_2$. We have

$$\begin{aligned} g'(t) = &-\left\langle j \,,\, u\mathrm{div}_{\varphi^v_{0,t}(x)} v \left| d_x \varphi^v_{0,t} \right|^{-1} \right\rangle_2 + \left\langle \xi \circ \varphi^v_{0,t} \,,\, u \right\rangle_2 \\ &- \int_0^t \left\langle \xi \circ \varphi^v_{0,s} \left| d_{\varphi^v_{0,s}} \varphi^v_{s,t} \right|^{-1} \,,\, u\mathrm{div}_{\varphi^v_{0,t}} v \right\rangle_2 ds \\ = &\left\langle \xi \circ \varphi^v_{0,t} - \tilde{\jmath}_t \mathrm{div}_{\varphi^v_{0,t}} v \,,\, u \right\rangle_2 . \end{aligned}$$

Since $f(t+\varepsilon) = \left\langle j \circ \varphi^v_{t,t+\varepsilon} \,,\, u \circ \varphi^v_{t,0} \left| d\varphi^v_{t,0} \right| \right\rangle_2$, we have

$$f'(t) = \left\langle Djv \,,\, u \circ \varphi^v_{t,0} \left| d\varphi^v_{t,0} \right| \right\rangle_2 = \left\langle (Djv) \circ \varphi^v_{0,t} \,,\, u \right\rangle_2 .$$

Therefore, computing the integral of the difference and using the definition of ξ,

$$\left\langle j \circ \varphi^v_{0,t} - \tilde{\jmath}_t \,,\, u \right\rangle_2 = \int_0^t \left\langle j \circ \varphi^v_{0,s} - \tilde{\jmath}_s \,,\, u\mathrm{div}_{\varphi^v_{0,t}} v \right\rangle_2 ds \le |u|_2 \, |v|_{\mathcal{B}} \int_0^t \left| j \circ \varphi^v_{0,s} - \tilde{\jmath}_s \right|_2 ds.$$

Taking the supremum of the left-hand term over continuously differentiable u with L^2-norm equal to 1 yields

$$\left| j \circ \varphi^v_{0,t} - \tilde{\jmath}_t \right|_2 \le |v|_{\mathcal{B}} \int_0^t \left| j \circ \varphi^v_{0,s} - \tilde{\jmath}_s \right|_2 ds,$$

which implies $\left| j \circ \varphi^v_{0,t} - \tilde{\jmath}_t \right|_2 = 0$ for all t. □

An interesting consequence of this is the following lemma.

Lemma 1. *For any $j \in L^2(\Omega, \mathbb{R}^d)$, one has $j \in D^*_j$ and, for $v \in D_j$,*

$$\langle Djv \,,\, j \rangle_2 = -\frac{1}{2} \left\langle |j|^2 \,,\, \mathrm{div} v \right\rangle_2 .$$

Proof. Indeed, let $v \in D_j$. Consider the function

$$f(t) = \int_\Omega \left| j \circ \varphi^v_{0,t}(x) \right|^2 dx.$$

Since $f(t) = \left\langle |j|^2 \,,\, |d\varphi_{t,0}| \right\rangle_2$, we have $f'(0) = -\langle |j|^2 \,,\, \mathrm{div} v \rangle_2$. Using, on the other hand, expression (17) yields $f'(0) = 2\langle Djv \,,\, j \rangle_2$. □

Interpretation of the pregeodesic equations. The property that $w = (v, z) \in W$ belongs to E_j, which states that, for all $u \in C^\infty_c(\Omega, \mathbb{R}^d)$,

$$\sigma^2 \langle z, u \rangle_2 + \langle j, \mathrm{div}(u \otimes v) \rangle_2 = 0$$

is equivalent to $v \in \mathcal{D}_j$ and $\sigma^2 z - Dj.v = 0$. Consider now some tangent vector $\gamma \in T_j \mathcal{J}_W$, and study the property that, for some $w = (v, z) \in W$, one has $\overline{p}(\gamma) = w$. This implies, in particular, that, for all $(v', z') \in E_j$,

$$|v + v'|^2_{\mathcal{B}} + \sigma^2 \, |z + z'|^2_2 \ge |v|^2_{\mathcal{B}} + \sigma^2 \, |z|^2_2 ,$$

which is in turn equivalent to the following: for all $v' \in \mathcal{D}_j$, $\langle v\,,\, v'\rangle_{\mathcal{B}} + \langle z\,,\, Dj.v'\rangle_2 = 0$. This implies that $z \in \mathcal{D}_j^*$ and that $\langle v\,,\, v'\rangle_{\mathcal{B}} + \langle Dj^* z\,,\, v'\rangle_{\mathcal{B}} = 0$, so that $v = -Dj^*z + \gamma_\perp$, where $\gamma_\perp$ is the projection of v onto $\mathcal{D}_j^\perp$. Note that this orthogonal component does not depend on the choice of (v, z) from the equivalence class defining γ (hence the notation). We thus may conclude that $(v, z) = \overline{p}(\gamma)$ if and only if $z \in \mathcal{D}_j^*$ and

$$v = -Dj^*z + \gamma_\perp.$$

The first conclusion we may draw from this is that, whenever $\mathcal{D}_j$ is dense in $\mathcal{B}$, v_t is uniquely determined by z_t and the condition $(\mathrm{v}_t, \mathrm{z}_t) = \overline{p}(\frac{d\mathrm{j}}{dt})$. It is given by $\mathrm{v}_t = -D\mathrm{j}_t^* \mathrm{z}_t$. This is true, for example, when $\mathrm{j}_t \in H^1(\Omega, \mathbb{R}^d)$ at all times, since, in this case $\mathcal{D}_{\mathrm{j}_t} = \mathcal{B}$ (notice that, by Theorem 4, this is true along a geodesic as soon as j_0 and j_1 belong to $H^1(\Omega, \mathbb{R}^d)$).

However, this is not the general situation. As an example, consider the case when j is the indicator function of a subdomain Ω_1 of Ω with smooth boundary. If v is a vector field on Ω and u is a smooth function on Ω, we have

$$\langle j\,,\, \mathrm{div} uv\rangle = \int_{\partial\Omega_1} u\langle v\,,\, \nu_1\rangle_{\mathbb{R}^k} d\sigma_1,$$

where ν_1 is the outward normal to $\partial\Omega_1$ and σ_1 is the surface measure on $\partial\Omega_1$. This implies that djv may be identified to a singular measure supported to $\partial\Omega_1$, which does not belong to L^2 unless it vanishes. Thus, $\mathcal{D}_j$ consists exactly of vector fields on Ω which belong to $\mathcal{B}$ and have vanishing normal components on $\partial\Omega_1$. For $a \in \mathbb{R}^k$ and $x \in \mathbb{R}$, denote by $K_x a$ the element of $\mathcal{B}$ such that $\langle K_x a\,,\, u\rangle_{\mathcal{B}} = \langle u(x)\,,\, a\rangle_{\mathbb{R}^k}$. Then, $K_x \nu(x)$ belongs to $D_j^\perp$ for any $x \in \partial\Omega$, and so does any linear combination of these vector fields. We see that in this case $D_j^\perp$ is nontrivial.

This discussion implies that the pregeodesic condition for a path may be written

$$\begin{cases} \mathrm{j}_t = \mathrm{j}_0 \circ \varphi_{t,0}^{\mathrm{v}} + \sigma^2 \displaystyle\int_0^t \mathrm{z}_s \circ \varphi_{t,s}^{\mathrm{v}} ds, \\ \mathrm{z}_t = \mathrm{z}_0 \circ \varphi_{t,0}^{\mathrm{v}} \left| d\varphi_{t,0}^{\mathrm{v}} \right|, \\ \mathrm{z}_t \in D_j^*, \text{ and } \mathrm{v}_t - D\mathrm{j}_t^* \mathrm{z}_t \in D_{j_t}^\perp. \end{cases} \tag{18}$$

These equations are not complete yet (in the sense that they cannot be solved from the initial values (j_0, v_0, z_0)) since they provide no information on the choice of $\mathrm{v}_t - D\mathrm{j}_t^* \mathrm{z}_t$ at time t (unless of course $D_{j_t}^\perp = \{0\}$). We need to specify the mode of propagation of this singular component along a geodesic. The following computation provides a hint on possible ways to achieve this. Assume that j is pregeodesic and $(\mathrm{v}_t, \mathrm{z}_t) = \overline{p}(\frac{d\mathrm{j}}{dt})$. In such a case, we have

$$|\mathrm{z}_s|_2^2 = \int_\Omega |z_0 \circ \varphi_{s,0}^{\mathrm{v}}|^2 |d\varphi_{s,0}^{\mathrm{v}}|^2 dx = \int_\Omega |z_0|^2 |d\varphi_{0,s}^{\mathrm{v}}|^{-1} dy$$

and

$$z_0(x) = \frac{1}{\sigma^2}(j_1 \circ \varphi_{0,1}^{\mathrm{v}}(x) - j_0(x)) \int_0^1 |d\varphi_{0,s}^{\mathrm{v}}|^{-1} ds.$$

Thus

$$\int_0^1 |\mathrm{z}_s|_2^2\, ds = \frac{1}{\sigma^4}\int_\Omega \frac{\left|j_1\circ\varphi^{\mathrm{v}}_{0,1} - j_0\right|^2}{\int_0^1 |d\varphi^{\mathrm{v}}_{0,s}|^{-1} ds}.$$

Making the change of variables $y = \varphi^{\mathrm{v}}_{0,1}(x)$ yields

$$\int_0^1 |z_s|_2^2\, ds = \frac{1}{\sigma^4}\int_\Omega \frac{\left|j_1 - j_0\circ\varphi^{\mathrm{v}}_{1,0}\right|^2}{\int_0^1 |d_{\varphi^{\mathrm{v}}_{1,0}(x)}\varphi^{\mathrm{v}}_{0,s}|^{-1}|d\varphi^{\mathrm{v}}_{1,0}(x)|^{-1} ds},$$

i.e.,

$$\int_0^1 |\mathrm{z}_s|_2^2\, ds = \frac{1}{\sigma^4}\int_\Omega \frac{\left|j_1 - j_0\circ\varphi^{\mathrm{v}}_{1,0}\right|^2}{\int_0^1 |d\varphi^{\mathrm{v}}_{1s}|^{-1} ds},$$

and the geodesic energy is given by

$$\int_0^1 |\mathrm{v}_s|_{\mathcal{B}}^2\, ds + \frac{1}{\sigma^2}\int_\Omega \frac{\left|j_1 - j_0\circ\varphi^{\mathrm{v}}_{1,0}\right|^2}{\int_0^1 |d\varphi^{\mathrm{v}}_{1,s}|^{-1} ds}. \tag{19}$$

We can obtain more precise information on the geodesic by studying variations of this expression with respect to v. This will be handled below, under a smoothness assumption on j_0. Before this, we need some notation for the reproducing kernel on $\mathcal{B}$. They will be useful throughout the paper.

Kernels for the inner-product on $\mathcal{B}$.

PROPOSITION 5. *There exists a continuous operator K (resp., K_∇) on $L^1(\Omega, \mathbb{R}^k)$ (resp., $L^1(\Omega, \mathbb{R})$) with values in $\mathcal{B}$ such that, for all $u \in L^1(\Omega, \mathbb{R}^k)$ (resp., $u \in L^1(\Omega, \mathbb{R})$), for all $v \in \mathcal{B}$,*

$$\langle Ku\,,\, v\rangle_{\mathcal{B}} = \langle u, v\rangle_2,$$

and

$$\langle K_\nabla u\,,\, v\rangle_{\mathcal{B}} = -\langle u, \mathrm{div} v\rangle_2.$$

Proof of Proposition 5. Let $u \in L^1(\Omega, \mathbb{R}^k)$. Since we assume that $\mathcal{B}$ is continuously embedded in C_0^1, the linear form defined on $\mathcal{B}$ by $v \mapsto \langle u, v\rangle_2$ is continuous because $|\langle u, v\rangle_2| \le |u|_1\,|v|_\infty$. Therefore, there exists a unique element in $\mathcal{B}$, denoted Ku, such that, for all $v \in \mathcal{B}$, $\langle Ku\,,\, v\rangle_{\mathcal{B}} = \langle u, v\rangle_2$ and continuity comes from the inequality $\langle Ku\,,\, v\rangle_{\mathcal{B}} \le |u|_1\,|v|_\infty \le \mathrm{cst}\;\; |u|_1\,|v|_{\mathcal{B}}$.

The same proof holds for K_∇, since $|\mathrm{div} v|_\infty$ is also controlled by $|v|_{\mathcal{B}}$. ∎

It can be remarked that, for smooth u, $K_\nabla u = K(\nabla u)$.

Remark 4. When j is smooth (e.g., $j \in H^1(\Omega, \mathbb{R}^d)$), the operator Dj^* introduced in the previous paragraph is given by $Dj^* z = K(dj^*.z)$, in which dj^* is the standard matrix adjoint of dj. Indeed, we have in this case

$$\langle z\,,\, Dj.v\rangle_2 = \langle z\,,\, dj.v\rangle_2 = \langle dj^*.z\,,\, v\rangle_2 = \langle K(dj^*.z)\,,\, v\rangle_{\mathcal{B}}.$$

Characterization with a smooth endpoint. We study the effect of small variations in v on the geodesic energy (19), under the additional hypothesis that $j_0 \in H^1(\Omega, \mathbb{R}^d)$. Thus, fix $h \in L^2([0,1], \mathcal{B})$, and consider a perturbation $\mathrm{v} + \varepsilon h$ of v. We compute the corresponding variation of the geodesic energy. The variation of the first term being $2\int_0^1 \langle \mathrm{v}_t\,,\, h_t\rangle_{\mathcal{B}} dt$, we can focus on the second term, namely,

$$U^\varepsilon \triangleq \frac{1}{\sigma^2}\int_\Omega \frac{\left|j_0 \circ \varphi_{1,0}^{\mathrm{v}+\varepsilon h} - j_1\right|^2}{\int_0^1 |d\varphi_{1s}^{\mathrm{v}+\varepsilon h}|^{-1} ds} dx.$$

The variations of U^ε are given in Lemma 2, which is proved in Appendix E.

Lemma 2. *We have, at $\varepsilon = 0$,*

(20)
$$\frac{dU^\varepsilon}{d\varepsilon} = \sigma^2 \int_0^1 \left\langle \left(K_\nabla(q_t^{\mathrm{v}}\,|\mathrm{z}_t|^2) + K(|\mathrm{z}_t|^2\,\nabla q_t^{\mathrm{v}})\right) + 2K([d\varphi_{t,0}^{\mathrm{v}}]^* d_{\varphi_{t,0}^{\mathrm{v}}} j_0^* \mathrm{z}_t), h_t \right\rangle_{\mathcal{B}} dt,$$

with $q_t^{\mathrm{v}} = \int_0^t |d\varphi_{t,s}^{\mathrm{v}}|^{-1} ds$.

We can deduce from this our additional condition for a regular path to be a minimizing geodesic: for almost all $t \in [0,1]$,

$$\mathrm{v}_t + \frac{1}{2}\left(K D_t^{\mathrm{v}} + K_\nabla C_t^{\mathrm{v}}\right)_{\mathcal{B}} = 0,$$

where

$$D_t^{\mathrm{v}} \triangleq \sigma^2\,|\mathrm{z}_t|^2\,\nabla q_t^{\mathrm{v}} + 2[d\varphi_{t,0}^{\mathrm{v}}]^* d_{\varphi_{t,0}^{\mathrm{v}}} j_0^* \mathrm{z}_t,$$

and

$$C_t^{\mathrm{v}} \triangleq \sigma^2 q_t^{\mathrm{v}}\,|\mathrm{z}_t|^2\,.$$

It may be interesting to check that this condition boils down to the one we have obtained before for smooth trajectories, namely,

$$\mathrm{v}_t + K(dj_t^* \mathrm{z}_t) = 0.$$

It suffices to notice that, for pregeodesic trajectories, $j_t = j_0 \circ \varphi_{t,0}^{\mathrm{v}} + \sigma^2 \mathrm{z}_t q_t^{\mathrm{v}}$ and that, when z_t is smooth,

$$K D_t^{\mathrm{v}} + K_\nabla C_t^{\mathrm{v}} = K(D_t^{\mathrm{v}} + \nabla C_t^{\mathrm{v}}).$$

We now define geodesic paths (not necessarily minimizing).

Definition 10. *Let $j_0 \in H^1(\Omega, \mathbb{R}^d)$. A regular path $\mathrm{j} \in H^1([0,1], \mathcal{J}_W)$ starting at j_0 is called a geodesic path if and only if there exists $\mathrm{w} = (\mathrm{v}, \mathrm{z}) \in L^2([0,1], W)$ such that the following equations are satisfied almost everywhere in t:*

$$\text{(21)}\quad \begin{cases} \mathrm{j}_t = \mathrm{j}_0 \circ \varphi_{t,0}^{\mathrm{v}} + \sigma^2 \displaystyle\int_0^t \mathrm{z}_s \circ \varphi_{t,s}^{\mathrm{v}} ds, \\ \mathrm{z}_t = \mathrm{z}_0 \circ \varphi_{t,0}^{\mathrm{v}}\,\left|d\varphi_{t,0}^{\mathrm{v}}\right|, \\ \mathrm{v}_t + K([d\varphi_{t,0}^{\mathrm{v}}]^* d_{\varphi_{t,0}^{\mathrm{v}}} j_0^* \mathrm{z}_t) + \dfrac{\sigma^2}{2}\left(K_\nabla(q_t^{\mathrm{v}}\,|\mathrm{z}_t|^2) + K(|\mathrm{z}_t|^2\,\nabla q_t^{\mathrm{v}})\right) = 0, \end{cases}$$

with $q^{\mathrm{v}}_t = \int_0^t |d\varphi^{\mathrm{v}}_{t,s}|^{-1} ds$.

These equations are complete; it will be shown in section 7 that initial conditions (j_0, z_0) uniquely specify the solutions. It is interesting to check that geodesics as defined in (21) also are pregeodesics. For this, we first show that, for all times t, $z_t \in \mathcal{D}^*_{j_t}$. Noting that the first equation in (21) may also be written

$$\mathrm{j}_t = \mathrm{j}_0 \circ \varphi^{\mathrm{v}}_{t,0} + \sigma^2 \mathrm{z}_t q_t$$

it is clear that $\mathcal{D}_{j_t} = D_{z_t}$, and $z_t \in D^*_{z_t}$ is proved in Lemma 1. The same lemma also provides the fact that, for $w \in \mathcal{D}_{z_t}$,

$$\langle z_t \, , \, Dj_t w \rangle_2 = \left\langle z_t \, , \, d(\mathrm{j}_0 \circ \varphi^{\mathrm{v}}_{t,0}) w \right\rangle_2 + \sigma^2 \left\langle |z_t|^2 \, \nabla q_t \, , \, w \right\rangle_2 - \frac{\sigma^2}{2} \left\langle |z_t|^2 \, , \, \mathrm{div}(q^{\mathrm{v}}_t w) \right\rangle,$$

and this is equal to $-\langle \mathrm{v}_t \, , \, w \rangle_{\mathcal{B}}$ by definition of K and K_∇. We thus obtain the fact that $v_t + Dj^*_t z_t \in D^\perp_{j_t}$ as required.

We shall prove existence of solutions for a broader class of evolution equations, extending the range of initial values v_0. Consider the term $\mathrm{u}_t = K([d\varphi^{\mathrm{v}}_{t,0}]^* d_{\varphi^{\mathrm{v}}_{t,0}} j^*_0 \mathrm{z}_0 \circ \varphi^{\mathrm{v}}_{t,0} \left| d\varphi^{\mathrm{v}}_{t,0} \right|)$ which appears in the third equation of (21). We have, letting $\omega_0 = -dj^*_0 z_0$, and, for $w \in \mathcal{B}$,

$$\begin{aligned}
\langle \mathrm{u}_t \, , \, w \rangle_{\mathcal{B}} &= \left\langle [d\varphi^{\mathrm{v}}_{t,0}]^* d_{\varphi^{\mathrm{v}}_{t,0}} j_0 \mathrm{z}_0 \circ \varphi^{\mathrm{v}}_{t,0} \left| d\varphi^{\mathrm{v}}_{t,0} \right| , \, w \right\rangle_{L^2} \\
&= \left\langle d_{\varphi^{\mathrm{v}}_{t,0}} j_0 \mathrm{z}_0 \circ \varphi^{\mathrm{v}}_{t,0} \left| d\varphi^{\mathrm{v}}_{t,0} \right| , \, d\varphi^{\mathrm{v}}_{t,0} w \right\rangle_{L^2} \\
&= \left\langle \omega_0 \, , \, (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t} \right\rangle_{L^2}.
\end{aligned}$$

We know, by Appendix C, that $\varphi^{\mathrm{v}}_{0,t}$ belongs to $C^p(\overline{\Omega})$ as soon as $\mathcal{B}$ is continuously embedded in $C^p_0(\Omega, \mathbb{R}^k)$, which implies in this case (with $p \geq 1$) that

$$\left| (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t} \right|_{p-1,\infty} \leq \mathrm{Const} \, |w|_{\mathcal{B}} \, ,$$

the constant depending on $|v|_{1,\mathcal{B}}$. But this implies in turn that, if the L^2 inner product is replaced by the action of any continuous functional, ω_0, on $C^{p-1}_0(\Omega, \mathbb{R}^k)$, which will be denoted

$$\left(\omega_0, (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t}\right),$$

there exists an element of $\mathcal{B}$ that we shall still denote u_t such that

$$\langle u_t \, , \, w \rangle_{\mathcal{B}} = \left(\omega_0, (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t}\right).$$

With this notation, we may formulate the following definition.

Definition 11. *Let* $j_0 \in L^2(\Omega, \mathbb{R}^d)$. *Let* ω_0 *be a continuous linear functional on* $C^{p-1}(\Omega, \mathbb{R}^k)$ *and* $z_0 \in L^2(\Omega, \mathbb{R}^d)$. *A regular path* $\mathrm{j} \in H^1([0,1], \mathcal{J}_W)$ *starting at* j_0 *with initial direction* (ω_0, z_0) *is called a generalized geodesic if and only if, for all* $u \in \mathcal{D}_{j_0}$, *one has*

$$(\omega_0, u) + \langle z_0 \, , \, Dj_0 u \rangle = 0,$$

and there exists $\mathrm{w} = (\mathrm{v}, \mathrm{z}) \in L^2([0,1], W)$ *such that the following equations are satisfied almost everywhere in* t*:*

$$(22)\qquad \begin{cases} \mathrm{j}_t = \mathrm{j}_0 \circ \varphi^{\mathrm{v}}_{t,0} + \sigma^2 \displaystyle\int_0^t \mathrm{z}_s \circ \varphi^{\mathrm{v}}_{t,s} ds, \\ \mathrm{z}_t = \mathrm{z}_0 \circ \varphi^{\mathrm{v}}_{t,0} \left| d\varphi^{\mathrm{v}}_{t,0} \right|, \\ \mathrm{v}_t - \mathrm{u}^{\mathrm{v}}_t + \dfrac{\sigma^2}{2} \Big(K_\nabla (q^{\mathrm{v}}_t \left|\mathrm{z}_t\right|^2) + K(\left|\mathrm{z}_t\right|^2 \nabla q^{\mathrm{v}}_t) \Big) = 0, \\ \forall w \in \mathcal{B}\ \langle \mathrm{u}^{\mathrm{v}}_t \,,\, w \rangle_{\mathcal{B}} = \big(\omega_0, (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t}\big) \end{cases}$$

with $q^{\mathrm{v}}_t = \int_0^t |d\varphi^{\mathrm{v}}_{t,s}|^{-1} ds$.

Recall that when j_0 is smooth, the only choice is $\omega_0 = dj_0^* z_0$, and if z_0 is also smooth, the system may be written under the simple form

$$(23)\qquad \begin{cases} \mathrm{j}_t = \mathrm{j}_0 \circ \varphi^{\mathrm{v}}_{t,0} + \sigma^2 \displaystyle\int_0^t \mathrm{z}_s \circ \varphi^{\mathrm{v}}_{t,s} ds, \\ \mathrm{z}_t = \mathrm{z}_0 \circ \varphi^{\mathrm{v}}_{t,0} \left| d\varphi^{\mathrm{v}}_{t,0} \right|, \\ \mathrm{v}_t + \sigma^2 K(dj_t^* z_t) = 0. \end{cases}$$

As an example of the nonsmooth applications we have in mind, assume that j_0 is a binary, plane image, which is the indicator function of the interior of a connected open subset Ω_1 of Ω with smooth boundary $\partial\Omega_1$. We have seen that any element $w \in \mathcal{D}_{j_0}$ should be tangent to $\partial\Omega_1$ and that in this case $Dj_0 w = 0$ and $\mathcal{D}^*_{j_0} = L^2(\Omega, \mathbb{R})$. We therefore may choose z_0 arbitrarily in L^2, and (ω_0, w) should vanish for $w \in \mathcal{D}_{j_0}$, which is true, for example, when ω_0 is defined by

$$(\omega_0, w) = \int_{\partial\Omega_1} \langle w \,,\, \nu_1 \rangle d\sigma_1,$$

where ν_1 is the outward normal to $\partial\Omega_1$ and σ_1 is the surface measure on $\partial\Omega_1$.

6. Existence of minimizing geodesics. The next theorem states that minimizing geodesics always exist between two elements of $\mathcal{J}_W$.

THEOREM 6. *Assume that* $\mathcal{B}$ *is compactly embedded in* $C^1_0(\Omega, \mathbb{R}^d)$*, and let* $j_0, j_1 \in \mathcal{J}_W$*. Then* $G_{\mathcal{J}_W}(j_0, j_1)$ *is nonempty.*

Proof. Let $(\mathrm{j}^n)_{n\in\mathbb{N}}$ be a minimizing family of paths in $H^1([0,1], \mathcal{J}_W)$ from j_0 to j_1; for any $n \in \mathbb{N}$, let $\mathrm{w}^n_t \triangleq \overline{p}(\frac{d\mathrm{j}^n}{dt})$ so that $(\mathrm{w}^n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^2([0,1], W)$. Up to the extraction of a subsequence, we can assume that w^n converges weakly to a w^∞ in $L^2([0,1], W)$. By lower semicontinuity, we have

$$\int_0^1 |\mathrm{w}^\infty_t|^2_W dt \le d_{\mathcal{J}_W}(j_0, j_1).$$

By a time change argument, which is classical in the proof that minimizing geodesics travel at constant speed (see [12]), we may furthermore assume that $|\mathrm{w}^n_t|_W$ is uniformly bounded by, say, $d_{\mathcal{J}_W}(j_0, j_1) + 1$. Denoting $w^n = (\mathrm{v}^n, \mathrm{z}^n)$, consider $\mathrm{j}'_t \triangleq j_0 \circ \varphi^\infty_{t,0} + \sigma^2 \int_0^t \mathrm{z}^\infty_s \circ \varphi^\infty_{t,s} ds$, where φ^∞ is the flow associated to v^∞. Since j'

is a regular path, it is sufficient to prove that $\mathrm{j}_1' = j_1$. However, if φ^n denotes the flow associated with v^n, we know, from Theorem 9, that $\varphi^n_{1,0}$ converges uniformly to $\varphi^\infty_{1,0}$ so that $j_0 \circ \varphi^n_{t,0} \to j_0 \circ \varphi^\infty_{t,0}$ in $L^2(\Omega, \mathbb{R}^d)$. Now, let $u \in C_c^\infty(\Omega, \mathbb{R}^d)$. We have $\int_0^1 \langle \mathrm{z}^n_s \circ \varphi^n_{1,s}, u\rangle_2 ds = \int_0^1 \langle \mathrm{z}^n_s, u \circ \varphi^n_{s,1} \left|d\varphi^n_{s,1}\right|\rangle_2 ds$. Since u has bounded derivatives and using Theorem 9 implies the uniform convergence of φ^n_{s1} to φ^∞_{s1} and the pointwise convergence of the derivatives (because of the uniform boundedness of $|v^n_s|_{\mathcal{B}}$), we have

$$\int_0^1 \langle \mathrm{z}^n_s, u \circ \varphi^n_{s,1} \left|d\varphi^n_{s,1}\right|\rangle_2 ds - \int_0^1 \langle \mathrm{z}^n_s, u \circ \varphi^\infty_{s,1} \left|d\varphi^\infty_{s,1}\right|\rangle_2 ds \to 0. \tag{24}$$

Moreover, from the weak convergence of z^n to z^∞, we get

$$\int_0^1 \langle \mathrm{z}^n_s, u \circ \varphi^\infty_{s,1} \left|d\varphi^\infty_{s,1}\right|\rangle_2 ds \to \int_0^1 \langle \mathrm{z}^\infty_s, u \circ \varphi^\infty_{s,1} \left|d\varphi^\infty_{s,1}\right|\rangle_2 ds, \tag{25}$$

so that finally $\langle \mathrm{j}_1 - \mathrm{j}_1', u\rangle_2 = 0$ for any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$. Hence $\mathrm{j}' \in H^1([0,1], \mathcal{J}_W)$ and the result is proved. □

7. Initial value problem for the geodesic equation. We have the following theorem.

THEOREM 7. *Assume that $\mathcal{B}$ is continuously embedded in $C_0^p(\Omega, \mathbb{R}^p)$ for $p \geq 3$. Then, for all $T > 0$, there exists a unique solution $(\mathrm{v}, \mathrm{j}, \mathrm{z})$ of (21) over $[0, T]$, with initial values $j_0 \in H^1(\Omega, \mathbb{R}^d)$, $z_0 \in L^2(\Omega, \mathbb{R}^d)$, and $\omega_0 \in C^{p-1}([0,1], \mathbb{R}^k)'$ (where $C^{p-1}([0,1], \mathbb{R}^k)'$ denotes the topological dual of $C^{p-1}([0,1], \mathbb{R}^k)$ with the norm $|\omega| \triangleq \sup_{|v|_{p-1,\infty} \leq 1} (\omega, v)$) which continuously depends on these initial conditions.*

Continuity of the solution $(\mathrm{v}, \mathrm{j}, \mathrm{z})$ as a function of (j_0, z_0) is meant according to $H^1(\Omega, \mathbb{R}^d) \times L^2(\Omega, \mathbb{R}^d)$-norms for the initial conditions, $L^2([0,T], W)$-norm for (v, z), and $C([0,1], L^2(\Omega, \mathbb{R}^d))$-norm for j.

8. Proof of Theorem 7. To prove Theorem 7, we show the existence of solutions for short time and then extend them to all time. Fix $T > 0$. For a given $\mathrm{v} \in L^2([0,T], \mathcal{B})$, let $\Psi(\mathrm{v}) \in L^2([0,T], \mathcal{B})$ be defined by

$$\begin{cases} \Psi(\mathrm{v})_t = \mathrm{u}^v_t - \dfrac{\sigma^2}{2}\left(K_\nabla(q^{\mathrm{v}}_t \left|\mathrm{z}^{\mathrm{v}}_t\right|^2) + K(\left|\mathrm{z}^{\mathrm{v}}_t\right|^2 \nabla q^{\mathrm{v}}_t)\right), \\ \mathrm{z}^{\mathrm{v}}_t = \mathrm{z}_0 \circ \varphi^{\mathrm{v}}_{t,0} \left|d\varphi^{\mathrm{v}}_{t,0}\right|, \\ \langle \mathrm{u}^v_t \,,\, w\rangle_{\mathcal{B}} = \left(\omega_0, (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t}\right). \end{cases} \tag{26}$$

To estimate the Lipschitz coefficient of Ψ, we introduce $\mathrm{v}, \mathrm{v}' \in L^1([0,T], \mathcal{B})$ and compute the variation of each term in $\Psi(\mathrm{v})_t - \Psi(\mathrm{v}')_t$. Fix $w \in \mathcal{B}$ with $|w|_{\mathcal{B}} = 1$. We have

$$\begin{aligned} \langle \Psi(\mathrm{v})_t \,,\, w\rangle_{\mathcal{B}} =& \frac{\sigma^2}{2}\left\langle \left|z^v_t\right|^2 , \, q^{\mathrm{v}}_t \mathrm{div}(w)\right\rangle_2 - \frac{\sigma^2}{2}\left\langle \left|z^v_t\right|^2 , \, dq^{\mathrm{v}}_t w\right\rangle_2 + \left(\omega_0, (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t}\right) \\ =& \left(\omega_0, (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t}\right) + \frac{\sigma^2}{2}\left\langle \left|z_0\right|^2 , \, \left|d\varphi^{\mathrm{v}}_{0,t}\right|^{-1} q^{\mathrm{v}}_t \circ \varphi^{\mathrm{v}}_{0,t} \mathrm{div}(w) \circ \varphi^{\mathrm{v}}_{0,t}\right\rangle_2 \\ & - \frac{\sigma^2}{2}\left\langle \left|z_0\right|^2 , \, \left|d\varphi^{\mathrm{v}}_{0,t}\right|^{-1} d_{\varphi^{\mathrm{v}}_{0,t}} q^{\mathrm{v}}_t w \circ \varphi^{\mathrm{v}}_{0,t}\right\rangle_2. \end{aligned} \tag{27}$$

We have

$$q_t^{\rm v} \circ \varphi_{0,t}^{\rm v}(x) = \int_0^t \left| d_{\varphi_{0,t}^{\rm v}(x)} \varphi_{t,s}^{\rm v} \right|^{-1} ds = \int_0^t \left| d_{\varphi_{0,s}^{\rm v}(x)} \varphi_{s,t}^{\rm v} \right| ds = \int_0^t \frac{\left| d_x \varphi_{0,t}^{\rm v} \right|}{\left| d_x \varphi_{0,s}^{\rm v} \right|} ds,$$

and letting $\xi_{s,t}^{\rm v} = \frac{\left| d_x \varphi_{0,t}^{\rm v} \right|}{\left| d_x \varphi_{0,s}^{\rm v} \right|}$ and $\lambda_t^{\rm v}(w) = (d\varphi_{0,t}^{\rm v})^{-1} w \circ \varphi_{0,t}^{\rm v}$,

(28)

$$\langle \Psi({\rm v})_t \,,\, w \rangle_{\mathcal{B}} = \frac{\sigma^2}{2} \int_0^t \left\langle |z_0|^2 \,,\, (\left| d\varphi_{0,s}^{\rm v} \right|^{-1} {\rm div}(w) \circ \varphi_{0,t}^{\rm v} - \left| d\varphi_{0,t}^{\rm v} \right|^{-1} \left\langle \nabla \xi_{s,t}^{\rm v} \,,\, \lambda_t^{\rm v}(w) \right\rangle) \right\rangle ds$$
$$+ (\omega_0, \lambda_t^{\rm v}(w)).$$

This implies

$$\begin{aligned}
|\Psi({\rm v}')_t - \Psi({\rm v})_t|_{\mathcal{B}} \le \frac{\sigma^2}{2} |z_0|_2^2 \sup \Bigg\{ \int_0^t \Bigg(& \left| d\varphi_{0,s}^{\rm v} \right|^{-1} {\rm div}(w) \circ \varphi_{0,t}^{\rm v} \\
& - \left| d\varphi_{0,s}^{{\rm v}'} \right|^{-1} {\rm div}(w) \circ \varphi_{0,t}^{{\rm v}'} \Bigg) ds : |w|_{\mathcal{B}} = 1 \Bigg\} \\
+ \frac{\sigma^2}{2} |z_0|_2^2 \sup \Bigg\{ \int_0^t \Bigg(& \left| d\varphi_{0,t}^{\rm v} \right|^{-1} \left\langle \nabla \xi_{s,t}^{\rm v} \,,\, \lambda_t^{\rm v}(w) \right\rangle \\
& - \left| d\varphi_{0,t}^{{\rm v}'} \right|^{-1} \left\langle \nabla \xi_{s,t}^{{\rm v}'} \,,\, \lambda_t^{{\rm v}'}(w) \right\rangle \Bigg) ds : |w|_{\mathcal{B}} = 1 \Bigg\} \\
+ |\omega_0| \sup \Bigg\{ & \left| \lambda_t^{\rm v}(w) - \lambda_t^{{\rm v}'}(w) \right|_{p-1,\infty} : |w|_{\mathcal{B}} = 1 \Bigg\}.
\end{aligned}$$

The problem is thus reduced to the estimation of variations, with respect to v, of $\lambda_t^{\rm v}(w)$, $\nabla \xi_{s,t}^{\rm v}$ and of $\left| d\varphi_{0,s}^{\rm v} \right|^{-1} {\rm div}(w) \circ \varphi_{0,t}^{\rm v}$. They involve differentials of $\varphi^{\rm v}$, $\varphi^{{\rm v}'}$, and w up to the second degree. The inclusion of $\mathcal{B}$ in $C^3([0,1], \mathbb{R}^k)$ and an application of Lemmas 7 and 11 in the appendix directly lead to the estimate

$$|\Psi({\rm v})_t - \Psi({\rm v}')_t|_{\mathcal{B}} \le C \left(\sigma^2 |z_0|_2^2 + |\omega_0| \right) |{\rm v} - {\rm v}'|_{1,T} \, e^{C' \max(|{\rm v}|_{1,T}, |{\rm v}'|_{1,T})}, \tag{29}$$

and finally

$$\begin{aligned}
|\Psi({\rm v}) - \Psi({\rm v}')|_{2,T} &\le C\sqrt{T} \left(\sigma^2 |z_0|_2^2 + |\omega_0| \right) |{\rm v} - {\rm v}'|_{1,T} \, e^{C' \max(|{\rm v}|_{1,T}, |{\rm v}'|_{1,T})} \\
&\le CT \left(\sigma^2 |z_0|_2^2 + |\omega_0| \right) |{\rm v} - {\rm v}'|_{2,T} \, e^{C' \sqrt{T} \max(|{\rm v}|_{2,T}, |{\rm v}'|_{2,T})}.
\end{aligned} \tag{30}$$

Therefore, Ψ is q-Lipschitz with $q < 1$ for T small enough, and its unique fixed point yields a unique solution of (21). This is stated below.

LEMMA 3. *There exists a time $T > 0$ depending only on $|z_0|_2$ and $|j_0|_{H^1}$ such that a unique solution of* (21) *exists on* $[0, T]$.

We now show that this solution can be extended to all times. For this, we prove that there exists a unique fixed point for Ψ at all times. Denote by Ψ_T this mapping when defined on $L^2([0,T], \mathcal{B})$. Clearly, if v is a fixed point of Ψ_T, its restriction to $[0, S]$ is a fixed point of Ψ_S. Thus, if T_0 is the largest T such that Ψ_S has a unique fixed point ${\rm v}^S$ in $L^2([0,S], \mathcal{B})$ for any $S < T$, then each ${\rm v}^T$ for $T < T_0$ is an extension of ${\rm v}^S$ whenever $S \le T$. We can show that $T_0 = \infty$ by showing that, if $T_0 < \infty$, then there exists $\varepsilon > 0$ (depending only on T_0 and the initial conditions) such that, for all $T < T_0$, there exists a unique extension of v^T to $[T, T+\varepsilon]$. Fix such a T; the issue of extending a fixed point of Ψ_T on $[T, T+\varepsilon]$ can be rephrased as a fixed point problem for small time with the following notation. For $\mathrm{v} \in L^2([0,T], \mathcal{B})$ and $\mathrm{v}' \in L^2([0,\varepsilon], \mathcal{B})$, define $\mathrm{v} \vee \mathrm{v}' \in L^2([0, T+\varepsilon], \mathcal{B})$, equal to v on $[0,T]$ and equal to $(t \mapsto \mathrm{v}'(t-T))$ on $[T, T+\varepsilon]$. Introduce the function $\Psi^\varepsilon : L^2([0,\varepsilon],\mathcal{B}) \to L^2([0,\varepsilon],\mathcal{B})$ defined by

$$\Psi^\varepsilon(\mathrm{v})(t) = \Psi_{T+\varepsilon}(\mathrm{v}^T \vee \mathrm{v})(t-T).$$

For $t > T$,

$$q_t^{\mathrm{v}^T \vee \mathrm{v}} = q_1^{\mathrm{v}^T} + q_{t-T}^{\mathrm{v}},$$

$\mathrm{z}_t = \mathrm{z}_T \circ \varphi^{\mathrm{v}}_{Tt} |d\varphi^{\mathrm{v}}_{Tt}|^{-1}$, and

$$\begin{aligned}\langle \mathrm{u}_t^{\mathrm{v}}, w\rangle_{\mathcal{B}} &= \left(\omega_0, (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t}\right)\\ &= \left(\omega_0, (d\varphi^{\mathrm{v}}_{0,T})^{-1} (d_{\varphi^{\mathrm{v}}_{0,T}} \varphi^{\mathrm{v}}_{Tt})^{-1} w \circ \varphi^{\mathrm{v}}_{Tt} \circ \varphi^{\mathrm{v}}_{0,T}\right)\\ &= \left(\omega_T, d\varphi^{\mathrm{v}}_{Tt})^{-1} w \circ \varphi^{\mathrm{v}}_{Tt}\right)\end{aligned}$$

with $(\omega_T, w) = \left(\omega_0, (d\varphi^{\mathrm{v}}_{0,T})^{-1} w \circ \varphi^{\mathrm{v}}_{0,T}\right)$. It is clear that the study of Ψ^ε can follow exactly the lines of the study of Ψ_T, yielding a unique fixed point if ε is small enough, the size of admissible ε being controlled by the L^2-norms of z_T and the norm of ω_T as a linear form on $C_0^{p-1}(\Omega, \mathbb{R}^k)$. These norms can in turn be bounded by the L^2-norms of z_0 and the norm of ω_0, respectively, multiplied by a continuous function of $\max(|\varphi^{\mathrm{v}^T}_{0,T}|_{1,\infty}, |\varphi^{\mathrm{v}^T}_{T,0}|_{1,\infty})$. Proving that this is uniformly bounded for $T < T_0$ is therefore sufficient to get the contradiction we aim for, that is, that the solution can be uniquely extended beyond T_0.

So, everything relies on proving the uniform boundedness of $\varphi^{\mathrm{v}^T}_{0,T}$, $\varphi^{\mathrm{v}^T}_{T,0}$, and their derivatives over Ω. By Lemmas 7 and 9, these quantities are bounded by functions of $|\mathrm{v}^T|_{1,T}$ so that we have to prove that these can be bounded uniformly in T. However, it suffices to use the facts that v^T satisfies a geodesic equation and that geodesics travel at constant speed. More precisely, defining, for $t \le T < T_0$,

$$\psi_t = \left|\mathrm{v}_t^T\right|_{\mathcal{B}}^2 + \sigma^2 \left|\mathrm{z}_t\right|_2^2,$$

we have (recall that this does not depend on T as soon as $T \ge t$) $\psi_t \equiv \psi(0)$ so that

$$\left|\mathrm{v}^T\right|_{1,T} \le T\psi(0)$$

for all T. It is well known that *minimizing* geodesics have constant speed, but we must check that this property remains true for all the solutions of (22). This is proved in the appendix and is stated, for further reference, in the next lemma.

LEMMA 4. *If (j, v, z) is a solution of system* (22) *on $[0, T[$, then $|v_t|_{\mathcal{B}}^2 + \sigma^2 |z_t|_2^2$ is constant with respect to time.*

To prove the continuity of the solution, let (v, j, z) and (v', j', z') be two solutions of system (22) with initial conditions (ω_0, z_0) and (ω_0', z_0'), respectively. Using, in particular, the computation leading from (28) to (29), it is not to difficult to obtain the estimate

$$\begin{aligned}|v_t - v_t'|_{\mathcal{B}} &\le C\left(|\omega_0 - \omega_0'| + \frac{\sigma^2}{2}\left||z_0|^2 - |z_0'|^2\right|\right) e^{C|\mathrm{v}|_{1,T}}\\ &\quad + C\left(\sigma^2 |z_0'|_2^2 + |\omega_0'|\right) |\mathrm{v} - \mathrm{v}'|_{1,T}\, e^{C|\mathrm{v}'|_{1,T}}.\end{aligned}$$

As we just have shown, $|\mathrm{v}|_{1,T} = T\,|\mathrm{v}_0|_{\mathcal{B}}$, and this is smaller (up to a universal multiplicative constant) than $|\omega_0|$ so that

$$|v_t - v_t'|_{\mathcal{B}} \leq C\left(|\omega_0 - \omega_0'| + \frac{\sigma^2}{2}\left||z_0|^2 - |z_0'|^2\right|\right)e^{CT|\omega_0|} + C\left(\sigma^2\,|z_0'|_2^2 + |\omega_0'|\right)|\mathrm{v} - \mathrm{v}'|_{1,T}\,e^{CT|\omega_0'|}.$$

Gronwall's lemma now allows us to conclude that, for some constant C which may now depend on $T, |\omega_0|, |\omega_0'|, |z_0|_2$, and $|z_0'|_2$,

$$|v_t - v_t'|_{\mathcal{B}} \leq C\left(|\omega_0 - \omega_0'| + \frac{\sigma^2}{2}\left||z_0|^2 - |z_0'|^2\right|\right). \tag{31}$$

9. Normal coordinates in H^1. We now consider the question, which motivated this paper, of whether the previous construction could be used as an indexing device for characterizing the deformations and variations of an object relative to a prototype.

Fix an image $j_0 \in H^1(\Omega, \mathbb{R}^d)$. The computationally simplest way to describe an image j in a neighborhood of j_0 is by the difference $j - j_0$. However, one cannot be satisfied with this representation which takes no account of the metric we have placed on $\mathcal{J}_W$. Among local charts related to the metric, normal coordinates on a Riemannian manifold are radial flattenings of this manifold onto its tangent space in the sense that radial lines in this space correspond to geodesics on the manifold. They provide a very efficient linear representation of the manifold and of its metric. Existence of such coordinates is a standard theorem in finite dimensions, and our purpose is to check how much of this result remains valid in our infinite dimensional framework.

In the previous sections, another candidate for local coordinates has emerged, which turns out to be closely related (it is in fact dual) to normal coordinates. We have proved that, for a fixed $j_0 \in H^1(\Omega, \mathbb{R}^d)$, one can associate to any $z_0 \in L^2(\Omega, \mathbb{R}^d)$ a unique solution of system (21). We introduce the function $M_{j_0} : L^2 \to L^2$, which assigns to $z_0 \in L^2$ the "image" j_1, where j_t is the solution of (21) at time t.

The following theorem shows that M_{j_0} shares some features of local coordinates on $\mathcal{J}_W$.

THEOREM 8. *Let $\mathcal{B}_{H^1}(0,\varepsilon)$ denote the open ball in $H^1(\Omega, \mathbb{R}^d)$ containing all $z_0 \in H^1(\Omega, \mathbb{R}^d)$ such that $|z_0|_{H^1} < \varepsilon$. Then, for all $j_0 \in H^1(\Omega, \mathbb{R}^d)$, there exists $\varepsilon > 0$ such that M_{j_0} restricted to $\mathcal{B}_{H^1}(0,\varepsilon)$ is continuous and one-to-one onto its image, equipped with the L^2-topology.*

Proof of Theorem 8. Continuity of $M_{j_0} : L^2(\Omega, \mathbb{R}^d) \to L^2(\Omega, \mathbb{R}^d)$ is a consequence of Theorem 7, and it trivially implies the continuity of the restriction $M_{j_0} : H^1(\Omega, \mathbb{R}^d) \to L^2(\Omega, \mathbb{R}^d)$ for the $H^1(\Omega, \mathbb{R}^d)$-topology. We show that this map is one-to-one in a neighborhood of 0. We first have the following lemma.

LEMMA 5. *Let $j_0, z_0, \tilde{z}_0 \in H^1(\Omega, \mathbb{R}^d)$, with $\max(|z_0|_{H^1}, |\tilde{z}_0|_{H^1}) \leq 1$. Denote $\tilde{\mathrm{v}}$ the time-dependent vector field along the solution of (21) with initial condition $(j_0, \tilde{z}_0)$. Then, there exist a constant C and a function ε which depend only on j_0 such that, for $t > 0$,*

$$\left|(M_{j_0}(t\tilde{z}_0) - M_{j_0}(tz_0)) \circ \varphi_{0,t}^{\tilde{\mathrm{v}}} - t[\sigma^2(\tilde{z}_0 - z_0) + dj_0 K(dj_0^*(\tilde{z}_0 - z_0))]\right|_2 \leq Ct\,|\tilde{z}_0 - z_0|_2\,\varepsilon(t),$$

and $\lim_{t\to 0}\varepsilon(t) = 0$.

The proof of Lemma 5 is given in Appendix G. To prove Theorem 8, we first remark that

$$\left|\sigma^2(\tilde z_0 - z_0) + dj_0 K(dj_0^*(\tilde z_0 - z_0))\right|_2 \geq \sigma^2 \left|\tilde z_0 - z_0\right|_2$$

so that

$$\left|(M_{j_0}(t\tilde z_0) - M_{j_0}(tz_0)) \circ \varphi_{0,t}^{\tilde{\mathrm{v}}}\right|_2 \geq t\sigma^2 \left|\tilde z_0 - z_0\right|_2 \left(1 - \frac{C}{\sigma^2}\varepsilon(j_0, t)\right),$$

and the lower bound is nonvanishing as soon as t is small enough. □

Remark that we have, for $j_1, j_2 \in H^1(\Omega, \mathbb{R}^d)$, the inequality

$$d(j_1, j_2) \leq \frac{1}{\sigma}\left|j_1 - j_2\right|_2$$

since the right-hand side is an upper bound of the length of the curve $\mathrm{j}_t = (1-t)j_1 + tj_2$ (since choosing $\mathrm{v} \equiv 0$ and $\sigma^2 \mathrm{z} \equiv j_2 - j_1$, we have $\mathrm{w}_t \triangleq (v_t, z_t) \in \frac{\partial j_t}{\partial t}$ and $\sigma^2 \int_0^1 \left|z_t\right|_2^2 = \left|j_2 - j_1\right|_2^2/\sigma^2$). So continuity of M_{j_0} for the d-topology on its image is also true.

According to Lemma 5, normal coordinates (which are time derivatives at $t = 0$ of geodesics) are related to M by the relation (we use the standard exponential notation)

$$\exp_{j_0}(Sz_0) = M_{j_0}(z_0),$$

where S is defined by

$$Sz \triangleq \sigma^2 z + Dj_0 K(Dj_0^* z).$$

This indicates that a good approximation of the metric in terms of the z-coordinate would be

$$\left|z_1 - z_2\right|_{j_0}^2 = \langle z_1 - z_2\,,\, S(z_1 - z_2)\rangle_2,$$

which satisfies

$$\left|z_1\right|_{j_0} = d(j_0, M_{j_0}(z_1))$$

in a neighborhood of 0.

10. Experiments. In this section, we propose a preliminary set of experiments to illustrate the information contained in the z-coordinate described above. Experiments in Figures 1, 2, and 3 were conducted in two steps: given two images j_0 and j_1, we first computed the minimizing geodesic between them, yielding a trajectory $(\mathrm{j}_t, \mathrm{z}_t, \mathrm{v}_t)$ and the corresponding flow φ_t^{v}. Then, using j_0 again, and the obtained value z_0 on the minimizing geodesic, we computed the solution of (21) until time $t = 1$. The obtained values $(\mathrm{j}'_t, \mathrm{z}'_t, \mathrm{v}'_t)$ could then be compared with those which have been computed along the geodesics. In Figure 4, the initial j_0 is the same as in Figures 2 and 3, but the z_0 is the average of the two so that it does not correspond to any precomputed geodesic in the image space. The result is quite interesting, because it still possesses characteristics of a human face and can be compared to the result of a simple linear combination of the target images in Figures 2 and 3.

The numerical implementation of both operations (minimization of the geodesic energy and integration of (21)) must be done with some care in order, in particular, to avoid instabilities due to the conservation part of the energy. Details will be provided in a forthcoming paper.

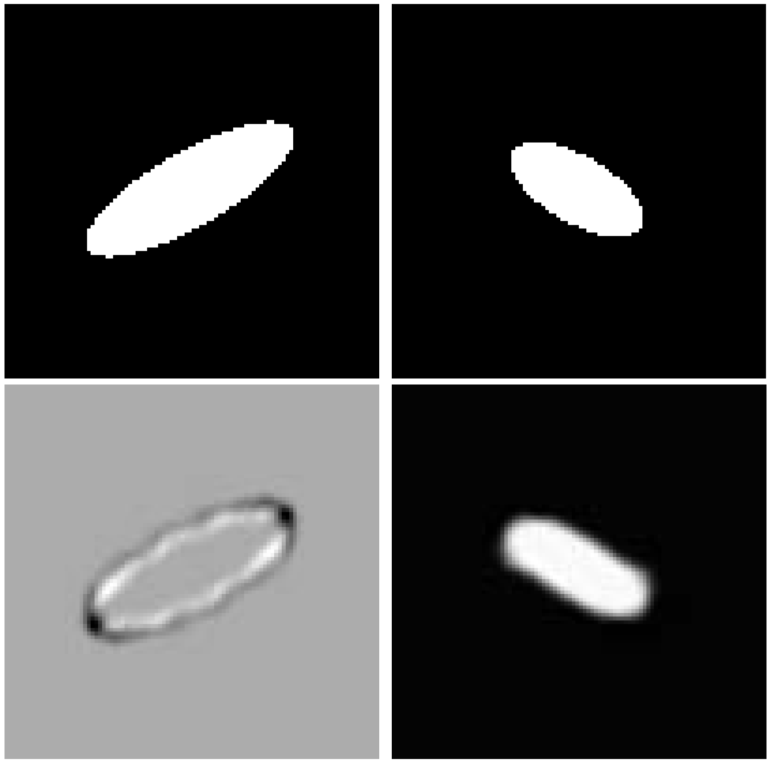

FIG. 1. *From top to bottom and from left to right: Initial image, target image, z-coordinate, reconstructed target image.*

Appendix A. Proofs of Propositions 2 and 3.

A.1. Proof of Proposition 2. The proof relies on a sequence of standard measurability arguments, of which we sketch only the main steps. First let $(w_n)_{n\in\mathbb{N}}$ be a Hilbert basis of W. Since, for any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$ and $w = (v, z) \in W$, $j \to l_{j,u}(w)$ (which has been defined in (6) by $\sigma^2\langle z, u\rangle_2 + \langle j, \mathrm{div}(u \otimes v)\rangle_2$) is continuous from $L^2(\Omega, \mathbb{R}^d)$ to $\mathbb{R}$, the map

$$j \mapsto w_{j,u} \triangleq \sum_{n\geq 0} l_{j,u}(w_n) w_n$$

is measurable from $L^2(\Omega, \mathbb{R}^d)$ to W. By construction, we have, for $w \in W$,

$$\langle w\,,\, w_{ju}\rangle_W = \sum_{n\geq 0} l_{j,u}(w_n)\langle w\,,\, w_n\rangle_W = l_{j,u}(w).$$

Thus, for $\gamma \in T_j \mathcal{J}_W$, we have

$$\overline{p}(\gamma) = \mathrm{Argmin}\left\{|w|_W : \langle w\,,\, w_{j,u}\rangle = \langle \gamma\,,\, u\rangle \text{ for all } u \in C_c^\infty(\Omega, \mathbb{R}^d)\right\}.$$

Introducing a family $(u_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega, \mathbb{R}^d)$ which is dense in $H_0^1(\Omega, \mathbb{R}^d)$, the previous expression may be replaced by

$$\overline{p}(\gamma) = \mathrm{Argmin}\left\{|w|_W : \langle w\,,\, w_{j,u_n}\rangle = \langle \gamma\,,\, u_n\rangle \text{ for all } n \geq 0\right\}.$$

Fig. 2. *From top to bottom and from left to right: Initial image, target image, z-coordinate, reconstructed target image.*

For $N \in \mathbb{N}$ and $\lambda > 0$, we define

$$\overline{p}^{N,\lambda}(\gamma) = \text{Argmin}\left\{ |w|_W^2 + \lambda \sum_{n=0}^{N} \left(\langle w\,,\, w_{j,u_n} \rangle_W - \langle \gamma\,,\, u_n \rangle \right)^2 \right\}. \tag{32}$$

Clearly, we must have $\overline{p}^{N,\lambda}(\gamma) = \sum_{i=1}^{N} x_i w_{j,u_i}$, where

$$x = \underset{x' \in \mathbb{R}^{N+1}}{\text{Argmin}} \left\{ \left| \sum_{n=0}^{N} x'_n w_{j,u_n} \right|_W^2 + \lambda \sum_{n=0}^{N} \left(\sum_{n'=1}^{N} x'_{n'} \left\langle w_{j,u_{n'}}\,,\, w_{j,u_n} \right\rangle_W - \langle \gamma\,,\, u_i \rangle \right)^2 + \frac{1}{\lambda} |x'|^2 \right\}.$$

For $\lambda > 0$, the optimal x is given by $x = (A + I/\lambda)^{-1} y$, where $y \in \mathbb{R}^{N+1}$ is such that $y_i = \langle \gamma\,,\, u_i \rangle$ and A is an $(N+1) \times (N+1)$ matrix with coefficients given

FIG. 3. *From top to bottom and from left to right: Initial image, target image, z-coordinate, reconstructed target image.*

by $a_{n,n'} = \left\langle w_{j,u_{n'}}, w_{j,u_n} \right\rangle_W$. This implies that, if γ_t is a measurable path, the function $t \mapsto \overline{p}^{N,\lambda}(\gamma_t)$ is measurable. The measurability of $\overline{p}(\gamma_t)$ is a consequence of the pointwise convergence of $\overline{p}^{N,N}(\gamma_t)$ to $\overline{p}(\gamma_t)$, which comes from the following argument: for all N and λ, we have $\left|\overline{p}^{N,\lambda}(\gamma)\right|_W \leq |\overline{p}(\gamma)|_W$, since the last term in (32) vanishes for $w = \overline{p}(\gamma)$. For the same reason,

$$\sum_{n=0}^{N} \left(\left\langle \overline{p}^{N,\lambda}(\gamma), w_{j,u_n} \right\rangle - \langle \gamma, u_n \rangle\right)^2 \leq \frac{1}{\lambda} |\overline{p}(\gamma)|_W ,$$

which implies that for all n, $\left\langle \overline{p}^{N,N}(\gamma), w_{j,u_n} \right\rangle \to \langle \gamma, u_n \rangle$ when N tends to infinity. Moreover, for any weakly converging subsequence extracted from $\overline{p}^{N,N}(\gamma)$ (which forms a weakly compact set in W), we have, and denoting w^* its limit, $|w^*|_W \leq \liminf \left|\overline{p}^{N,N}(\gamma)\right|_W \leq |\overline{p}(\gamma)|_W$, and, for all n, $\langle w^*, w_{j,u_n} \rangle = \langle \gamma, u_n \rangle$ by weak convergence, which is only possible when $w^* = \overline{p}(\gamma)$.

Hence $t \mapsto \overline{p}(\gamma_t)$ is measurable if γ_t is measurable, and the proof of Proposition 2 is ended.

FIG. 4. *From top to bottom and from left to right: Initial image, target image, z-coordinate, obtained by averaging the z-coordinate in Figures* 2 *and* 3, *and obtained target image.*

A.2. Proof of Proposition 3. We deduce from Proposition 2 that it is sufficient to prove the next proposition.

PROPOSITION 6. *Let* $w \in L^2([0,1], W)$ *such that for any* $u \in C_c^\infty(\Omega \times]0,1[, \mathbb{R}^d)$ *we have*

$$\int_0^1 \left(\sigma^2 \langle z_t, u_t \rangle_2 + \langle j_t, \operatorname{div}(u_t \otimes v_t) \rangle_2 \right) \, dt = 0. \tag{33}$$

Then almost everywhere in t, $w_t \in E_{j_t}$.

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be a family in $C_c^\infty(\Omega, \mathbb{R}^d)$ dense in $C_c^\infty(\Omega, \mathbb{R}^d)$ for the $H^1(\Omega, \mathbb{R}^d)$-norm. If we prove that for any $n \in \mathbb{N}$, the function c_n defined by $c_n(t) \triangleq \sigma^2 \langle z_t, u_n \rangle_2 + \langle j_t, \operatorname{div}(u_n \otimes v_t) \rangle_2$ is vanishing almost everywhere, then by density, there exists a negligible set $\mathcal{N}$ such that for any $t \in [0,1] \setminus \mathcal{N}$ and any $u \in C_c^\infty(\Omega, \mathbb{R}^d)$

$$\sigma^2 \langle z_t, u \rangle_2 + \langle j_t, \operatorname{div}(u \otimes v_t) \rangle_2 = 0,$$

which implies Proposition 6. Hence, let us consider $n \in \mathbb{N}$. For any $f \in C_c^\infty([0,1],\mathbb{R})$, if $u(t,x) \triangleq f(t)u_n(x)$, we have from (33) that

$$\int_0^1 c_n(t)f(t)dt = 0$$

so that, by standard arguments, we get $c_n = 0$ almost everywhere. ◻

Appendix B. Proof of Theorem 2. We start the ($\Leftarrow$) part in the case $L^2([0,1],W)$.

Lemma 6. *Let* $\mathrm{w} = (\mathrm{z},\mathrm{v}) \in L^2([0,1],W)$. *Let us define for any* $t \in [0,1]$

$$\mathrm{j}_t \triangleq \mathrm{j}_0 \circ \varphi_{t,0} + \sigma^2 \int_0^t \mathrm{z}_s \circ \varphi_{t,s} ds,$$

where φ_t *is the flow at time* t *associated with* v. *Then* j *is regular.*

Proof. Let us notice first that

$$\mathrm{j}_{t+h} = \mathrm{j}_t \circ \varphi_{t+h,t} + \sigma^2 \int_t^{t+h} \mathrm{z}_s \circ \varphi_{t+h,s} dt. \tag{34}$$

From equality (34), the continuity in $\mathcal{J}_W$ of j is straightforward.

It is sufficient to prove that for any $u \in C_c^\infty(\Omega\times]0,1[,\mathbb{R}^d)$, we have

$$-\int_0^1 \left\langle \mathrm{j}_t, \frac{\partial u}{\partial t}\right\rangle_2 dt = \int_0^1 \left(\sigma^2\langle \mathrm{z}_t, u_t\rangle_2 + \langle \mathrm{j}_t, \mathrm{div}(u_t \otimes \mathrm{v}_t)\rangle_2\right) dt. \tag{35}$$

Indeed, if (35) is proved, if for any $t \in [0,1]$ we denote $\gamma_t \triangleq (\mathrm{j}_t, \overline{\mathrm{w}}_t)$, we have for any $u \in C_c^\infty(\Omega,\mathbb{R}^d)$, $t \to \langle \gamma_t\, ,\, u\rangle = \sigma^2\langle \mathrm{z}_t, u\rangle_2 + \langle \mathrm{j}_t, \mathrm{div}(u\otimes \mathrm{v}_t)\rangle_2$ measurable, and $|\gamma_t|_{\mathrm{j}_t} \le |\mathrm{w}_t|_W$ so that $\int_0^1 |\gamma_t|_t^2 dt \le \int_0^1 |\mathrm{w}_t|_W^2 dt < +\infty$, and the lemma is proved.

We have

$$-\int_0^1 \left\langle \mathrm{j}_t, \frac{\partial u}{\partial t}\right\rangle_2 dt = -\lim_{h\to 0}\int_0^1 \left\langle \mathrm{j}_t, \frac{u_t - u_{t-h}}{h}\right\rangle_2 dt = \lim_{h\to 0}\int_0^1 \left\langle \frac{\mathrm{j}_{t+h}-\mathrm{j}_t}{h}, u_t\right\rangle_2 dt$$

so that

$$-\int_0^1 \left\langle \mathrm{j}_t, \frac{\partial u}{\partial t}\right\rangle_2 = \lim_{h\to 0}\frac{1}{h}\int_0^1 \left(\langle \mathrm{j}_t\circ\varphi_{t+h,t} - \mathrm{j}_t, u_t\rangle_2 dt + \sigma^2\int_t^{t+h}\langle \mathrm{z}_s\circ\varphi_{t+h,s}, u_t\rangle_2 ds\right) dt.$$

However, $\langle \mathrm{j}_t\circ\varphi_{t+h,t} - \mathrm{j}_t, u_t\rangle_2 = \int_t^{t+h}\langle \mathrm{j}_t\circ\varphi_{s,t}, \mathrm{div}(u_t\otimes \mathrm{v}_s)\rangle_2 ds$ so that

$$-\int_0^1 \left\langle \mathrm{j}_t, \frac{\partial u}{\partial t}\right\rangle_2 dt = \lim_{h\to 0}\int_0^1 \frac{1}{h}\left(\int_t^{t+h}\langle \mathrm{j}_t\circ\varphi_{s,t}, \mathrm{div}(u_t\otimes \mathrm{v}_s)\rangle_2 + \sigma^2\langle \mathrm{z}_s\circ\varphi_{t+h,s}, u_t\rangle_2 ds\right) dt$$

$$= \lim_{h\to 0}\int_0^1 \frac{1}{h}\left(\int_t^{t+h}\langle \mathrm{j}_t, \mathrm{div}(u_t\otimes \mathrm{v}_s)\circ\varphi_{t,s}|d\varphi_{t,s}|\rangle_2 + \sigma^2\langle \mathrm{z}_s, u_t\circ\varphi_{s,t+h}|d\varphi_{s,t+h}|\rangle_2 ds\right) dt.$$

Since j_t is uniformly bounded on L^2 and $|\varphi_{t,s} - I|_{1,\infty} = \epsilon(|t-s|)$ (since $\mathcal{B}$ is continuously embedded in $C^1(\overline{\Omega},\mathbb{R}^k)$), there exists $C>0$ such that

$$\left|\int_0^1 \frac{1}{h}\int_t^{t+h}\langle \mathrm{j}_t, \mathrm{div}(u_t\otimes \mathrm{v}_s)\circ(\varphi_{t,s}|d\varphi_{t,s}| - I)\rangle_2 ds dt\right| \le C\epsilon(h)\int_0^1 |\mathrm{v}_t|_{1,\infty}\, dt \tag{36}$$

$$\le C'\epsilon(h)\left(\int_0^1 |\mathrm{w}_t|_W^2 dt\right)^{1/2}. \tag{37}$$

Now, using again the fact that j_t is uniformly bounded in L^2 and fact that $C([0,1],L^2(\Omega,\mathbb{R}^k))$ is dense in $L^2([0,1],L^2(\Omega,\mathbb{R}^k))$, we get

$$\left|\int_0^1 \frac{1}{h}\int_t^{t+h} \langle \mathrm{j}_t, \mathrm{div}(u_t\otimes \mathrm{v}_s) - \mathrm{div}(u_t\otimes \mathrm{v}_t)\rangle_2 dsdt\right| \tag{38}$$
$$\leq C\int_0^1 \frac{1}{h}\int_t^{t+h} |\mathrm{div}(u_t\otimes \mathrm{v}_s) - \mathrm{div}(u_t\otimes \mathrm{v}_t)|_2 dsdt$$
$$\to 0 \text{ when } h\to 0.$$

At this point we have proved that

$$\lim_{h\to 0}\frac{1}{h}\int_0^1 \langle \mathrm{j}_t\circ\varphi_{t+h,t} - \mathrm{j}_t, u_t\rangle_2 dt = \int_0^1 \langle \mathrm{j}_t, \mathrm{div}(u_t\otimes \mathrm{v}_t)\rangle_2 dt. \tag{39}$$

Still using the fact that $|\varphi_{t,s} - I|_{1,\infty} = \epsilon(|t-s|)$ and the fact that $|u_t|_{1,\infty}$ is uniformly bounded, we have

$$\sigma^2\int_0^1 \frac{1}{h}\int_t^{t+h} \langle \mathrm{z}_s, u_t\circ(\varphi_{s,t+h}|d\varphi_{s,t+h}| - I)\rangle_2 \; dsdt \leq C\sigma^2\epsilon(h)\int_0^1 |\mathrm{z}_s|_2 ds \tag{40}$$
$$\leq C\sigma\epsilon(h)\left(\int_0^1 |\mathrm{w}_t|_2^2\right)^{1/2}. \tag{41}$$

Finally, since $|u_t|_\infty$ is uniformly bounded, we get

$$\lim_{h\to 0}\left|\int_0^1 \frac{1}{h}\int_t^{t+h} \langle \mathrm{z}_s - \mathrm{z}_t, u_t\rangle_2 dsdt\right| \leq \lim_{h\to 0} C\int_0^1\int_t^{t+h} |\mathrm{z}_s - \mathrm{z}_t|_2 \; dsdt = 0.$$

Hence the proof of the lemma is ended. □

Let us consider the (⇒) part of Theorem 2 for $H^1([0,1],\mathcal{J}_W)$. Let $j\in H^1([0,1],\mathcal{J}_W)$ be a regular path, and let $\mathrm{w}_t = \overline{p}(\frac{\partial j}{\partial t})$ for any $t\in[0,1]$. We get from Proposition 2 that $\mathrm{w}\in L^2([0,1],W)$. Hence, let us define the new path j' by

$$\mathrm{j}'_t = \mathrm{j}_0\circ\varphi_{t,0} + \sigma^2\int_0^t \mathrm{z}_s\circ\varphi_{t,s} ds,$$

where φ is the flow associated with v. From the (⇐) part, we get that j' is regular and that $\frac{\partial \mathrm{j}'}{\partial t} = \frac{\partial j}{\partial t}$. Now let $u_0\in C_c^\infty(\Omega,\mathbb{R}^d)$. For any $f\in C_c^\infty(]0,1[,\mathbb{R})$ if $u(t,x) = u_0(x)f(t)$ for any $x\in\Omega$ and $t\in[0,1]$, we have from the integration by parts formula for a regular path

$$\int_0^1 r(t)f'(t)dt = 0,$$

where $r(t) = \langle \mathrm{j}_t, u\rangle_2 - \langle \mathrm{j}'_t, u\rangle_2$. Since r is continuous and $r(0) = 0$, we get $r\equiv 0$. Considering arbitrary u_0, we get finally $\mathrm{j}_t = \mathrm{j}'_t$ for any $t\in[0,1]$.

Since the (⇒) part for $C^1([0,1],\mathcal{J}_W)$ is a straightforward consequence of the definition of C^1 curves and of the (⇒) part for $H^1([0,1],\mathcal{J}_W)$, we consider the (⇐) part for $\mathrm{w}\in C([0,1],W)$. We get from the corresponding part for $L^2([0,1],W)$ that (35) is still true. For any $f\in C_c^\infty([0,1],\mathbb{R})$ and any $u\in C_c^\infty(\Omega,\mathbb{R}^d)$ we have

$$-\int_0^t f'(t)\langle \mathrm{j}_t, u\rangle_2 dt = \int_0^1 f(t)\left(\sigma^2\langle \mathrm{z}_t, u_t\rangle_2 + \langle \mathrm{j}_t, \mathrm{div}(u\otimes \mathrm{v}_t)\rangle_2\right) dt.$$

One easily checks that $t \to \sigma^2\langle \mathrm{z}_t, u_t\rangle_2 + \langle \mathrm{j}_t, \mathrm{div}(u \otimes \mathrm{v}_t)\rangle_2$ is continuous as well as $t \to \langle \mathrm{j}_t, u\rangle_2$ so that, considering smooth approximates of step functions, we deduce that

$$\langle \mathrm{j}_s, u\rangle_2 = \langle \mathrm{j}_0, u\rangle_2 + \int_0^s \left(\sigma^2\langle \mathrm{z}_t, u_t\rangle_2 + \langle \mathrm{j}_t, \mathrm{div}(u \otimes \mathrm{v}_t)\rangle_2\right) dt,$$

and the result is proved.

Appendix C. Regularity results for $\mathbf{A}_T$. In this section, we collect a few useful results on how the regularity of the diffeomorphism $\mathbf{A}_T(\mathrm{v}) = \varphi_T^{\mathrm{v}}$ may be related to the norm on $\mathcal{B}$, provided this norm can in turn control a sufficient number of derivatives; the first result deals with boundedness. In the following, we assume at least that $\mathcal{B}$ is continuously embedded in $C_0^1(\Omega, \mathbb{R}^k)$ so that $\mathbf{A}_T$ is well defined for all T. In this case,

$$\varphi_T^{\mathrm{v}}(x) = x + \int_0^T \mathrm{v}_s(\varphi_s^{\mathrm{v}}(x))ds.$$

If v_s had p space derivatives for all s, a formal differentiation of this equality yields

$$d^p\varphi_T^{\mathrm{v}} = d^p\mathrm{id} + \int_0^T d^p(\mathrm{v}_s \circ \varphi_s^{\mathrm{v}})ds. \tag{42}$$

This can be proved rigorously from rather standard arguments in the study of ODEs and is stated in the next lemma, for which we provide a proof for completeness, because of the small complication due to the fact that we have only an L^1 control with respect to the t variable, instead of the usual uniform one.

LEMMA 7. *If $p \geq 1$ and $\mathcal{B}$ is embedded in $C_0^p(\Omega, \mathbb{R}^k)$, then, for all $\mathrm{v} \in L^1([0,T], \Omega)$, φ^{v} is p times differentiable and, for all $q \leq p$,*

$$\frac{\partial}{\partial t} d^q\varphi_t^{\mathrm{v}} = d^q(\mathrm{v}_t \circ \varphi_t^{\mathrm{v}}).$$

Moreover, there exist constants C, C' such that, for all $\mathrm{v} \in L^1([0,T], \Omega)$,

$$\sup_{s\in[0,T]} |\varphi_s^{\mathrm{v}}|_{p,\infty} \leq Ce^{C'|\mathrm{v}|_{1,T}}. \tag{43}$$

Proof. For further reference, we first state Gronwall's lemma.

LEMMA 8 (Gronwall). *Asume that α and β are two positive, continuous functions on the interval $[0, c]$ and that*

$$w(t) \leq \alpha(t) + \int_0^t \beta(s)w(s)ds.$$

Then,

$$w(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \beta(u)du}ds.$$

The continuity of $x \mapsto \varphi^{\mathrm{v}}_{0,t}(x)$ is a direct consequence of this lemma since, for $x, y \in \Omega$,

$$\begin{aligned}|\varphi^{\mathrm{v}}_{0,t}(x) - \varphi^{\mathrm{v}}_{0,t}(y)| &= \left| x - y + \int_0^t \left(\mathrm{v}_s(\varphi^{\mathrm{v}}_{0,s}(x)) - \mathrm{v}_s(\varphi^{\mathrm{v}}_{0,s}(y))\right) ds \right| \\ &\leq |x-y| + \int_0^t \|\mathrm{v}_s\|_{1,\infty} |\varphi^{\mathrm{v}}_{0,s}(x) - \varphi^{\mathrm{v}}_{0,s}(y)| ds,\end{aligned}$$

and Gronwall's lemma implies

$$|\varphi^{\mathrm{v}}_{0,t}(x) - \varphi^{\mathrm{v}}_{0,t}(y)| \leq |x-y| \exp(C\,|\mathrm{v}|_{1,T}). \tag{44}$$

Assume $p = 1$ and pass now to the differential of $\varphi^{\mathrm{v}}_{0,t}$. Fix $x \in \Omega$ and introduce the linear differential equation, formally obtained in (42) for $p = 1$,

$$\frac{\partial W_t}{\partial t} = d_{\varphi^{\mathrm{v}}_{0,t}(x)} \mathrm{v}_t W_t \tag{45}$$

with initial condition $W(0) = \delta \in \mathbb{R}^k$. We skip the argument ensuring the existence and uniqueness of a solution of this equation on $[0, 1]$ and proceed to identifying it as $W_t = d_x \varphi^{\mathrm{v}}_{0,t} \delta$. Denote

$$a_\varepsilon(t) = \left(\varphi^{\mathrm{v}}_{0,t}(x + \varepsilon\delta) - \varphi^{\mathrm{v}}_{0,t}(x)\right)/\varepsilon - W_t.$$

For $\alpha > 0$, introduce

$$\mu_t(\alpha) = \max\left\{|d_x \mathrm{v}_t - d_y \mathrm{v}_t| : x, y \in \Omega, |x-y| \leq \alpha\right\}.$$

The function $d_x \mathrm{v}_t \in C^1_0(\Omega)$ being uniformly continuous on the compact set $\overline{\Omega}$, we have $\lim_{\alpha \to 0} \mu_t(\alpha) = 0$. We may write

$$\begin{aligned}a_\varepsilon(t) &= \frac{1}{\varepsilon}\int_0^t \left(\mathrm{v}_s(\varphi^{\mathrm{v}}_{0,s}(x+\varepsilon\delta)) - \mathrm{v}_s(\varphi^{\mathrm{v}}_{0,s}(x))\right) ds - \int_0^t d_{\varphi^{\mathrm{v}}_{0,s}(x)} \mathrm{v}_s W_s ds \\ &= \int_0^t d_{\varphi^{\mathrm{v}}_{0,s}(x)} \mathrm{v}_s a_\varepsilon(s) ds \\ &+ \frac{1}{\varepsilon}\int_0^t (\mathrm{v}_s(\varphi^{\mathrm{v}}_{0,s}(x+\varepsilon\delta)) - \mathrm{v}_s(\varphi^{\mathrm{v}}_{0,s}(x)) - \varepsilon d_{\varphi^{\mathrm{v}}_{0,s}(x)} \mathrm{v}_s(\varphi^{\mathrm{v}}_{0,s}(x+\varepsilon\delta) - \varphi^{\mathrm{v}}_{0,s}(x))) ds.\end{aligned}$$

Since for all $y, y' \in \Omega$

$$|\mathrm{v}_t(y') - \mathrm{v}_t(y) - d_y \mathrm{v}_t(y'-y)| \leq \mu_s(|y'-y|)\,|y'-y|,$$

we may write

$$|a_\varepsilon(t)| \leq \int_0^t |\mathrm{v}_s|_{1,\infty} |a_\varepsilon(s)|\, ds + C(\mathrm{v})\,|\delta| \int_0^1 \mu_s(\varepsilon C(\mathrm{v})\,|\delta|) ds$$

for some constant $C(\mathrm{v})$ which depends only on v. The fact that $a_\varepsilon(t)$ tends to 0 when $\varepsilon \to 0$ now is a direct consequence of Gronwall's lemma and of the fact that

$$\lim_{\alpha \to 0} \int_0^1 \mu_s(\alpha) ds = 0,$$

which is true by the dominated convergence theorem, since μ_s pointwise converges to 0 and $\mu_s(\alpha) \leq 2\,|\mathrm{v}|_{1,\infty}$. This proves Lemma 7 in the case $p = 1$. The rest of the proof is by induction: let $q_0 \leq p$, $q_0 > 1$, and assume that the result is proved for all $q < q_0$:

$$\frac{\partial}{\partial t} d^q \varphi_t^{\mathrm{v}} = d^q(\mathrm{v}_t \circ \varphi_t^{\mathrm{v}}).$$

This implies that for $\delta_1, \ldots, \delta_q \in \mathbb{R}^k$, we may write

$$\frac{\partial}{\partial t} d^q \varphi_t^{\mathrm{v}}(\delta_1, \ldots, \delta_q) = d_{\varphi_{0,t}^{\mathrm{v}}} \mathrm{v}_t d^q \varphi_t^{\mathrm{v}}(\delta_1, \ldots, \delta_q) + \sum_{l=2}^{q} d^l \mathrm{v}_t(\delta_1^{(l)}, \ldots, \delta_l^{(l)}),$$

each vector $\delta_k^{(l)}$ being a linear combination (with universal coefficients) of terms of the kind $d^{l'} \varphi_{0,t}^{\mathrm{v}}(\delta_{i_1}, \ldots, \delta_{i_{l'}})$ with $l' \leq q + 1 - l$ (this result on the differentials of the composition of two functions can be easily proved by induction). This is a linear equation in $d^q \varphi_t^{\mathrm{v}}(\delta_1, \ldots, \delta_q)$, which is valid for $q = q_0 - 1$, and the proof of its validity for q_0 follows exactly the same lines as for $p = 1$.

This expression also shows (using Gronwall's lemma) that $|d^q \varphi_t^{\mathrm{v}}|_\infty$ may be bounded by an expression of the kind

$$|\mathrm{v}|_{1,T}\, \tilde{C}\left(|d\varphi_t^{\mathrm{v}}|_\infty, \ldots, \left|d^{q-1}\varphi_t^{\mathrm{v}}\right|_\infty\right) \exp\left(C\,|\mathrm{v}|_{1,T}\right),$$

where $\tilde{C}$ is a polynomial, which in turn implies (43). □

The same estimate is true for $(\varphi^{\mathrm{v}})^{-1}$.

LEMMA 9. *If* $p \geq 1$ *and* $\mathcal{B}$ *is continuously embedded in* $C_0^p(\Omega, \mathbb{R}^k)$, *there exist constants* C, C' *such that, for all* $\mathrm{v} \in L^1([0,T], \Omega)$,

$$\sup_{s \in [0,T]} \left|(\varphi_s^{\mathrm{v}})^{-1}\right|_{p,\infty} \leq C e^{C' |\mathrm{v}|_{1,T}}.$$

Lemma 9 is a consequence of Lemma 7 and of the fact that $(\varphi_t^{\mathrm{v}})^{-1} = \varphi_t^{\mathrm{w}}$ with $\mathrm{w}_s = -\mathrm{v}_{t-s}$ on $[0,\, t]$.

We now pass to sufficient conditions for Lipschitz continuity of $\mathbf{A}_T$. For this, let $\mathrm{v}, \mathrm{v}' \in L^1([0,T], \mathcal{B})$. For $\xi \in [0,1]$, denote $\mathrm{v}^\xi = (1-\xi)\mathrm{v} + \xi \mathrm{v}'$ and $\varphi^\xi = \varphi^{\mathrm{v}^\xi}$.

LEMMA 10.

$$\frac{\partial}{\partial \xi} \varphi_{s,t}^{\mathrm{v}^\xi}(x) = \int_s^t d_{\varphi_{s,u}^{\mathrm{v}^\xi}(x)} \varphi_{ut}^{\mathrm{v}^\xi}(\mathrm{v}'_u - \mathrm{v}_u) \circ \varphi_{s,u}^{\mathrm{v}^\xi}(x) du. \tag{46}$$

Proof. Let us first start with a formal differentiation of

$$\frac{\partial \varphi_{s,t}^{\mathrm{v}^\xi}}{\partial t} = v_t^\xi \circ \varphi_{s,t}^{\mathrm{v}^\xi}$$

with respect to ξ, which yields

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \xi} \varphi_{s,t}^{\mathrm{v}^\xi} = (v'_t - v_t) \circ \varphi_t^{\mathrm{v}^\xi} + d_{\varphi_{s,t}^{\mathrm{v}^\xi}} v_t^\xi \frac{d}{d\xi} \varphi_{s,t}^{\mathrm{v}^\xi},$$

which naturally leads us to introduce the solution of the differential equation

$$\frac{\partial}{\partial t} W_t = (v'_t - v_t) \circ \varphi_t^{\mathrm{v}^\xi} + d_{\varphi_{s,t}^{\mathrm{v}^\xi}} v_t^\xi W_t \tag{47}$$

with initial condition $W_s = 0$. Noting that we have already encountered this equation without the constant term in (45), the solution of which is of the form $d_x\varphi^{\mathrm{v}^\xi}_{0,t}\delta$, a standard argument by variation of the constant shows that the solution of (47) is given by the right-hand term of (46). Therefore, the proof boils down to show that the interversion of derivatives underlying the formal argument above can be made rigorous.

For this, it clearly suffices to consider the problem in the vicinity of $\xi = 0$. The proof in fact follows the same lines as the proof of Lemma 7: letting W_t be the solution of (47), we let

$$a_\xi(t) = \left(\varphi^{\mathrm{v}^\xi}_{s,t}(x) - \varphi^{\mathrm{v}}_{s,t}(x)\right)/\xi - W_t$$

and express it under the form, letting $h_u = v'_u - v_u$,

$$\begin{aligned} a_\xi(t) = &\int_s^t d_{\varphi^{\mathrm{v}}_{s,u}}\mathrm{v}_u a_\xi(u)du + \int_s^t (h_u(\varphi^{\mathrm{v}^\xi}_{s,u}(x)) - h_u(\varphi^{\mathrm{v}}_{s,u}(x)))du \\ &+ \frac{1}{\xi}\int_s^t \left(\mathrm{v}_u(\varphi^{\mathrm{v}^\xi}_{s,u}(x)) - \mathrm{v}_u(\varphi^{\mathrm{v}}_{s,u}(x)) - \xi d_{\varphi^{\mathrm{v}}_{s,u}(x)}\mathrm{v}_u\left(\varphi^{\mathrm{v}^\xi}_{s,u}(x) - \varphi^{\mathrm{v}}_{s,u}(x)\right)\right)du. \end{aligned}$$

The proof can proceed exactly as that of Lemma 7, provided it has been shown that $|\varphi^{\mathrm{v}^\xi}_{s,u}(x) - \varphi^{\mathrm{v}}_{s,u}(x)|$ tends to 0 with ξ, which is again a direct consequence of Gronwall's lemma and of the inequality

$$\left|\varphi^{\mathrm{v}^\xi}_{s,t}(x) - \varphi^{\mathrm{v}}_{s,t}(x)\right| \leq \int_s^t |\mathrm{v}_u|_{1,\infty}\left|\varphi^{\mathrm{v}^\xi}_{s,u}(x) - \varphi^{\mathrm{v}}_{s,u}(x)\right| du + \xi\int_s^t |h_u|_\infty\, du. \qquad \square$$

This lemma implies, in particular, that

$$\varphi^{\mathrm{v}'}_{s,t}(x) - \varphi^{\mathrm{v}}_{s,t}(x) = \int_0^1\int_s^t d_{\varphi^{\mathrm{v}^\xi}_{s,u}(x)}\varphi^{\mathrm{v}^\xi}_{ut}(\mathrm{v}'_u - \mathrm{v}_u)\circ\varphi^{\mathrm{v}^\xi}_{s,u}(x)dud\xi, \tag{48}$$

which almost immediately leads to the following result (by computing differentials and applying Lemma 7).

Lemma 11. *Assume that $\mathcal{B}$ is continuously embedded in $C^p_0(\Omega)$. If $\mathrm{v}, \mathrm{v}' \in L^1([0,T],\mathcal{B})$, we have, for $t \leq T$,*

$$\left|\varphi^{\mathrm{v}}_t - \varphi^{\mathrm{v}'}_t\right|_{p-1,\infty} \leq C_p\,|\mathrm{v}-\mathrm{v}'|_{1,t}\, e^{C_p(|\mathrm{v}|_{1,t}+|\mathrm{v}'|_{1,t})}$$

for some constant C_p which depends only on p.

The same results apply on $L^2([0,1],\mathcal{B})$, since $|\mathrm{v}|_{1,T} \leq \sqrt{T}\,|\mathrm{v}|_{2,T}$, but, in this space, weak continuity is true under more general conditions.

Theorem 9 (Trouvé, Dupuis, et al). *Assume that $\mathcal{B}$ is continuously embedded in $C^p_0(\Omega,\mathbb{R}^k)$. Then the map*

$$\begin{aligned} \tilde{\mathbf{A}}_T:\quad L^2([0,T],\mathcal{B}) &\to C^p([0,T]\times\overline{\Omega},\mathbb{R}^k), \\ \mathrm{v} &\mapsto \varphi^{\mathrm{v}}_{\cdot}(.) \end{aligned}$$

is continuous for the weak *topology on $L^2([0,1],\mathcal{B})$ and the norm $|.|_{T,p-1,\infty}$ on $C^p([0,T]\times\overline{\Omega},\mathbb{R}^k)$ defined by $|\varphi|_{T,p-1,\infty} = \mathrm{ess.sup}(|\varphi_t|_{p-1,\infty}, t\in[0,T])$.*

Moreover, assume that the embedding is compact, that v^n *converges weakly to* v*, and that there exists a constant* A *such that, for all* n *and almost all* $s \in [0,1]$*,* $|v_s^n|_{\mathcal{B}} \leq A$*. Then, for all* $x \in \Omega$ *and* $t \in [0,T]$*,*

$$d_x^p \varphi_t^{\mathrm{v}^n} \to d_x^p \varphi_t^{\mathrm{v}}.$$

Recall that v_n converges to v in the weak topology on $L^2([0,1],\mathcal{B})$ if and only if, for all $\mathrm{w} \in L^2([0,1],\mathcal{B})$,

$$\lim_{n\to\infty} \int_0^1 \int_\Omega \langle \mathrm{v}_n(t)\,,\,\mathrm{w}(t)\rangle_{\mathcal{B}} dt = \int_0^1 \int_\Omega \langle \mathrm{v}(t)\,,\,\mathrm{w}(t)\rangle_{\mathcal{B}} dt.$$

Proof. The proof of this theorem, which is sketched here for completeness, relies on the remark that, since v_n weakly converges, it is bounded in $L^2([0,1],\mathcal{B})$, and Lemma 7 readily implies that (φ^{v_n}) and their space derivatives up to order $p-1$ are equicontinuous sequences in space. Equicontinuity in time comes by applying the Cauchy–Schwarz inequality to

$$d^q \varphi_t^{\mathrm{v}} - d^q \varphi_s^{\mathrm{v}} = \int_s^t d^q(\mathrm{v}_u \circ \varphi_u^{\mathrm{v}}) du.$$

Ascoli's theorem implies compactness of (φ^{v_n}) for the $|.|_{T,p-1,\infty}$-topology, and it remains to identify a limit of any converging subsequence as φ^{v}. Denoting this limit by ψ, one deduces from

$$\varphi_{0,t}^{\mathrm{v}^n}(x) = \int_0^t \mathrm{v}_s^n(\varphi_{0,s}^{\mathrm{v}^n}(x)) ds$$

and the convergence of $\varphi_{0,t}^{\mathrm{v}^n}$ to φ_t the fact that

$$\psi_t(x) = \int_0^t \mathrm{v}_s^n(\psi_s(x)) ds + o(n),$$

and the conclusion comes after the remark that $\mathrm{w} \mapsto \int_0^t \mathrm{w}_s(\psi_s(x)) ds$ is a continuous linear functional on $L^2([0,1],\mathcal{B})$ so that the weak convergence of v^n to v implies that

$$\psi_t(x) = \int_0^t \mathrm{v}_s(\psi_s(x)) ds$$

and $\psi_t = \varphi_{0,t}^{\mathrm{v}}$.

We now prove the pointwise convergence of the pth derivative. We know that

$$\frac{d}{dt} d_x^p \varphi_t^{\mathrm{v}} = d_{\varphi_t^{\mathrm{v}}} \mathrm{v} d_x^p \varphi_t^{\mathrm{v}} + Q_t^{\mathrm{v}}(x),$$

where $Q_t^{\mathrm{v}}(x)$ depends on the derivatives of v evaluated at $\varphi_t^{\mathrm{v}}(x)$ and on the $p-1$ first space derivatives of φ_t^{v}. We may therefore write

$$\begin{aligned} d_x^p \varphi_t^{\mathrm{v}} - d_x^p \varphi_t^{\mathrm{v}^n} = \int_0^t d_{\varphi_s^{\mathrm{v}^n}} \mathrm{v}(d_x^p \varphi_s^{\mathrm{v}} - d_x^p \varphi_s^{\mathrm{v}^n}) ds + \int_0^t (d_{\varphi_s^{\mathrm{v}}} \mathrm{v} - d_{\varphi_s^{\mathrm{v}^n}} \mathrm{v}^n) d_x^p \varphi_s^{\mathrm{v}} ds \\ + \int_0^t (Q_s^{\mathrm{v}}(x) - Q_s^{\mathrm{v}^n}(x)) ds. \end{aligned}$$

The first integral may be bounded by $C(|\mathrm{v}^n|_{1,T})\int_0^t \left|d_x^p\varphi_s^{\mathrm{v}} - d_x^p\varphi_s^{\mathrm{v}^n}\right| ds$, and the result will be a consequence of Gronwall's lemma, provided we show that the remaining terms tend to 0. Consider the second integral, which may be written

$$\int_0^t (d_{\varphi_s^{\mathrm{v}}}\mathrm{v} - d_{\varphi_s^{\mathrm{v}}}\mathrm{v}^n) d_x^p\varphi_s^{\mathrm{v}} ds + \int_0^t (d_{\varphi^{\mathrm{v}_s}}\mathrm{v}^n - d_{\varphi_s^{\mathrm{v}^n}}\mathrm{v}^n) d_x^p\varphi_s^{\mathrm{v}} ds.$$

The first term tends to 0 because

$$w \mapsto \int_0^t d_{\varphi_s^{\mathrm{v}}(x)} w d_x^p\varphi_s^{\mathrm{v}} ds$$

is a continuous linear functional on $L^2([0,t],\mathcal{B})$ and v^n weakly converges to v in this space. To estimate the second one, introduce, for $A, \varepsilon > 0$, the number

$$C(A,\varepsilon) = \max\left\{|d_x w - d_y w| : x,,y \in \Omega, |x-y| \leq \varepsilon, |w|_{\mathcal{B}} \leq A\right\}.$$

The compact embedding assumption implies that, A being fixed, $C(A,\varepsilon)$ tends to 0 when ε tends to 0. Using this notation, we have

$$\begin{aligned}\int_0^t (d_{\varphi^{\mathrm{v}_s}}\mathrm{v}^n - d_{\varphi_s^{\mathrm{v}^n}}\mathrm{v}^n) d_x^p\varphi_s^{\mathrm{v}} ds &\leq \int_0^t C\left(|\mathrm{v}_s^n|_{\mathcal{B}}, |\varphi_s^{\mathrm{v}} - \varphi_s^{\mathrm{v}_n}|_\infty\right) |d_x^p\varphi_s^{\mathrm{v}}|_\infty ds \\ &\leq \int_0^t C\left(A, |\varphi_s^{\mathrm{v}} - \varphi_s^{\mathrm{v}_n}|_\infty\right) |\mathrm{v}|_{\mathcal{B}} ds,\end{aligned}$$

where $A = \text{ess.sup}\left\{|\mathrm{v}_s^n|_{\mathcal{B}}, n \geq 0, s \in [0,1]\right\}$. The last upper bound now tends to 0, by dominated convergence.

Finally, a generic term of Q_t^{v} being

$$d^k_{\varphi_t^{\mathrm{v}}(x)}\mathrm{v}_t(d^{i_1}\varphi_t^{\mathrm{v}}, \ldots, d^{i_k}\varphi_t^{\mathrm{v}}),$$

we can use the same argument to prove its pointwise convergence. ∎

Appendix D. Action of diffeomorphisms on images. The next theorem provides results concerning the regularity of the action of diffeomorphisms on $L^2(\Omega,\mathbb{R}^d)$ and $H^1(\Omega,\mathbb{R}^d)$.

THEOREM 10.

(i) *Let φ be a diffeomorphism of Ω such that φ and φ^{-1} have uniformly bounded first derivatives on Ω. Then, if $i \in L^2(\Omega,\mathbb{R}^d)$ (resp., $i \in H^1(\Omega,\mathbb{R}^d)$), also $i \circ \varphi \in L^2(\Omega,\mathbb{R}^d)$ (resp., $i \circ \varphi \in H^1(\Omega,\mathbb{R}^d)$ and $d_x(i\circ\varphi) = d_{\varphi(x)}i.d_x\varphi$).*

(ii) *Moreover, for all $M > 0$ and for all $i \in L^2(\Omega,\mathbb{R}^d)$, there exists a function $\varepsilon_M(i,\eta)$ such that, for all φ, φ' such that*

$$\max\left(|\varphi|_{1,\infty}, \left|\varphi^{-1}\right|_{1,\infty}, |\varphi'|_{1,\infty}, \left|\varphi'^{-1}\right|_{1,\infty}\right) \leq M,$$

we have

$$|i\circ\varphi' - i\circ\varphi|_2 \leq \varepsilon_M(i, |\varphi - \varphi'|_{1,\infty}),$$

and $\varepsilon_M(i,\eta) \to 0$ when $\eta \to 0$. The same statement is true for $i \in H^1(\Omega,\mathbb{R}^d)$, the $L^2(\Omega,\mathbb{R}^d)$-norm being replaced by the $H^1(\Omega,\mathbb{R}^d)$-norm.

Proof. We start with (i) and give the proof for $H^1(\Omega, \mathbb{R}^d)$, since it contains exactly the arguments which are valid for $L^2(\Omega, \mathbb{R}^d)$. Fix φ and let L_φ be defined by $L_\varphi(i) = i \circ \varphi$. The vector space $C_d^\infty = C^\infty(\Omega, \mathbb{R}^d)$ of restrictions to Ω of infinitely differentiable functions on $\mathbb{R}^k$ taking values in $\mathbb{R}^d$ is dense in $H^1(\Omega, \mathbb{R}^d)$ [10]. The linear map L_φ is continuous from C_d^∞ (with the topology induced by $H^1(\Omega, \mathbb{R}^d)$) to $H^1(\Omega, \mathbb{R}^d)$; indeed, for $i \in C_d^\infty$,

$$|L_\varphi(i)|_{H^1}^2 = |i \circ \varphi|_2^2 + |d_\varphi i\, d\varphi|_2^2 \le |i|_2^2 \left|d\varphi^{-1}\right|_\infty + |di|_2^2 |d\varphi|_\infty^2 |d\varphi^{-1}|_\infty \le C\, |i|_{H^1}^2$$

since the first derivatives of φ and φ^{-1} are bounded. Thus, L_φ restricted to C_d^∞ can be extended to a continuous function $\tilde{L}_\varphi$ on $H^1(\Omega, \mathbb{R}^d)$. If $i \in H^1(\Omega, \mathbb{R}^d)$ and i_n is a sequence of elements of C_d^∞ which converges to i when n tends to infinity (so that $i_n \circ \varphi \to \tilde{L}_\varphi(i)$ in $H^1(\Omega, \mathbb{R}^d)$), then, because convergence in $H^1(\Omega, \mathbb{R}^d)$ implies convergence in $L^2(\Omega, \mathbb{R}^d)$, a subsequence of i_n can be extracted which converges almost everywhere to i and such that $i_n \circ \varphi$ converges almost everywhere to $\tilde{L}_n(i)$. If $N \subset \Omega$ has null Lebesgue measure, then it is also the case for $\varphi^{-1}(N)$ (by boundedness of $\left|d\varphi^{-1}\right|$), so that $i_n \circ \varphi$ also converges almost everywhere to $i \circ \varphi$, yielding $\tilde{L}_\varphi = L_\varphi$. Now, since the map $i \to di$ is obviously continuous from H^1 to L^2, so is $i \to d(L_\varphi(i))$. But, since this map coincides with $i \to d_\varphi i\, d\varphi$ on C_d^∞, and this last map is also continuous on $H^1(\Omega, \mathbb{R}^d)$ (by the previous computation), we get equality over all $H^1(\Omega, \mathbb{R}^d)$, again by density of C_d^∞.

For (ii), we first consider the $L^2(\Omega, \mathbb{R}^d)$ case. Let i, φ', φ, and M be as in the theorem, and fix $s \in C^\infty(\Omega, \mathbb{R}^d)$; we have

$$|i \circ \varphi' - i \circ \varphi|_2 \le |i \circ \varphi' - s \circ \varphi'|_2 + |s \circ \varphi - s \circ \varphi'|_2 + |i \circ \varphi - s \circ \varphi|_2 .$$

First notice that

$$|i \circ \varphi' - s \circ \varphi'|_2^2 = \int_\Omega \left|d{\varphi'}^{-1}\right| |i - s|^2 dx \le C \left|{\varphi'}^{-1}\right|_{1,\infty} |i - s|_2^2$$

for some constant C. For the middle term, we have

$$\begin{aligned} |s \circ \varphi - s \circ \varphi'|_2 &\le \int_0^1 \left|d_{\varphi + t(\varphi' - \varphi)} s(\varphi' - \varphi)\right|_2 dt \\ &\le |\varphi' - \varphi|_\infty \int_0^1 \left|d_{\varphi + t(\varphi' - \varphi)} s\right|_2 dt \\ &\le C(M)\, |ds|_2\, |\varphi' - \varphi|_\infty . \end{aligned}$$

We thus get

$$|i \circ \varphi' - i \circ \varphi|_2 \le C(M) \left(|i - s|_2 + |ds|_2\, |\varphi' - \varphi|_\infty\right).$$

Letting

$$\varepsilon_M(i, \eta) \triangleq C(M) \inf_{s \in C^\infty(\Omega)} \left(|i - s|_2 + |ds|_2\, \eta\right)$$

yields the conclusion of the theorem in the $L^2(\Omega, \mathbb{R}^d)$ case, the $H^1(\Omega, \mathbb{R}^d)$ case being handled similarly. ∎

Appendix E. Proof of Lemma 2. We must compute the derivative at $\varepsilon = 0$ of

$$U^\varepsilon = \frac{1}{\sigma^2}\int_\Omega \frac{\left|j_0 \circ \varphi_{1,0}^{\mathrm{v}+\varepsilon h} - j_1\right|^2}{\int_0^1 |d\varphi_{1,s}^{\mathrm{v}+\varepsilon h}|^{-1} ds} dx.$$

First, we notice the equation

$$\sigma^2 z_t(x) = \frac{j_1 \circ \varphi_{t,1}^{\mathrm{v}} - j_0 \circ \varphi_{t,0}^{\mathrm{v}}}{\int_0^1 |d\varphi_{t,s}^{\mathrm{v}}|^{-1} ds}, \tag{49}$$

which implies that (differentiating at $\varepsilon = 0$)

$$\frac{dU^\varepsilon}{d\varepsilon} = -2\left\langle z_1\,,\, d_{\varphi_{1,0}^{\mathrm{v}}} j_0 \frac{d}{d\varepsilon}\varphi_{1,0}^{\mathrm{v}+\varepsilon h}\right\rangle_2 - \sigma^2 \int_0^1 \left\langle |z_1|^2\,,\, \frac{d}{d\varepsilon}\left|d\varphi_{1,s}^{\mathrm{v}+\varepsilon h}\right|^{-1}\right\rangle_2 ds.$$

Starting with the first term and using Lemma 10, we have

$$\begin{aligned}
\left\langle z_1\,,\, d_{\varphi_{1,0}^{\mathrm{v}}} j_0 \frac{d}{d\varepsilon}\varphi_{1,0}^{\mathrm{v}+\varepsilon h}\right\rangle_2 &= -\int_0^1 \left\langle d_{\varphi_{1,0}^{\mathrm{v}}} j_0^* z_1\,,\, d_{\varphi_{1t}^{\mathrm{v}}}\varphi_{t,0}^{\mathrm{v}} h_t \circ \varphi_{1t}^{\mathrm{v}}\right\rangle_2 dt \\
&= -\int_0^1 \left\langle d_{\varphi_{t,0}^{\mathrm{v}}} j_0^* z_1 \circ \varphi_{t1}^{\mathrm{v}}\, |d\varphi_{t1}^{\mathrm{v}}|\,,\, d\varphi_{t,0}^{\mathrm{v}} h_t\right\rangle_2 dt \\
&= -\int_0^1 \left\langle (d\varphi_{t,0}^{\mathrm{v}})^* d_{\varphi_{t,0}^{\mathrm{v}}} j_0^* z_t\,,\, h_t\right\rangle_2 dt \\
&= -\int_0^1 \left\langle K\left((d\varphi_{t,0}^{\mathrm{v}})^* d_{\varphi_{t,0}^{\mathrm{v}}} j_0^* z_t\right)\,,\, h_t\right\rangle_{\mathcal{B}} dt
\end{aligned}$$

because of the identity $z_t = z_1 \circ \varphi_{t1}^{\mathrm{v}}\, |d\varphi_{t1}^{\mathrm{v}}|$.

We now pass to the second term, for which we use the equality

$$\left|d\varphi_{t,s}^{\mathrm{v}+\varepsilon h}\right|^{-1} = \exp\left[\int_s^t \mathrm{div}(\mathrm{v}_u + \varepsilon h_u) \circ \varphi_{t,u}^{\mathrm{v}+\varepsilon h} du\right],$$

which is a consequence of Lemma 7 and standard computations on the derivative of the determinant. This implies that

$$\begin{aligned}
\frac{d}{d\varepsilon}\left(\left|d\varphi_{t,s}^{\mathrm{v}+\varepsilon h}\right|^{-1}\right) &= \left|d\varphi_{t,s}^{\mathrm{v}}\right|^{-1} \int_s^t \mathrm{div}(h_u) \circ \varphi_{t,u}^{\mathrm{v}} du \\
&\quad + \left|d\varphi_{t,s}^{\mathrm{v}}\right|^{-1} \int_s^t d_{\varphi_{t,u}^{\mathrm{v}}} \mathrm{div}(\mathrm{v}_u) \int_t^u d_{\varphi_{t\tau}^{\mathrm{v}}} \varphi_{\tau u}^{\mathrm{v}} h_\tau \circ \varphi_{t\tau}^{\mathrm{v}} d\tau du \\
&= \left|d\varphi_{t,s}^{\mathrm{v}}\right|^{-1} \int_s^t \mathrm{div}(h_u) \circ \varphi_{t,u}^{\mathrm{v}} du \\
&\quad - \left|d\varphi_{t,s}^{\mathrm{v}}\right|^{-1} \int_s^t \int_s^\tau d_{\varphi_{t,u}^{\mathrm{v}}} \mathrm{div}(\mathrm{v}_u) d_{\varphi_{t\tau}^{\mathrm{v}}} \varphi_{\tau u}^{\mathrm{v}} h_\tau \circ \varphi_{t\tau}^{\mathrm{v}} du d\tau.
\end{aligned}$$

We may notice that

$$\left\langle \nabla\, |d\varphi_{\tau s}^{\mathrm{v}}|^{-1}\,,\, \xi\right\rangle = |d\varphi_{\tau s}^{\mathrm{v}}|^{-1} \int_s^\tau d_{\varphi_{\tau u}^{\mathrm{v}}} (\mathrm{div} v_u) d\varphi_{\tau u}^{\mathrm{v}}(\xi) du$$

to identify the last term as

$$\left|d\varphi^{\mathrm{v}}_{t,s}\right|^{-1}\int_s^t \left|d_{\varphi^{\mathrm{v}}_{t\tau}}\varphi^{\mathrm{v}}_{\tau s}\right| \left\langle \nabla_{\varphi^{\mathrm{v}}_{t\tau}} |d\varphi^{\mathrm{v}}_{\tau s}|^{-1}\,,\, h_\tau\circ\varphi^{\mathrm{v}}_{t\tau}\right\rangle d\tau$$

so that

$$\begin{aligned}\frac{d}{d\varepsilon}\left(\left|d\varphi^{\mathrm{v}+\varepsilon h}_{t,s}\right|^{-1}\right) &= \left|d\varphi^{\mathrm{v}}_{t,s}\right|^{-1}\int_s^t \operatorname{div}(h_u)\circ\varphi^{\mathrm{v}}_{t,u}du\\ &\quad-\int_s^t |d\varphi^{\mathrm{v}}_{t\tau}|^{-1}\left\langle \nabla_{\varphi^{\mathrm{v}}_{t\tau}} |d\varphi^{\mathrm{v}}_{\tau s}|^{-1}\,,\, h_\tau\circ\varphi^{\mathrm{v}}_{t\tau}\right\rangle d\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 \left\langle |z_1|^2\,,\,\frac{d}{d\varepsilon}\left|d\varphi^{\mathrm{v}+\varepsilon h}_{1,s}\right|^{-1}\right\rangle_2 ds &= \int_0^1\int_s^1 \left\langle |z_1|^2\,,\, |d\varphi^{\mathrm{v}}_{1s}|^{-1}\operatorname{div}(h_u)\circ\varphi^{\mathrm{v}}_{1u}\right\rangle_2 duds\\ &\quad-\int_0^1\int_s^1\left\langle |z_1|^2\,,\,|d\varphi^{\mathrm{v}}_{1u}|^{-1}\left\langle \nabla_{\varphi^{\mathrm{v}}_{1u}}|d\varphi^{\mathrm{v}}_{us}|^{-1}\,,\,h_u\circ\varphi^{\mathrm{v}}_{1u}\right\rangle\right\rangle_2 duds\\ &=\int_0^1\int_s^1\left\langle |z_u|^2\,,\,|d\varphi^{\mathrm{v}}_{u1}|^{-1}\left|d_{\varphi^{\mathrm{v}}_{u1}}\varphi^{\mathrm{v}}_{1s}\right|^{-1}\operatorname{div}(h_u)\right\rangle_2 duds\\ &\quad-\int_0^1\int_s^1\left\langle |z_u|^2\,,\,\left\langle\nabla|d\varphi^{\mathrm{v}}_{us}|^{-1}\,,\,h_u\right\rangle\right\rangle_2 duds.\end{aligned}$$

Introducing

$$q^{\mathrm{v}}_u \triangleq \int_0^u |d\varphi^{\mathrm{v}}_{us}|^{-1}ds,$$

this may be written

$$\begin{aligned}\int_0^1 \left\langle |z_1|^2\,,\,\frac{d}{d\varepsilon}\left|d\varphi^{\mathrm{v}+\varepsilon h}_{1,s}\right|^{-1}\right\rangle_2 ds &= \int_0^1\left\langle q^{\mathrm{v}}_u|z_u|^2\,,\,\operatorname{div}(h_u)\right\rangle_2 du\\ &\quad-\int_0^1\left\langle |z_u|^2\,,\,\langle\nabla q^{\mathrm{v}}_u\,,\,h_u\rangle\right\rangle_2 du\\ &=-\int_0^1\left\langle K_\nabla(q^{\mathrm{v}}_u|z_u|^2)\,,\,h_u\right\rangle_{\mathcal{B}}du\\ &\quad-\int_0^1\left\langle K(|z_u|^2\,\nabla q^{\mathrm{v}}_u)\,,\,h_u\right\rangle_{\mathcal{B}}du.\end{aligned}$$

Now, defining functions

$$C^{\mathrm{v}}_t \triangleq \sigma^2 q^{\mathrm{v}}_t\,|z_t|^2 \tag{50}$$

and

$$D^{\mathrm{v}}_t \triangleq \sigma^2\,|z_t|^2\,\nabla q^{\mathrm{v}}_t + 2[d\varphi^{\mathrm{v}}_{t,0}]^* d_{\varphi^{\mathrm{v}}_{t,0}} j_0^* z_t, \tag{51}$$

Proposition 5 implies

$$\frac{dU^\varepsilon}{d\varepsilon} = \int_0^1 \langle h_t\,,\,K.D^{\mathrm{v}}_t + K_\nabla C^{\mathrm{v}}_t\rangle_{\mathcal{B}}dt,$$

which is the conclusion of Lemma 2.

Appendix F. Proof of Lemma 4. We prove that solutions of system (22) travel at constant speed and therefore compute the derivative of $|v_t|^2_{\mathcal{B}} + \sigma^2 |z_t|_2^2$ for such a solution. Starting with the second term, we have $z_t = z_0 \circ \varphi^{\mathrm{v}}_{t,0} \left|d\varphi^{\mathrm{v}}_{t,0}\right|$, which implies, after a change of variables,

$$|z_t|_2^2 = \int_\Omega |z_0|_2^2 \left|d\varphi^{\mathrm{v}}_{0,t}\right|^{-1} dx.$$

Using the identity

$$\left|d\varphi^{\mathrm{v}}_{s,t}\right| = \exp\left(\int_s^t \mathrm{div}(v_u) \circ \varphi^{\mathrm{v}}_{s,u} du\right), \tag{52}$$

we obtain

$$\frac{d}{dt} |z_t|_2^2 = -\int_\Omega |z_0|_2^2 \left|d\varphi^{\mathrm{v}}_{0,t}\right|^{-1} \mathrm{div}(v_t) \circ \varphi^{\mathrm{v}}_{0,t} dx. \tag{53}$$

To study the variation of $|\mathrm{v}_t|^2_{\mathcal{B}}$, we start with the computation of the derivative of $\langle \mathrm{v}_t \, , \, w\rangle$ for a fixed $w \in \mathcal{B}$. Applying formula (28) for a solution of (22) yields

$$\begin{aligned}\langle \mathrm{v}_t \, , \, w\rangle_{\mathcal{B}} = \frac{\sigma^2}{2} \int_0^1 \Big\langle |z_0|^2 \, , \, (\left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} \mathrm{div}(w) \circ \varphi^{\mathrm{v}}_{0,t} - \left|d\varphi^{\mathrm{v}}_{0,t}\right|^{-1} \left\langle \nabla \xi^{\mathrm{v}}_{s,t} \, , \, \lambda^{\mathrm{v}}_t(w)\right\rangle) \Big\rangle ds \\ + (\omega_0, \lambda^{\mathrm{v}}_t(w))\end{aligned}$$

with $\xi^{\mathrm{v}}_{s,t} = \left|d\varphi^{\mathrm{v}}_{0,t}\right| / \left|d\varphi^{\mathrm{v}}_{0,s}\right|$ and $\lambda^{\mathrm{v}}_t(w) = (d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi^{\mathrm{v}}_{0,t}$. From formula (52), we have

$$\xi^{\mathrm{v}}_{s,t} = \exp\left(\int_s^t (\mathrm{div} v_u) \circ \varphi^{\mathrm{v}}_{0u} du\right),$$

which implies that

$$d\xi^{\mathrm{v}}_{s,t} = \xi^{\mathrm{v}}_{s,t} \int_s^t d_{\varphi^{\mathrm{v}}_{0u}} (\mathrm{div} v_u) d\varphi^{\mathrm{v}}_{0u} du$$

so that

$$\begin{aligned}\langle \mathrm{v}_t \, , \, w\rangle_{\mathcal{B}} &= (\omega_0, \lambda^{\mathrm{v}}_t(w)) + \frac{\sigma^2}{2} \int_0^t \Big\langle |z_0|^2 \, , \, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} \mathrm{div}(w) \circ \varphi^{\mathrm{v}}_{0,t} \Big\rangle ds \\ &\quad - \frac{\sigma^2}{2} \int_0^t \int_s^t \Big\langle |z_0|^2 \, , \, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} d_{\varphi^{\mathrm{v}}_{0u}} (\mathrm{div} v_u) d\varphi^{\mathrm{v}}_{0u} \lambda^{\mathrm{v}}_t(w) \Big\rangle du ds \\ &= (\omega_0, \lambda^{\mathrm{v}}_t(w)) + \frac{\sigma^2}{2} \int_0^t \Big\langle |z_0|^2 \, , \, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} \mathrm{div}(w) \circ \varphi^{\mathrm{v}}_{0,t} \Big\rangle ds \\ &\quad - \frac{\sigma^2}{2} \int_0^t \int_0^u \Big\langle |z_0|^2 \, , \, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} d_{\varphi^{\mathrm{v}}_{0u}} (\mathrm{div} v_u) d\varphi^{\mathrm{v}}_{0u} \lambda^{\mathrm{v}}_t(w) \Big\rangle ds du.\end{aligned}$$

We now compute the time differential of each term which appears in this expression. Denote $\overline{\lambda}^{\mathrm{v}}_t(w) = \frac{d}{dt} \lambda^{\mathrm{v}}_t(w)$. We have

$$\overline{\lambda}^{\mathrm{v}}_t(w) = \frac{d}{dt}\left((d\varphi^{\mathrm{v}}_{0,t})^{-1} w \circ \varphi_{0,t}\right) = -(d\varphi^{\mathrm{v}}_{0,t})^{-1} d_{\varphi^{\mathrm{v}}_{0,t}} v_t w \circ \varphi_{0,t} + (d\varphi^{\mathrm{v}}_{0,t})^{-1} d_{\varphi^{\mathrm{v}}_{0,t}} w v_t \circ \varphi_{0,t}.$$

Next, we have

$$\frac{d}{dt}\int_0^t \left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1}\mathrm{div}(w)\circ\varphi^{\mathrm{v}}_{0,t}\right\rangle ds = \left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,t}\right|^{-1}\mathrm{div}(w)\circ\varphi^{\mathrm{v}}_{0,t}\right\rangle + \int_0^t \left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1}\nabla_{\varphi^{\mathrm{v}}_{0,t}}(\mathrm{div}(w))v_t\circ\varphi^{\mathrm{v}}_{0,t}\right\rangle ds$$

and

$$\begin{aligned}&\frac{d}{dt}\int_0^t\int_0^u \left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} d_{\varphi^{\mathrm{v}}_{0u}}(\mathrm{div}v_u)d\varphi^{\mathrm{v}}_{0u}\lambda^{\mathrm{v}}_t(w)\right\rangle dsdu\\ &\quad=\int_0^t \left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} d_{\varphi^{\mathrm{v}}_{0,t}}(\mathrm{div}v_t)d\varphi^{\mathrm{v}}_{0,t}\lambda^{\mathrm{v}}_t(w)\right\rangle dsdu\\ &\quad+\int_0^t\int_0^u \left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} d_{\varphi^{\mathrm{v}}_{0u}}(\mathrm{div}v_u)d\varphi^{\mathrm{v}}_{0u}\overline{\lambda}^{\mathrm{v}}_t(w)\right\rangle dsdu.\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}\frac{d}{dt}\langle \mathrm{v}_t\,,\, w\rangle_{\mathcal{B}} &= \left(\omega_0, \overline{\lambda}^{\mathrm{v}}_t(w)\right) + \frac{\sigma^2}{2}\left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,t}\right|^{-1}\mathrm{div}(w)\circ\varphi^{\mathrm{v}}_{0,t}\right\rangle\\ &+\frac{\sigma^2}{2}\int_0^t \left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} d_{\varphi^{\mathrm{v}}_{0,t}}(\mathrm{div}w)v_t\circ\varphi^{\mathrm{v}}_{0,t}\right\rangle dsdu\\ &-\frac{\sigma^2}{2}\int_0^t \left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} d_{\varphi^{\mathrm{v}}_{0,t}}(\mathrm{div}v_t)d\varphi^{\mathrm{v}}_{0,t}\lambda^{\mathrm{v}}_t(w)\right\rangle dsdu\\ &-\frac{\sigma^2}{2}\int_0^t\int_0^u \left\langle |z_0|^2\,,\, \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} d_{\varphi^{\mathrm{v}}_{0u}}(\mathrm{div}v_u)d\varphi^{\mathrm{v}}_{0u}\overline{\lambda}^{\mathrm{v}}_t(w)\right\rangle dsdu.\end{aligned}$$

A little care must be taken in writing, as we did above, $\frac{d}{dt}(\omega_0, \lambda^{\mathrm{v}}_t(w)) = (\omega_0\overline{\lambda}^{\mathrm{v}}_t(w))$, since this requires proving that $(\lambda^{\mathrm{v}}_{t+\varepsilon}(w) - \lambda^{\mathrm{v}}_t(w))/\varepsilon$ converges to $\overline{\lambda}^{\mathrm{v}}_t(w)$ for the $(p-1,\infty)$-norm. This is indeed true in our case, because of the fact that $w \in \mathcal{B}$ allows us to control the uniform norm of its differentials up to order p, and the differentials of φ^{v}_t up to the same order are solutions of a linear differential equation which ensures their uniform continuity.

We now use the identity (which is justified below)

$$\frac{d}{dt}\,|\mathrm{v}_t|^2_{\mathcal{B}} = 2\lim_{\varepsilon\to 0}\langle \mathrm{v}_{t+\varepsilon} - \mathrm{v}_t\,,\, \mathrm{v}_t\rangle_{\mathcal{B}}/\varepsilon, \tag{54}$$

which implies that, to compute the time differential of $|\mathrm{v}_t|^2_{\mathcal{B}}$, it suffices to use the obtained expression for the derivative of $\langle \mathrm{v}_t\,,\, w\rangle_{\mathcal{B}}$ with $w = \mathrm{v}_t$ and multiply it by 2. Since $\overline{\lambda}^{\mathrm{v}}_t(v_t) = 0$, and because of (53), we see that all terms cancel, yielding $\frac{d}{dt}(|\mathrm{v}_t|^2_{\mathcal{B}} + \sigma^2\,|z_t|^2_2) = 0$.

To show (54), one writes

$$(|\mathrm{v}_t|^2_{\mathcal{B}} - |\mathrm{v}_t|^2_{\mathcal{B}} - \langle \mathrm{v}_{t+\varepsilon} - \mathrm{v}_t\,,\, \mathrm{v}_t\rangle_{\mathcal{B}})/\varepsilon = |\mathrm{v}_{t+\varepsilon} - \mathrm{v}_t|^2_{\mathcal{B}}\,/\varepsilon,$$

and the result is obtained by proving that, for $w \in \mathcal{B}$,

$$|\langle \mathrm{v}_{t+\varepsilon} - \mathrm{v}_t\,,\, w\rangle_{\mathcal{B}}| = O(\varepsilon)\,|w|_{\mathcal{B}}\,,$$

which can be done by a direct estimation of $\frac{d}{dt}\langle \mathrm{v}_t\,,\, w\rangle_{\mathcal{B}}$.

Appendix G. Proof of Lemma 5. It suffices to prove this result for smooth $z_0, \tilde z_0, j_0$. It is straightforward that $M_{j_0}(tz_0) = \mathrm{j}_t$, where j is the solution of (21) with initial conditions (j_0, z_0). Let $\tilde{\mathrm{j}}_t = M_{j_0}(t\tilde z_0)$. Introduce also the corresponding $(\mathrm{v}_t, \mathrm{z}_t)$ and $(\tilde{\mathrm{v}}_t, \tilde{\mathrm{z}}_t)$.

Introduce the notation $\eta = \tilde{\mathrm{j}} - \mathrm{j}$, $\zeta = \tilde{\mathrm{z}} - \mathrm{z}$, and $\alpha = \tilde{\mathrm{v}} - \mathrm{v}$. Since we have assumed smooth trajectories, we may write

$$\frac{\partial \mathrm{j}_t}{\partial t} = \sigma^2 \mathrm{z}_t - d\mathrm{j}_t \mathrm{v}_t$$

and

$$\frac{\partial \mathrm{z}_t}{\partial t} = -\mathrm{div}(\mathrm{z}_t \otimes \mathrm{v}_t)$$

and similar equations for the trajectory with initial condition $(j_0, \tilde z_0)$. Computing the differences along both trajectories yields

$$\begin{cases} \frac{\partial \eta_t}{\partial t} + d\eta_t \tilde{\mathrm{v}}_t = \sigma^2 \zeta_t - d\mathrm{j}_t \alpha_t, \\ \frac{\partial \zeta_t}{\partial t} + \mathrm{div}(\zeta_t \otimes \tilde{\mathrm{v}}_t) = -\mathrm{div}(\mathrm{z}_t \otimes \alpha_t). \end{cases} \tag{55}$$

Since

$$\begin{aligned}\frac{\partial}{\partial t}\left[\left|d\varphi^{\tilde{\mathrm{v}}}_{0,t}\right| \zeta_t \circ \varphi^{\tilde{\mathrm{v}}}_{0,t}\right] &= \left|d\varphi^{\tilde{\mathrm{v}}}_{0,t}\right| \left(\frac{\partial \zeta_t}{\partial t} + \mathrm{div}(\zeta_t \otimes \tilde{\mathrm{v}}_t)\right) \circ \varphi^{\tilde{\mathrm{v}}}_{0,t} \\ &= -\left|d\varphi^{\tilde{\mathrm{v}}}_{0,t}\right| \mathrm{div}(\mathrm{z}_t \otimes \alpha_t) \circ \varphi^{\tilde{\mathrm{v}}}_{0,t},\end{aligned}$$

the second term yields

$$\zeta_s \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} = \left|d\varphi^{\tilde{\mathrm{v}}}_{0,s}\right|^{-1} (\tilde z_0 - z_0) - \left(\int_0^s \left|d\varphi^{\tilde{\mathrm{v}}}_{s,u}\right| \mathrm{div}(\mathrm{z}_u \otimes \alpha_u) \circ \varphi^{\tilde{\mathrm{v}}}_{s,u} du\right) \circ \varphi^{\tilde{\mathrm{v}}}_{0,s},$$

and the first one implies

$$\eta_t \circ \varphi^{\tilde{\mathrm{v}}}_{0,t} = \sigma^2 \int_0^t \zeta_s \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} ds - \int_0^t [d\mathrm{j}_s \alpha_s] \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} ds.$$

Replacing ζ in the last equation gives

(56)

$$\begin{aligned}\eta_t \circ \varphi^{\tilde{\mathrm{v}}}_{0,t} = {}& t[\sigma^2(\tilde z_0(.) - z_0(.)) - dj_0(\tilde v_0 - v_0)] \\ & - \sigma^2 \int_0^t \left(\int_0^s \left|d\varphi^{\tilde{\mathrm{v}}}_{s,u}\right| \mathrm{div}(\mathrm{z}_u \otimes \alpha_u) \circ \varphi^{\tilde{\mathrm{v}}}_{s,u} du\right) \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} ds \\ & - \int_0^t \left\{[d\mathrm{j}_s \alpha_s] \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} - dj_0(\tilde v_0 - v_0)\right\} ds + \int_0^t \left(\left|d\varphi^{\tilde{\mathrm{v}}}_{0,s}\right|^{-1} - 1\right)(\tilde z_0 - z_0) ds\end{aligned}$$

so that Lemma 5 reduces to evaluating the L^2-norm of the last three integrals. We shall use the fact that, for a function $f \in L^2([0,1] \times \Omega, \mathbb{R}^d)$,

$$\left|\int_0^t f_s ds\right|_2 \leq \int_0^t |f_s|_2 \, ds.$$

For $t \in [0,1]$, we also have, from (31), with $\omega_0 = dj_0^* z_0$ and $\omega_0' = dj_0^* z_0'$ (here and in the following, we denote by const any quantity which depends only on j_0, z_0 and $\tilde{z}_0$),

$$|\alpha_t|_{\mathcal{B}} \leq \text{const } |z_0 - \tilde{z}_0|_2 . \tag{57}$$

This implies that

$$\begin{aligned}
&\left|\int_0^t \left(\int_0^s \left|d\varphi^{\tilde{\mathrm{v}}}_{s,u}\right| \operatorname{div}(\mathrm{z}_u \otimes \alpha_u) \circ \varphi^{\tilde{\mathrm{v}}}_{s,u} du\right) \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} ds\right|_2 \\
&\quad \leq \int_0^t \int_0^s \left|\left|d_{\varphi^{\tilde{\mathrm{v}}}_{0,s}} \varphi^{\tilde{\mathrm{v}}}_{s,u}\right| \operatorname{div}(\mathrm{z}_u \otimes \alpha_u) \circ \varphi^{\tilde{\mathrm{v}}}_{0,u}\right|_2 duds \\
&\quad = \int_0^t \int_0^s \left|\left|d\varphi^{\tilde{\mathrm{v}}}_{0,s}\right|^{-1} \operatorname{div}(\mathrm{z}_u \otimes \alpha_u)\right|_2 ds \\
&\quad \leq \text{const } \int_0^t \int_0^s |\operatorname{div}(\mathrm{z}_u \otimes \alpha_u)|_2 \, duds \\
&\quad \leq \text{const } \int_0^t \int_0^s |\mathrm{z}_u|_{H^1} \, |\alpha_u|_{\mathcal{B}} \, duds.
\end{aligned}$$

The relation $\mathrm{z}_t = z_0 \circ \varphi^{\mathrm{v}}_{t,0} \left|d\varphi^{\mathrm{v}}_{t,0}\right|$ implies that $|\mathrm{z}_u|_{H^1} \leq \text{const } |z_0|_{H^1}$ so that

$$\left|\int_0^t \left(\int_0^s \left|d\varphi^{\tilde{\mathrm{v}}}_{s,u}\right| \operatorname{div}(\mathrm{z}_u \otimes \alpha_u) \circ \varphi^{\tilde{\mathrm{v}}}_{s,u} du\right) \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} ds\right|_2 \leq \text{const } t^2 \, |\tilde{z}_0 - z_0|_2 .$$

A similar estimate is valid for the last integral in (56), since $||d\varphi^{\mathrm{v}}_{0,s}|^{-1} - 1|_\infty \leq \text{const } s$. We finally consider the second integral in this equation.

Since

$$\mathrm{j}_s = j_0 \circ \varphi^{\mathrm{v}}_{s,0} + \sigma^2 z_0 \circ \varphi^{\mathrm{v}}_{s,0} \int_0^s |d\varphi^{\mathrm{v}}_{u0}| \circ \varphi^{\mathrm{v}}_{s,u} du,$$

we have, letting $\gamma_s = \varphi^{\mathrm{v}}_{s,0} \circ \varphi^{\tilde{\mathrm{v}}}_{0,s}$,

$$d_{\varphi^{\tilde{\mathrm{v}}}_{0,s}} \mathrm{j}_s \, \alpha \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} = d_{\gamma_s} j_0 \, d_{\varphi^{\tilde{\mathrm{v}}}_{0,s}} \varphi^{\mathrm{v}}_{s,0} \, \alpha \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} + R_s,$$

and it is easy to check that $|R_t|_2 \leq \text{const } t \, |z_0|_{H^1} \, |\tilde{z}_0 - z_0|_2$. We need to estimate

$$\begin{aligned}
&\int_0^t \left(d_{\gamma_s} j_0 \, d_{\varphi^{\tilde{\mathrm{v}}}_{0,s}} \varphi^{\mathrm{v}}_{s,0} \, \alpha_s \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} - dj_0 \alpha_0\right) ds \\
&\quad = \int_0^t \left(d_{\gamma_s} j_0 (d_{\varphi^{\tilde{\mathrm{v}}}_{0,s}} \varphi^{\mathrm{v}}_{s,0} \, \alpha_s \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} - \alpha_0)\right) ds + \int_0^t (d_{\gamma_s} j_0 \, \alpha_0 - dj_0 \, \alpha_0) \, ds.
\end{aligned} \tag{58}$$

Start with the first term, for which we must bound, for the L^∞-norm, the difference $d_{\varphi^{\tilde{\mathrm{v}}}_{0,s}} \varphi^{\mathrm{v}}_{s,0} \, \alpha \circ \varphi^{\tilde{\mathrm{v}}}_{0,s} - \alpha_0$ or, equivalently,

$$d\varphi^{\mathrm{v}}_{s,0} \alpha_s - \alpha_0 \circ \varphi^{\tilde{\mathrm{v}}}_{s,0}.$$

It is simple to check, from (57) and estimates we have used several times on the variations of the diffeomorphisms, that $(d\varphi^{\mathrm{v}}_{s,0} - I)\alpha_s$ and $\alpha_0 \circ \varphi^{\tilde{\mathrm{v}}}_{s,0} - \alpha_0$ are bounded by const $s \, |\tilde{z}_0 - z_0|_2$. We now proceed to an upper bound for $\alpha_s - \alpha_0$, for which we need to return to the expression obtained in (28), which yields

$$\begin{aligned}
\langle \mathrm{v}_s - \mathrm{v}_0, w \rangle_{\mathcal{B}} = \frac{\sigma^2}{2} \int_0^s &\Big\langle |z_0|^2 , (|d\varphi^{\mathrm{v}}_{0u}|^{-1} \operatorname{div}(w) \circ \varphi^{\mathrm{v}}_{0,s} \\
&- \left|d\varphi^{\mathrm{v}}_{0,s}\right|^{-1} \langle \nabla \xi^{\mathrm{v}}_{us} \, , \, \lambda^{\mathrm{v}}_s(w) \rangle) \Big\rangle \, du + \langle z_0 \, , \, dj_0(\lambda^{\mathrm{v}}_s(w) - w) \rangle_2
\end{aligned}$$

so that

$$\begin{aligned}\langle \alpha_s - \alpha_0\,,\, w\rangle_{\mathcal{B}} &= \frac{\sigma^2}{2}\int_0^s \Big\langle |\tilde z_0|^2\,,\, \big(\big|d\varphi^{\tilde v}_{0u}\big|^{-1}\operatorname{div}(w)\circ\varphi^{\tilde v}_{0,s} - \big|d\varphi^{\tilde v}_{0,s}\big|^{-1}\langle \nabla\xi^{\tilde v}_{us}\,,\,\lambda^{\tilde v}_s(w)\rangle\big)\Big\rangle du\\ &\quad - \frac{\sigma^2}{2}\int_0^s \Big\langle |z_0|^2\,,\, \big(\big|d\varphi^{\mathrm v}_{0u}\big|^{-1}\operatorname{div}(w)\circ\varphi^{\mathrm v}_{0,s} - \big|d\varphi^{\mathrm v}_{0,s}\big|^{-1}\langle \nabla\xi^{\mathrm v}_{us}\,,\,\lambda^{\mathrm v}_s(w)\rangle\big)\Big\rangle du\\ &\quad + \big\langle \tilde z_0\,,\, dj_0(\lambda^{\tilde v}_s(w) - w)\big\rangle_2 - \langle z_0\,,\, dj_0(\lambda^{\mathrm v}_s(w)-w)\rangle_2.\end{aligned}$$

The difference of the first two integrals takes the form

$$\frac{\sigma^2}{2}\int_0^s \big(\big\langle \tilde z_0\,,\, Q^{\tilde v}_{us}(w)\big\rangle - \langle z_0\,,\, Q^{\mathrm v}_{us}(w)\rangle\big)\,du \tag{59}$$

with $Q^{\mathrm v}_{us}(w) = |d\varphi^{\mathrm v}_{0u}|^{-1}\operatorname{div}(w)\circ\varphi^{\mathrm v}_{0,s} - \big|d\varphi^{\mathrm v}_{0,s}\big|^{-1}\langle\nabla\xi^{\mathrm v}_{us}\,,\,\lambda^{\mathrm v}_s(w)\rangle$. From Lemmas 7 and 11, and from (57), we obtain the fact that $\big|Q^{\tilde v}_{us}(w) - Q^{\mathrm v}_{us}(w)\big| \le \text{const }\ |\tilde z_0 - z_0|_2\,|w|_{\mathcal B}$ so that the quantity in (59) is bounded by const $s\,|\tilde z_0 - z_0|_2$. Writing

$$\begin{aligned}\big\langle \tilde z_0\,,\, dj_0(\lambda^{\tilde v}_s(w) - w)\big\rangle_2 - \langle z_0\,,\, dj_0(\lambda^{\mathrm v}_s(w) - w)\rangle_2 &= \big\langle \tilde z_0 - z_0\,,\, dj_0(\lambda^{\tilde v}_s(w) - w)\big\rangle_2\\ &\quad + \big\langle z_0\,,\, dj_0(\lambda^{\tilde v}_s(w) - \lambda^{\mathrm v}_s(w))\big\rangle_2\end{aligned}$$

and using $\big|\lambda^{\tilde v}_s(w) - w\big|_\infty \le$ const s (which is deduced from Lemma 7 and a computation of the differential of $\lambda^{\tilde v}_s(w)$ with respect to s) and $\big|\lambda^{\tilde v}_s(w) - \lambda^{\mathrm v}_s(w)\big|_\infty \le$ const $s\,|\tilde z_0 - z_0|_2$ (from Lemma 11 and (57)), we finally conclude that

$$\Big|d_{\varphi^{\tilde v}_{0,s}}\varphi^{\mathrm v}_{s,0}\,\alpha\circ\varphi^{\tilde v}_{0,s} - \alpha_0\Big|_\infty \le \text{const }s\,|\tilde z_0 - z_0|_2\,,$$

which implies that the first integral in the right-hand term of (58) is bounded by const $t^2\,|\tilde z_0 - z_0|_2$.

Consider now the last term of (58), namely,

$$\int_0^t (dj_0\circ\gamma_s - dj_0)\,\alpha_0 ds.$$

Since $|\alpha_0|_\infty \le C\,|\tilde z_0 - z_0|_2$, we must estimate $|d_{\gamma_s} j_0 - dj_0|_2$. By Theorem 10, this is a function of the kind

$$\varepsilon_M(dj_0, |\gamma_s - \mathrm{Id}|_\infty) = \varepsilon_M(dj_0, \big|\varphi^{\tilde v}(s) - \varphi^{\mathrm v}(s)\big|_\infty),$$

where M depends only on $|j_0|_{H^1}\,, |z_0|_2\,, |\tilde z_0|_2$. Since $|\varphi^{\tilde v}_{0,s} - \varphi^{\mathrm v}_{0,s}|_\infty = O(s)$, we get (with another function ε)

$$\int_0^t (dj_0\circ\gamma_s - dj_0)\,\alpha_0 ds \le \varepsilon(j_0, t)t\,|\tilde z_0 - z_0|_2\,.$$

We need finally to consider the last line of (56) which can be easily bounded from above by $\varepsilon(j_0,t)t\,|\tilde z_0 - z_0|_2$. We now can collect the estimates we have obtained to conclude the proof of Lemma 5.

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