N-soliton collision in the Manakov model

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Abstract

We investigate soliton collisions in the Manakov model which is a system of coupled nonlinear Schrödinger equations integrable via the inverse scattering method. Computing the asymptotic forms of the general N-soliton solution as $t \to \mp\infty$, we elucidate a mechanism which factorizes an N-soliton collision into nonlinear superposition of $\binom{N}{2}$ pair collisions in arbitrary order. This removes the misunderstanding that multi-particle effects exist in the Manakov model. As a byproduct, we also obtain a new nontrivial relation among determinants and extended determinants.

1 Introduction

In recent years, there has been a surge of interest in some systems of coupled nonlinear Schrödinger (coupled NLS) equations for their relevance in nonlinear optics [1, 2, 3]. Among those systems, this paper focuses on the following system of coupled NLS equations:

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2\|\boldsymbol{q}\|^2\boldsymbol{q} = \boldsymbol{0}, \qquad \boldsymbol{q} = (q_1, q_2, \ldots, q_m). \tag{1.1}$$

Here $\|\boldsymbol{q}\|^2 \equiv \boldsymbol{q}\cdot\boldsymbol{q}^\dagger = \sum_{j=1}^{m}|q_j|^2$, where the superscript $\dagger$ denotes Hermitian conjugation. The subscripts t and x denote the partial differentiations by t and x, respectively. It is well known that (1.1) is a completely integrable system [4, 5]. We call (1.1) the Manakov model, since the two-component ($m = 2$) case of (1.1) was solved for the first time by Manakov [4] via the inverse scattering method (ISM). The extension of the ISM to the general m-component case is straightforward [5]. Nevertheless, the value of m is extremely important when we consider soliton solutions. The $m = 2$ case is merely a special case of the general m case. The most interesting is the case where the total number of solitons, say N, is equal to the number of components, m. Indeed, in the $N = m$ case, the coefficient vectors for the sech-type envelope of solitons (see, *e.g.* $\boldsymbol{u}_1$ in (2.12)), which we call after normalization *polarization vectors*, are just enough to span the vector space $\mathbb{C}^m$ in which $\boldsymbol{q}$ lives. Soliton solutions in the other cases ($N > m$ or $N < m$) are obtained from those in this case through a special choice of soliton parameters or the operation of a unitary transformation. In this paper, we consider the most general case where N and m are arbitrary positive integers. Then, the most interesting $N = m$ case is automatically included.

Although the integrability of the Manakov model (1.1) has been established formally via the ISM, multi-soliton dynamics in the model still remains to be clarified. There are two reasons for this. One reason is the vector nature of (1.1) which supports the internal degrees of freedom of solitons. As a consequence, even a two-soliton collision is highly nontrivial in the Manakov model (1.1) [6, 7]: it not only displaces the soliton centers in dependence on the initial polarization vectors but also changes the polarization vectors. Therefore, the effect of an N-soliton collision can never be written as the *algebraic* sum of those of pair collisions (at least, seen in the original physical quantities). This is quite different from the NLS case ((1.1) with $m = 1$) in which the effect of an N-soliton collision is written as the *algebraic* sum of those of pair collisions (the order of pair collisions does not matter) [8, 9, 10]. The other reason is that Manakov gave a rather misleading description of an N-soliton collision in his paper [4]. We quote the corresponding part, the first two sentences of the last paragraph of section 2, from [4]:

> "Comparison of relations (17) and (18) indicates that an N-soliton collision does not, in general, reduce to a pair collision. This is clear, for example, from the fact that the expression for $\boldsymbol{S}_k^+$ contains $\boldsymbol{S}_j^+$ with $j > k$, which depend on the initial parameters of all the remaining solitons." (♣)

Here equations (17) are given by

$$
\begin{aligned}
\boldsymbol{S}_N^+ &= \left\{\prod_{n<N} \alpha_{11}^{-1}(\zeta_N, \zeta_n)\right\} \hat{\alpha}^T(\zeta_N, \zeta_1, \boldsymbol{S}_1^-) \ldots \hat{\alpha}^T(\zeta_N, \zeta_{N-1}, \boldsymbol{S}_{N-1}^-)\boldsymbol{S}_N^-, \\
\boldsymbol{S}_i^+ &= \left\{\prod_{k>i} \alpha_{11}(\zeta_i, \zeta_k)\right\}\left\{\prod_{n<i} \alpha_{11}^{-1}(\zeta_i, \zeta_n)\right\} \hat{\alpha}^*(\zeta_i^*, \zeta_{i+1}, \boldsymbol{S}_{i+1}^+) \ldots \hat{\alpha}^*(\zeta_i^*, \zeta_N, \boldsymbol{S}_N^+) \\
&\quad \cdot \hat{\alpha}^T(\zeta_i, \zeta_1, \boldsymbol{S}_1^-) \ldots \hat{\alpha}^T(\zeta_i, \zeta_{i-1}, \boldsymbol{S}_{i-1}^-)\boldsymbol{S}_i^-, \qquad i = 1, 2, \ldots, N-1,
\end{aligned}
$$

while equations (18) are given by

$$
\begin{aligned}
\boldsymbol{S}_2^+ &= \alpha_{11}^{-1}(\zeta_2, \zeta_1)\, \hat{\alpha}^T(\zeta_2, \zeta_1, \boldsymbol{S}_1^-)\boldsymbol{S}_2^-, \\
\boldsymbol{S}_1^+ &= \alpha_{11}(\zeta_1, \zeta_2)\, \hat{\alpha}^*(\zeta_1^*, \zeta_2, \boldsymbol{S}_2^+)\boldsymbol{S}_1^-.
\end{aligned}
$$

We briefly explain the situation considered and the notation used in [4]. Equations (17) give the solution of the collision problem of N solitons (we name them solitons-1, 2, …, N), while equations (18) give that of two solitons. We note that equations (18) are obtained from (17) by setting $N = 2$. Equations (17) were derived through deep thinking and intuition based on the ISM. It is assumed that a soliton with a larger number moves faster along the x-axis. Namely, soliton-i *overtakes*[1] solitons-1, 2, …, $i-1$ and is *overtaken* by solitons-$i+1$, $i+2$, …, N as time t passes from $-\infty$ to $+\infty$. The velocity of soliton-i as well as its amplitude is determined by the complex parameter ζ_i which is time-independent. $\boldsymbol{S}_i$ are column vectors with two complex components, corresponding to the choice of $m = 2$ in [4]. The vector norm $\|\boldsymbol{S}_i\|$ $(= (\boldsymbol{S}_i^\dagger \cdot \boldsymbol{S}_i)^{\frac{1}{2}})$ determines the center position of soliton-i, while the normalized vector $\boldsymbol{S}_i/\|\boldsymbol{S}_i\|$ gives its polarization vector up to the operation of Hermitian conjugation. The superscripts $+$ and $-$ denote the final state ($t \to +\infty$) and the initial state ($t \to -\infty$), respectively. α_{11} is a scalar function and $\hat{\alpha}$ is a 2×2 matrix function in their arguments. We omit their explicit forms which are dispensable to the following discussion. The superscripts T and $*$ denote the operation of transposition and complex conjugation, respectively.

Although the meaning of (♣) is somewhat ambiguous, the most natural and reasonable interpretation seems to be the following:

[1] Throughout this paper, we use the term "overtake" also for head-on collisions.

> Let us try to explain (17) by assuming that the N solitons collide pairwise in accordance with (18). Then, the first equation in (17) can be read as follows. Soliton-N first overtakes soliton-$N-1$ with its initial state, i.e. soliton-$N-1$ which has not collided with other solitons. Next, soliton-N overtakes soliton-$N-2$ which has not collided with other solitons. ... Finally, it overtakes soliton-1 which has not collided with other solitons.
>
> Similarly, the second equation in (17) can be read as follows. Soliton-i overtakes solitons-$i-1$, $i-2$, ..., 1 in this order, all of which have not collided with other solitons. Next, soliton-i is overtaken by solitons-N, $N-1$, ..., $i+1$ with their final states, i.e. those which will not collide with other solitons.
>
> If we try to diagram these events, we instantly encounter a contradiction. This *indicates* that an N-soliton collision cannot be explained by a chain of pair collisions, since the matrices $\hat{\alpha}$ for different sets of arguments do not commute in general.

The logic in the above interpretation is not mathematically rigorous, but seemingly correct if the complex structure of (17) is taken into account. One may believe that an N-soliton collision in the Manakov model (1.1) does not reduce to a pair collision, and thus some multi-particle effects exist in the Manakov model.

The main goal of this paper is, however, to remove this misunderstanding. We show explicitly a mechanism which factorizes an N-soliton collision in the Manakov model (1.1) into *nonlinear* superposition of pair collisions. Here we have used the term "nonlinear" to indicate that the considered superposition is no longer additive. For definiteness, we explain in advance what we are going to prove in terms of the Manakov notation, which also gives the definition of *factorization* in this paper. We first interpret equations (18) as a nonlinear mapping with two complex parameters, $f(\zeta_2, \zeta_1)$, which maps the initial state $\{\boldsymbol{S}_2^-, \boldsymbol{S}_1^-\}$ into the final state $\{\boldsymbol{S}_2^+, \boldsymbol{S}_1^+\}$. Then, we can use the mapping $f(\zeta_j, \zeta_k)$ to evaluate in an N-soliton collision the effect of the two-soliton collision that soliton-j with a state $\boldsymbol{S}_j$ overtakes soliton-k with a state $\boldsymbol{S}_k$. For a given order of $\binom{N}{2}$ pair collisions, we consider the corresponding composition of $\binom{N}{2}$ mappings: $f(\zeta_j, \zeta_k)$, $N \geq j > k \geq 1$. Then, regardless of the order of pair collisions, the composed mapping maps the initial state $\{\boldsymbol{S}_N^-, \boldsymbol{S}_{N-1}^-, \dots, \boldsymbol{S}_1^-\}$ exactly into the final state $\{\boldsymbol{S}_N^+, \boldsymbol{S}_{N-1}^+, \dots, \boldsymbol{S}_1^+\}$ given by equations (17).

To prove this factorization, we part with the Manakov results, equations (17) and (18), for the following two reasons. One reason is that, although it seems to be ingenious and correct, the derivation of (17) in [4] is neither very rigorous nor understandable for the reader not familiar with the ISM. The other reason is that equations (17) are not tractable for our purpose. In this paper, we take a more straightforward way to get another formula for the asymptotic behavior of N solitons. We start from an explicit formula for the N-soliton solution of the matrix NLS equation derived via the ISM in [11]. Through a simple reduction, we obtain an explicit formula for the general N-soliton solution of the Manakov model (1.1) [12]. To make the paper self-contained, we first set $N = 2$ and compute the asymptotic forms of the two-soliton solution as $t \to \mp\infty$ in our notation. They define the collision laws of two solitons in the Manakov model, which are essentially the same as those given by equations (18). Next, we consider the general N case and compute the asymptotic forms of the N-soliton solution as $t \to \mp\infty$. To express polarization vectors appearing in the asymptotic forms concisely, we extend the definition of a determinant in such a way that the last column of an extended determinant consists of vectors. It represents a vector defined by the Laplace expansion with respect to the last column. We find a beautiful relation which casts the Hermitian product between such extended determinants into a product of conventional determinants. Using this relation and the Jacobi formula for determinants, we prove that an N-soliton collision in the Manakov model (1.1) is factorized into nonlinear superposition of $\binom{N}{2}$ pair collisions in arbitrary order.

This paper is organized as follows. In section 2, we derive an explicit formula for the N-soliton solution of the Manakov model [12]. In section 3, we obtain the collision laws of two solitons. In section 4, we compute the asymptotic forms of the N-soliton solution as $t \to \mp\infty$. In section 5, we elucidate a mechanism which factorizes an N-soliton collision into nonlinear superposition of pair collisions. Section 6 is devoted to concluding remarks, including some comments on the literature.

2 Explicit formula for the general N-soliton solution

In this section, through considering a reduction of a formula in [11]. we derive an explicit formula for the general N-soliton solution of the Manakov model (1.1).

In [11], under vanishing boundary conditions, we applied the ISM to nonlinear evolution equations associated with the generalized Zakharov–Shabat eigenvalue problem:

$$\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}_x = \begin{bmatrix} -\mathrm{i}\zeta I & Q \\ -Q^\dagger & \mathrm{i}\zeta I \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{2.1}$$

Here ζ is the spectral parameter. I is the $m \times m$ unit matrix. Q is a potential function which takes its value in $m \times m$ matrices. The first two of nonlinear evolution equations associated with (2.1) are the matrix NLS equation,

$$\mathrm{i}Q_t + Q_{xx} + 2QQ^\dagger Q = O, \tag{2.2}$$

and the matrix complex mKdV equation,

$$Q_t + Q_{xxx} + 3(Q_x Q^\dagger Q + QQ^\dagger Q_x) = O. \tag{2.3}$$

We mention that integrable space-discretizations of equations (2.2) and (2.3) were found recently [13]. The general N-soliton solution of (2.2) or (2.3) is expressed as [11]

$$Q(x,t) = -2\mathrm{i}(\underbrace{I\; I\; \cdots\; I}_{N})\, S^{-1} \begin{pmatrix} C_1(t)^\dagger\, \mathrm{e}^{-2\mathrm{i}\zeta_1^* x} \\ C_2(t)^\dagger\, \mathrm{e}^{-2\mathrm{i}\zeta_2^* x} \\ \vdots \\ C_N(t)^\dagger\, \mathrm{e}^{-2\mathrm{i}\zeta_N^* x} \end{pmatrix}, \tag{2.4}$$

where the $mN \times mN$ matrix S is given by

$$S_{jk} = \delta_{jk} I - \sum_{l=1}^{N} \frac{\mathrm{e}^{2\mathrm{i}(\zeta_l - \zeta_j^*)x}}{(\zeta_l - \zeta_k^*)(\zeta_l - \zeta_j^*)} C_j(t)^\dagger C_l(t), \quad 1 \le j,k \le N. \tag{2.5}$$

ζ_j are distinct eigenvalues in the upper-half plane of ζ ($\mathrm{Im}\, \zeta_j > 0$), each of which determines a bound state by the potential Q. $C_j(t)$ are $m \times m$ nonzero matrices whose time dependences are given by

$$C_j(t) = C_j(0)\mathrm{e}^{4\mathrm{i}\zeta_j^2 t}, \quad j = 1, 2, \ldots, N, \tag{2.6}$$

for the matrix NLS equation (2.2) and

$$C_j(t) = C_j(0)\mathrm{e}^{8\mathrm{i}\zeta_j^3 t}, \quad j = 1, 2, \ldots, N,$$

for the matrix complex mKdV equation (2.3), respectively.

Let us consider a reduction of the N-soliton solution of the matrix NLS equation to that of the Manakov model. We restrict the matrix Q to the following form:

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{q} \\ O \end{bmatrix}, \tag{2.7}$$

so that the matrix NLS equation (2.2) is reduced to the Manakov model (1.1). In this case, the matrices $C_j(t)^\dagger$ must take the same form as Q from their definition [11, 12]:

$$C_j(t)^\dagger = \begin{bmatrix} c_j^{(1)} & c_j^{(2)} & \cdots & c_j^{(m)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \equiv \mathrm{i} \begin{bmatrix} \boldsymbol{c}_j(t) \\ O \end{bmatrix}, \quad j = 1, 2, \ldots, N. \tag{2.8}$$

Conversely, if $C_j(t)^\dagger$ take the form (2.8), $Q(x,t)$ given by the formula (2.4) with (2.5) fits the form (2.7). Then the formula is compressed into a compact form [12]:

$$\boldsymbol{q}(x,t) = 2\sum_{j=1}^{N}\sum_{k=1}^{N}(T^{-1})_{jk}\mathrm{e}^{-2\mathrm{i}\zeta_k^* x}\boldsymbol{c}_k(t),$$

where the $N \times N$ matrix T is given by

$$T_{jk} = \delta_{jk} - \sum_{l=1}^{N}\frac{\mathrm{e}^{2\mathrm{i}(\zeta_l-\zeta_j^*)x}}{(\zeta_l-\zeta_k^*)(\zeta_l-\zeta_j^*)}\boldsymbol{c}_j(t)\cdot\boldsymbol{c}_l(t)^\dagger, \quad 1 \le j,k \le N.$$

Thanks to (2.6) and (2.8), the time dependence of $\boldsymbol{c}_j(t)$ is given by

$$\boldsymbol{c}_j(t) = \mathrm{e}^{-4\mathrm{i}\zeta_j^{*2}t}\boldsymbol{c}_j(0), \quad j = 1,2,\ldots,N.$$

The above set of equations gives a formula for the general N-soliton solution of the Manakov model (1.1) under the vanishing boundary conditions. Let us rewrite it to a form convenient to investigate the asymptotic behavior. We first rewrite it as

$$\boldsymbol{q}(x,t) = 2\sum_{j=1}^{N}\sum_{k=1}^{N}(W^{-1})_{jk}\mathrm{e}^{-\mathrm{i}[(\zeta_k+\zeta_k^*)x+2(\zeta_k^2+\zeta_k^{*2})t]}\boldsymbol{c}_k(0),$$

where the $N \times N$ matrix W is given by

$$\begin{aligned} W_{jk} \;=\; & \delta_{jk}\mathrm{e}^{-\mathrm{i}[(\zeta_j-\zeta_j^*)x+2(\zeta_j^2-\zeta_j^{*2})t]} - \sum_{l=1}^{N}\frac{\boldsymbol{c}_j(0)\cdot\boldsymbol{c}_l(0)^\dagger}{(\zeta_l-\zeta_k^*)(\zeta_l-\zeta_j^*)}\mathrm{e}^{\mathrm{i}[(\zeta_l-\zeta_l^*)x+2(\zeta_l^2-\zeta_l^{*2})t]} \\ & \times \mathrm{e}^{\mathrm{i}[(\zeta_l+\zeta_l^*)x+2(\zeta_l^2+\zeta_l^{*2})t]}\mathrm{e}^{-\mathrm{i}[(\zeta_j+\zeta_j^*)x+2(\zeta_j^2+\zeta_j^{*2})t]}, \quad 1 \le j,k \le N. \end{aligned}$$

Next, we introduce the following parametrization:

$$\begin{aligned} &\zeta_j = \xi_j + \mathrm{i}\eta_j \;\; (\xi_j \in \mathbb{R},\ \eta_j > 0), \\ &\boldsymbol{c}_j(0) = 2\eta_j\mathrm{e}^{-\alpha_j}\boldsymbol{u}_j \;\; (\alpha_j \in \mathbb{R},\ \|\boldsymbol{u}_j\| = 1), \end{aligned}$$

and employ the following abbreviations:

$$\begin{aligned} &\tau_j \equiv -\mathrm{i}[(\zeta_j-\zeta_j^*)x + 2(\zeta_j^2-\zeta_j^{*2})t] = 2\eta_j(x+4\xi_j t), \\ &\Theta_j \equiv (\zeta_j+\zeta_j^*)x + 2(\zeta_j^2+\zeta_j^{*2})t = 2\xi_j x + 4(\xi_j^2-\eta_j^2)t. \end{aligned}$$

Then, we obtain a simplest formula for the general N-soliton solution of the Manakov model (1.1):

$$\boldsymbol{q}(x,t) = 2\sum_{j=1}^{N}\sum_{k=1}^{N}(U^{-1})_{jk}\mathrm{e}^{-\mathrm{i}\Theta_k}\boldsymbol{u}_k, \tag{2.9}$$

where the $N \times N$ matrix U is given by

$$U_{jk} = \frac{\mathrm{e}^{\tau_j+\alpha_j}}{2\eta_j}\delta_{jk} + \sum_{l=1}^{N} \lambda_{jkl}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_j)}, \quad 1 \le j,k \le N, \tag{2.10}$$

with

$$\lambda_{jkl} = -\frac{2\eta_l(\boldsymbol{u}_j \cdot \boldsymbol{u}_l^\dagger)}{(\zeta_l - \zeta_k^*)(\zeta_l - \zeta_j^*)}. \tag{2.11}$$

If we set $N = 1$ in the above formula, we obtain the one-soliton solution:

$$\boldsymbol{q}(x,t) = 2\eta_1 \operatorname{sech}(\tau_1 + \alpha_1)\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1. \tag{2.12}$$

Therefore, we can determine the significance of each parameter as follows:

$2\eta_j$: amplitude of soliton-j
$-4\xi_j$: velocity of soliton-j's envelope
τ_j : coordinate for observing soliton-j's envelope
Θ_j : coordinate for observing soliton-j's carrier waves
$\boldsymbol{u}_j$: polarization vector of soliton-j $(\|\boldsymbol{u}_j\| = 1)$.

To be precise, in the case of two or more solitons, the *real* polarization vectors are not invariant and changed by a soliton collision. The vector $\boldsymbol{u}_j$ defines the *bare* polarization of soliton-j which is achieved when it becomes the rightmost soliton. This point will be seen later. In the following, we assume that all the soliton velocities are distinct so that every soliton collides with all others.

3 Two-soliton collision

In this section, we compute the asymptotic forms of the two-soliton solution as $t \to \mp\infty$, which define the collision laws of two solitons in the Manakov model (1.1).

We first write down the two-soliton solution given by (2.9) with $N = 2$. According to (2.10), the matrix U in this case takes the form,

$$U = \begin{bmatrix} \dfrac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1} + \displaystyle\sum_{l=1}^{2} \lambda_{11l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_1)} & \displaystyle\sum_{l=1}^{2} \lambda_{12l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_1)} \\ \displaystyle\sum_{l=1}^{2} \lambda_{21l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_2)} & \dfrac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2} + \displaystyle\sum_{l=1}^{2} \lambda_{22l}\mathrm{e}^{-(\tau_l+\alpha_l)+\mathrm{i}(\Theta_l-\Theta_2)} \end{bmatrix}.$$

Then, the two-soliton solution is given by

$$\boldsymbol{q}(x,t) = \frac{2}{\det U}\left\{\left[\frac{e^{\tau_2+\alpha_2}}{2\eta_2} + \sum_{l=1}^{2}\lambda_{22l}e^{-(\tau_l+\alpha_l)+i(\Theta_l-\Theta_2)} - \sum_{l=1}^{2}\lambda_{21l}e^{-(\tau_l+\alpha_l)+i(\Theta_l-\Theta_2)}\right]e^{-i\Theta_1}\boldsymbol{u}_1 \right.$$
$$\left. + \left[\frac{e^{\tau_1+\alpha_1}}{2\eta_1} + \sum_{l=1}^{2}\lambda_{11l}e^{-(\tau_l+\alpha_l)+i(\Theta_l-\Theta_1)} - \sum_{l=1}^{2}\lambda_{12l}e^{-(\tau_l+\alpha_l)+i(\Theta_l-\Theta_1)}\right]e^{-i\Theta_2}\boldsymbol{u}_2\right\}, \quad (3.1)$$

with

$$\det U = \frac{e^{\tau_1+\alpha_1}}{2\eta_1}\frac{e^{\tau_2+\alpha_2}}{2\eta_2} + \frac{e^{\tau_1+\alpha_1}}{2\eta_1}\sum_{l=1}^{2}\lambda_{22l}e^{-(\tau_l+\alpha_l)+i(\Theta_l-\Theta_2)} + \frac{e^{\tau_2+\alpha_2}}{2\eta_2}\sum_{l=1}^{2}\lambda_{11l}e^{-(\tau_l+\alpha_l)+i(\Theta_l-\Theta_1)}$$
$$+ e^{-(\tau_1+\alpha_1)}e^{-(\tau_2+\alpha_2)}\sum_{\{l_1,l_2\}=\{1,2\}}\begin{vmatrix} \lambda_{11l_1} & \lambda_{12l_1} \\ \lambda_{21l_2} & \lambda_{22l_2}\end{vmatrix}. \quad (3.2)$$

Here we have simplified the expression of $\det U$ by using the relations,

$$\begin{vmatrix} \lambda_{11l} & \lambda_{12l} \\ \lambda_{21l} & \lambda_{22l}\end{vmatrix} = 0, \quad l = 1, 2, \quad (3.3)$$

which can be proved straightforwardly.

Next, we assume that

$$\xi_1 (= \operatorname{Re}\zeta_1) > \xi_2 (= \operatorname{Re}\zeta_2),$$

and investigate the asymptotic behavior of $\boldsymbol{q}(x,t)$ as $t \to \mp\infty$. This is accomplished by picking the dominant terms from the numerator of (3.1) and its denominator (3.2). We here remark the relation $\tau_1/\eta_1 = \tau_2/\eta_2 + 8(\xi_1 - \xi_2)t$.

In the limit $t \to -\infty$, we have

$$\frac{\tau_1}{\eta_1} \ll \frac{\tau_2}{\eta_2}.$$

In this case, we have to consider the following two regions (1^-) and (2^-) separately. It is easily seen that $\boldsymbol{q} \simeq \boldsymbol{0}$ in the other regions.

(1^-) $\tau_1(\sim \text{finite}) \ll \tau_2(\to +\infty)$

The dominant terms are those which contain the factor e^{τ_2}. Using the relation $\lambda_{111} = 1/(2\eta_1)$, we obtain

$$\boldsymbol{q} \simeq \frac{2\frac{e^{\tau_2+\alpha_2}}{2\eta_2}e^{-i\Theta_1}\boldsymbol{u}_1}{\frac{e^{\tau_1+\alpha_1}}{2\eta_1}\frac{e^{\tau_2+\alpha_2}}{2\eta_2} + \frac{e^{\tau_2+\alpha_2}}{2\eta_2}\lambda_{111}e^{-(\tau_1+\alpha_1)}}$$
$$= 2\eta_1 \operatorname{sech}(\tau_1+\alpha_1)e^{-i\Theta_1}\boldsymbol{u}_1. \quad (3.4)$$

(2^-) $\tau_1(\to -\infty) \ll \tau_2(\sim \text{finite})$

The dominant terms are those which contain the factor $\mathrm{e}^{-\tau_1}$. Then we obtain

$$\boldsymbol{q} \simeq \frac{2\left[(\lambda_{221}-\lambda_{211})\mathrm{e}^{-(\tau_1+\alpha_1)}\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_1 + (\lambda_{111}-\lambda_{121})\mathrm{e}^{-(\tau_1+\alpha_1)}\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_2\right]}{\frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2}\lambda_{111}\mathrm{e}^{-(\tau_1+\alpha_1)} + \mathrm{e}^{-(\tau_1+\alpha_1)}\mathrm{e}^{-(\tau_2+\alpha_2)}\sum_{\substack{\{l_1,l_2\}\\=\{1,2\}}}\begin{vmatrix} \lambda_{11l_1} & \lambda_{12l_1} \\ \lambda_{21l_2} & \lambda_{22l_2}\end{vmatrix}}. \tag{3.5}$$

In terms of ϕ_{12} defined by

$$\mathrm{e}^{-2\phi_{12}} \equiv \frac{1}{\lambda_{111}\lambda_{222}}\sum_{\{l_1,l_2\}=\{1,2\}}\begin{vmatrix} \lambda_{11l_1} & \lambda_{12l_1} \\ \lambda_{21l_2} & \lambda_{22l_2}\end{vmatrix}, \tag{3.6}$$

we can rewrite the asymptotic form (3.5) as

$$\boldsymbol{q} \simeq 2\eta_2 \,\mathrm{sech}(\tau_2+\alpha_2+\phi_{12})\mathrm{e}^{-\mathrm{i}\Theta_2} \times \mathrm{e}^{\phi_{12}}\left[\left(\frac{\lambda_{221}-\lambda_{211}}{\lambda_{111}}\right)\boldsymbol{u}_1 + \left(1-\frac{\lambda_{121}}{\lambda_{111}}\right)\boldsymbol{u}_2\right]. \tag{3.7}$$

Here ϕ_{12} is always taken as real, since (3.6) is rewritten as (cf. (3.3) and (2.11))

$$\begin{aligned}
\mathrm{e}^{-2\phi_{12}} &= \frac{1}{\lambda_{111}\lambda_{222}}\begin{vmatrix} \lambda_{111}+\lambda_{112} & \lambda_{121}+\lambda_{122} \\ \lambda_{211}+\lambda_{212} & \lambda_{221}+\lambda_{222}\end{vmatrix} \\
&= (2\eta_1)(2\eta_2)\begin{vmatrix} \mathrm{i}\frac{\boldsymbol{u}_1\cdot\boldsymbol{u}_1^\dagger}{\zeta_1-\zeta_1^*} & \mathrm{i}\frac{\boldsymbol{u}_1\cdot\boldsymbol{u}_2^\dagger}{\zeta_2-\zeta_1^*} \\ \mathrm{i}\frac{\boldsymbol{u}_2\cdot\boldsymbol{u}_1^\dagger}{\zeta_1-\zeta_2^*} & \mathrm{i}\frac{\boldsymbol{u}_2\cdot\boldsymbol{u}_2^\dagger}{\zeta_2-\zeta_2^*}\end{vmatrix}\begin{vmatrix} \mathrm{i}\frac{2\eta_1}{\zeta_1-\zeta_1^*} & \mathrm{i}\frac{2\eta_1}{\zeta_1-\zeta_2^*} \\ \mathrm{i}\frac{2\eta_2}{\zeta_2-\zeta_1^*} & \mathrm{i}\frac{2\eta_2}{\zeta_2-\zeta_2^*}\end{vmatrix} \\
&= \left|\frac{\zeta_1-\zeta_2}{\zeta_1-\zeta_2^*}\right|^2\left\{1+\frac{(\zeta_1-\zeta_1^*)(\zeta_2-\zeta_2^*)}{|\zeta_1-\zeta_2^*|^2}\left|\boldsymbol{u}_1\cdot\boldsymbol{u}_2^\dagger\right|^2\right\}\ (>0).
\end{aligned}$$

In the limit $t \to +\infty$, we have

$$\frac{\tau_1}{\eta_1} \gg \frac{\tau_2}{\eta_2}.$$

In this case, we have to consider the following two regions (2^+) and (1^+) separately. It is easily seen that $\boldsymbol{q} \simeq \boldsymbol{0}$ in the other regions.

(2^+) $\tau_1(\to +\infty) \gg \tau_2(\sim \text{finite})$

The dominant terms are those which contain the factor e^{τ_1}. Using the relation $\lambda_{222} = 1/(2\eta_2)$, we obtain

$$\begin{aligned}
\boldsymbol{q} &\simeq \frac{2\frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_2}{\frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\frac{\mathrm{e}^{\tau_2+\alpha_2}}{2\eta_2} + \frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\lambda_{222}\mathrm{e}^{-(\tau_2+\alpha_2)}} \\
&= 2\eta_2 \,\mathrm{sech}(\tau_2+\alpha_2)\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_2.
\end{aligned} \tag{3.8}$$

(1$^+$) $\tau_1(\sim \text{finite}) \gg \tau_2(\to -\infty)$

The dominant terms are those which contain the factor $\mathrm{e}^{-\tau_2}$. With the help of (3.6), we obtain

$$\begin{aligned}\boldsymbol{q} &\simeq \frac{2\left[(\lambda_{222}-\lambda_{212})\mathrm{e}^{-(\tau_2+\alpha_2)}\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1 + (\lambda_{112}-\lambda_{122})\mathrm{e}^{-(\tau_2+\alpha_2)}\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_2\right]}{\frac{\mathrm{e}^{\tau_1+\alpha_1}}{2\eta_1}\lambda_{222}\mathrm{e}^{-(\tau_2+\alpha_2)} + \mathrm{e}^{-(\tau_1+\alpha_1)}\mathrm{e}^{-(\tau_2+\alpha_2)}\sum_{\substack{\{l_1,l_2\}\\=\{1,2\}}}\begin{vmatrix}\lambda_{11l_1} & \lambda_{12l_1}\\ \lambda_{21l_2} & \lambda_{22l_2}\end{vmatrix}}\\ &= 2\eta_1\,\mathrm{sech}(\tau_1+\alpha_1+\phi_{12})\mathrm{e}^{-\mathrm{i}\Theta_1}\times \mathrm{e}^{\phi_{12}}\left[\left(1-\frac{\lambda_{212}}{\lambda_{222}}\right)\boldsymbol{u}_1 + \left(\frac{\lambda_{112}-\lambda_{122}}{\lambda_{222}}\right)\boldsymbol{u}_2\right].\end{aligned} \tag{3.9}$$

Summing up (3.4) and (3.7), or (3.8) and (3.9) with slight simplification, we arrive at the following theorem.

Theorem 3.1. *The asymptotic forms of the two-soliton solution of the Manakov model* (*1.1*) *are as follows (see also Fig. 1).*
As $t \to -\infty$,

$$\boldsymbol{q} \simeq 2\eta_1\,\mathrm{sech}(\tau_1+\alpha_1)\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1 + 2\eta_2\,\mathrm{sech}(\tau_2+\alpha_2+\phi_{12})\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_{\{1\},2}.$$

As $t \to +\infty$,

$$\boldsymbol{q} \simeq 2\eta_1\,\mathrm{sech}(\tau_1+\alpha_1+\phi_{12})\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_{\{2\},1} + 2\eta_2\,\mathrm{sech}(\tau_2+\alpha_2)\mathrm{e}^{-\mathrm{i}\Theta_2}\boldsymbol{u}_2.$$

Here ϕ_{12} *and* $\boldsymbol{u}_{\{1\},2}$, $\boldsymbol{u}_{\{2\},1}$ *are given by*

$$\mathrm{e}^{-2\phi_{12}} = \left|\frac{\zeta_1-\zeta_2}{\zeta_1-\zeta_2^*}\right|^2\left\{1+\frac{(\zeta_1-\zeta_1^*)(\zeta_2-\zeta_2^*)}{|\zeta_1-\zeta_2^*|^2}\left|\boldsymbol{u}_1\cdot\boldsymbol{u}_2^\dagger\right|^2\right\},$$

and

$$\begin{aligned}\boldsymbol{u}_{\{1\},2} &= \mathrm{e}^{\phi_{12}}\frac{\zeta_1^*-\zeta_2^*}{\zeta_1-\zeta_2^*}\left\{\boldsymbol{u}_2-\frac{\zeta_1-\zeta_1^*}{\zeta_1-\zeta_2^*}\left(\boldsymbol{u}_2\cdot\boldsymbol{u}_1^\dagger\right)\boldsymbol{u}_1\right\},\\ \boldsymbol{u}_{\{2\},1} &= \mathrm{e}^{\phi_{12}}\frac{\zeta_2^*-\zeta_1^*}{\zeta_2-\zeta_1^*}\left\{\boldsymbol{u}_1-\frac{\zeta_2-\zeta_2^*}{\zeta_2-\zeta_1^*}\left(\boldsymbol{u}_1\cdot\boldsymbol{u}_2^\dagger\right)\boldsymbol{u}_2\right\},\end{aligned}$$

respectively.

Theorem 3.1 defines the collision laws of two solitons in the Manakov model, which we will use in section 5 to factorize an N-soliton collision into pair collisions. Here, we mention some important properties of the collision laws:

- The two-soliton collision does not change the amplitudes of solitons as well as the modulus of the Hermitian product between polarization vectors. In fact, recalling that $\|\boldsymbol{u}_1\| = \|\boldsymbol{u}_2\| = 1$, we can prove by direct computations that

$$\left\|\boldsymbol{u}_{\{1\},2}\right\| = \left\|\boldsymbol{u}_{\{2\},1}\right\| = 1, \quad \left|\boldsymbol{u}_1 \cdot \boldsymbol{u}_{\{1\},2}^{\dagger}\right| = \left|\boldsymbol{u}_{\{2\},1} \cdot \boldsymbol{u}_2^{\dagger}\right|, \quad \left|\boldsymbol{u}_1 \cdot \boldsymbol{u}_2^{\dagger}\right| = \left|\boldsymbol{u}_{\{2\},1} \cdot \boldsymbol{u}_{\{1\},2}^{\dagger}\right|.$$

  Although we omit here the tiresome proof, the most important relation $\left\|\boldsymbol{u}_{\{1\},2}\right\| = \left\|\boldsymbol{u}_{\{2\},1}\right\| = 1$ will be shown in section 5 in a more general context. This relation shows that the collision is elastic if we observe it in the conserved density $\|\boldsymbol{q}\|^2 = \sum_{j=1}^{m} |q_j|^2$.

- As a result of the collision, the polarization vectors rotate nontrivially on the unit sphere in $\mathbb{C}^m$. Thus, if we observe the collision in each component q_k, the collision looks as if it were inelastic.

- We have expressed ϕ_{12}, $\boldsymbol{u}_{\{1\},2}$ and $\boldsymbol{u}_{\{2\},1}$ in terms of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Then, the collision laws become symmetric with respect to interchange of the subscripts 1 and 2. This version of collision laws is very useful in studying the factorization of an N-soliton collision into pair collisions. For any fixed vector $\boldsymbol{u}_1$, we can invert the mapping $\boldsymbol{u}_2 \mapsto \boldsymbol{u}_{\{1\},2}$ using the following relation for the projection operator $\boldsymbol{u}_1^{\dagger}\boldsymbol{u}_1$ (cf. $(\boldsymbol{u}_1^{\dagger}\boldsymbol{u}_1)^2 = \boldsymbol{u}_1^{\dagger}\boldsymbol{u}_1$):

$$\left(I - \frac{\zeta_1 - \zeta_1^*}{\zeta_1 - \zeta_2^*}\boldsymbol{u}_1^{\dagger}\boldsymbol{u}_1\right)\left(I - \frac{\zeta_1 - \zeta_1^*}{\zeta_2^* - \zeta_1^*}\boldsymbol{u}_1^{\dagger}\boldsymbol{u}_1\right) = I.$$

  Then, we can express ϕ_{12}, $\boldsymbol{u}_2$ and $\boldsymbol{u}_{\{2\},1}$ in terms of $\boldsymbol{u}_1$ and $\boldsymbol{u}_{\{1\},2}$ as Manakov did in his paper [4]. However, the symmetry of $1 \leftrightarrow 2$ is lost to that version of collision laws and it is not useful in studying the factorization problem.

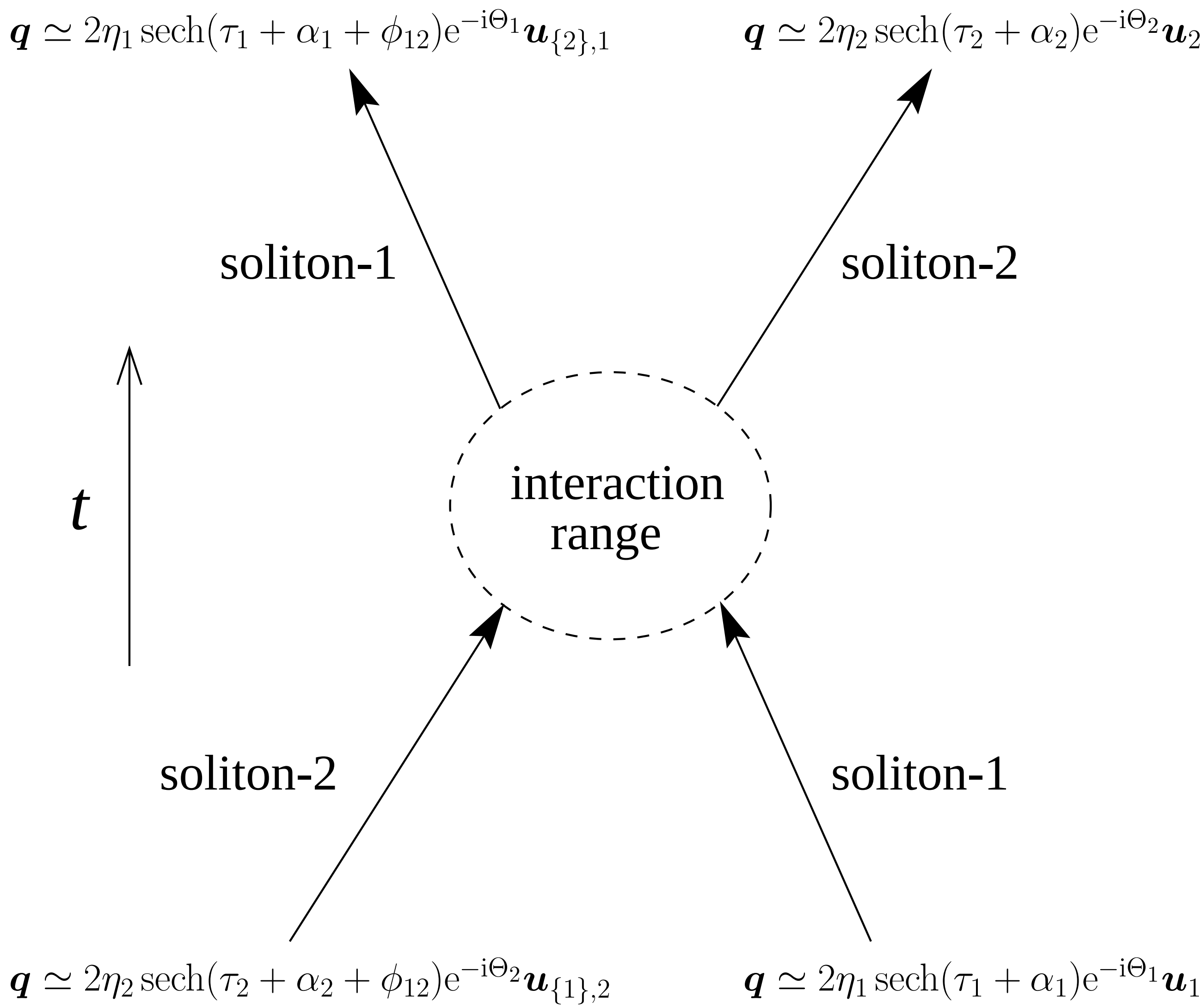

Figure 1: Two-soliton collision

4 Asymptotic behavior of the N-soliton solution

In this section, we compute the asymptotic forms of the N-soliton solution as $t \to \mp\infty$ and simplify them as much as possible.

We first rewrite the N-soliton solution (2.9) before considering the limits $t \to \mp\infty$. We use the tilde to denote cofactors. For instance, the cofactor $\tilde{U}_{kj}$ is obtained by deleting the k-th row and the j-th column from the determinant of U and multiplying it by $(-1)^{k+j}$. Using the definition of U (2.10) and multilinearity of determinants, we can rewrite (2.9) as

$$
\begin{aligned}
&\boldsymbol{q}(x,t)\\
&= \frac{2}{\det U}\sum_{j=1}^{N}\sum_{k=1}^{N}\tilde{U}_{kj}\mathrm{e}^{-\mathrm{i}\Theta_k}\boldsymbol{u}_k\\
&= \frac{2}{\det U}\left[(\tilde{U}_{11}+\cdots+\tilde{U}_{1N})\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1+\cdots+(\tilde{U}_{N1}+\cdots+\tilde{U}_{NN})\mathrm{e}^{-\mathrm{i}\Theta_N}\boldsymbol{u}_N\right]\\
&= \frac{2}{\det U}\left[\begin{vmatrix} 1 & 1 & \cdots & 1\\ U_{21} & U_{22} & \cdots & U_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ U_{N1} & U_{N2} & \cdots & U_{NN}\end{vmatrix}\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1+\cdots+\begin{vmatrix} U_{11} & U_{12} & \cdots & U_{1N}\\ \vdots & \vdots & & \vdots\\ U_{N-11} & U_{N-12} & \cdots & U_{N-1N}\\ 1 & 1 & \cdots & 1\end{vmatrix}\mathrm{e}^{-\mathrm{i}\Theta_N}\boldsymbol{u}_N\right]\\
&= \frac{2}{\det U}\left[\sum_{n=0}^{N-1}\sum_{2\le j_1<\cdots<j_n\le N}\left(\prod_{\substack{k=2\\ k\ne j_1,\dots,j_n}}^{N}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\sum_{l_1,\dots,l_n=1}^{N}\mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_{j_1})}\right.\\
&\qquad\times\cdots\times\mathrm{e}^{-(\tau_{l_n}+\alpha_{l_n})+\mathrm{i}(\Theta_{l_n}-\Theta_{j_n})}\begin{vmatrix} 1 & 1 & \cdots & 1\\ \lambda_{j_1 1 l_1} & \lambda_{j_1 j_1 l_1} & \cdots & \lambda_{j_1 j_n l_1}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_{j_n 1 l_n} & \lambda_{j_n j_1 l_n} & \cdots & \lambda_{j_n j_n l_n}\end{vmatrix}\mathrm{e}^{-\mathrm{i}\Theta_1}\boldsymbol{u}_1\\
&\qquad+\cdots+\sum_{n=0}^{N-1}\sum_{1\le j_1<\cdots<j_n\le N-1}\left(\prod_{\substack{k=1\\ k\ne j_1,\dots,j_n}}^{N-1}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\sum_{l_1,\dots,l_n=1}^{N}\mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_{j_1})}\\
&\qquad\left.\times\cdots\times\mathrm{e}^{-(\tau_{l_n}+\alpha_{l_n})+\mathrm{i}(\Theta_{l_n}-\Theta_{j_n})}\begin{vmatrix} \lambda_{j_1 j_1 l_1} & \cdots & \lambda_{j_1 j_n l_1} & \lambda_{j_1 N l_1}\\ \vdots & \ddots & \vdots & \vdots\\ \lambda_{j_n j_1 l_n} & \cdots & \lambda_{j_n j_n l_n} & \lambda_{j_n N l_n}\\ 1 & \cdots & 1 & 1\end{vmatrix}\mathrm{e}^{-\mathrm{i}\Theta_N}\boldsymbol{u}_N\right].
\end{aligned}
\tag{4.1}
$$

Similarly, we can rewrite the determinant of U as

$$\det U$$

$$
\begin{aligned}
&= \prod_{k=1}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} + \sum_{j_1=1}^{N} \left(\prod_{\substack{k=1\\k\neq j_1}}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{l_1=1}^{N} \mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_{j_1})} \lambda_{j_1 j_1 l_1} \\
&\quad + \sum_{1\le j_1<j_2\le N} \left(\prod_{\substack{k=1\\k\neq j_1,j_2}}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{l_1,l_2=1}^{N} \mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_{j_1})} \mathrm{e}^{-(\tau_{l_2}+\alpha_{l_2})+\mathrm{i}(\Theta_{l_2}-\Theta_{j_2})} \begin{vmatrix} \lambda_{j_1 j_1 l_1} & \lambda_{j_1 j_2 l_1} \\ \lambda_{j_2 j_1 l_2} & \lambda_{j_2 j_2 l_2} \end{vmatrix} \\
&\quad + \cdots + \sum_{l_1,\ldots,l_N=1}^{N} \mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_1)} \cdots \mathrm{e}^{-(\tau_{l_N}+\alpha_{l_N})+\mathrm{i}(\Theta_{l_N}-\Theta_N)} \begin{vmatrix} \lambda_{11l_1} & \cdots & \lambda_{1Nl_1} \\ \vdots & \ddots & \vdots \\ \lambda_{N1l_N} & \cdots & \lambda_{NNl_N} \end{vmatrix} \\
&= \sum_{n=0}^{N} \sum_{1\le j_1<\cdots<j_n\le N} \left(\prod_{\substack{k=1\\k\neq j_1,\ldots,j_n}}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \\
&\quad \times \sum_{l_1,\ldots,l_n=1}^{N} \mathrm{e}^{-(\tau_{l_1}+\alpha_{l_1})+\mathrm{i}(\Theta_{l_1}-\Theta_{j_1})} \cdots \mathrm{e}^{-(\tau_{l_n}+\alpha_{l_n})+\mathrm{i}(\Theta_{l_n}-\Theta_{j_n})} \begin{vmatrix} \lambda_{j_1 j_1 l_1} & \cdots & \lambda_{j_1 j_n l_1} \\ \vdots & \ddots & \vdots \\ \lambda_{j_n j_1 l_n} & \cdots & \lambda_{j_n j_n l_n} \end{vmatrix}. \qquad (4.2)
\end{aligned}
$$

Here, we note (cf. the definition of λ_{jkl} (2.11)) that the quantity

$$\frac{(\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger)}{\zeta_l-\zeta_{j'}^*}\lambda_{jkl} = -\frac{(\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger)2\eta_l(\boldsymbol{u}_j\cdot\boldsymbol{u}_l^\dagger)}{(\zeta_l-\zeta_{j'}^*)(\zeta_l-\zeta_k^*)(\zeta_l-\zeta_j^*)},$$

is invariant under interchange of subscripts j and j'. Thus we have

$$\frac{(\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger)}{\zeta_l-\zeta_{j'}^*}\lambda_{jkl} - \frac{(\boldsymbol{u}_j\cdot\boldsymbol{u}_l^\dagger)}{\zeta_l-\zeta_j^*}\lambda_{j'kl} = 0.$$

This shows that, if $\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger \neq 0$ or $\boldsymbol{u}_j\cdot\boldsymbol{u}_l^\dagger \neq 0$, two vectors $(\lambda_{j1l},\lambda_{j2l},\ldots,\lambda_{jNl})$ and $(\lambda_{j'1l},\lambda_{j'2l},\ldots,\lambda_{j'Nl})$ are linearly dependent. In the case where $\boldsymbol{u}_{j'}\cdot\boldsymbol{u}_l^\dagger = \boldsymbol{u}_j\cdot\boldsymbol{u}_l^\dagger = 0$, according to (2.11), both vectors become zero. Therefore, the determinants in (4.1) or (4.2) contribute only if $l_1,\ldots,l_n$ are distinct. This fact is a generalization of the relations (3.3).

Next, we assume that

$$\xi_1(=\operatorname{Re}\zeta_1) > \xi_2(=\operatorname{Re}\zeta_2) > \cdots > \xi_N(=\operatorname{Re}\zeta_N),$$

and investigate the asymptotic behavior of $\boldsymbol{q}(x,t)$ as $t\to\mp\infty$. This is accomplished by picking the dominant terms from the numerator of (4.1) and its denominator (4.2). We here remark the relations $\tau_j/\eta_j = \tau_k/\eta_k + 8(\xi_j-\xi_k)t$.

In the limit $t \to -\infty$, we have

$$\frac{\tau_1}{\eta_1} \ll \frac{\tau_2}{\eta_2} \ll \cdots \ll \frac{\tau_N}{\eta_N}.$$

In this case, we have to consider the following N regions (1^-)–(N^-) separately. It is easily seen that $\boldsymbol{q} \simeq \boldsymbol{0}$ in the other regions.

(1^-) $\tau_1(\sim \text{finite}) \ll \tau_2, \ldots, \tau_N(\to +\infty)$

The dominant terms are those which contain the factor $\mathrm{e}^{\tau_2+\cdots+\tau_N}$. Using the relation $\lambda_{111} = 1/(2\eta_1)$, we obtain

$$\begin{aligned}\boldsymbol{q} &\simeq \frac{2\left(\prod_{k=2}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right) \mathrm{e}^{-\mathrm{i}\Theta_1} \boldsymbol{u}_1}{\prod_{k=1}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} + \left(\prod_{k=2}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right) \mathrm{e}^{-(\tau_1+\alpha_1)} \lambda_{111}} \\ &= 2\eta_1 \operatorname{sech}(\tau_1+\alpha_1) \mathrm{e}^{-\mathrm{i}\Theta_1} \boldsymbol{u}_1. \end{aligned} \tag{4.3}$$

(n^-) $\tau_1, \ldots, \tau_{n-1}(\to -\infty) \ll \tau_n(\sim \text{finite}) \ll \tau_{n+1}, \ldots, \tau_N(\to +\infty), \qquad n = 2, \ldots, N-1$

The dominant terms are those which contain the factor $\mathrm{e}^{-\tau_1-\cdots-\tau_{n-1}+\tau_{n+1}+\cdots+\tau_N}$. Then, those in the numerator of (4.1) are

$$2 \sum_{j=1}^{n} \left(\prod_{k=n+1}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{\substack{\{l_1,\ldots,l_{n-1}\} \\ =\{1,\ldots,n-1\}}} \mathrm{e}^{-(\tau_1+\alpha_1)} \cdots \mathrm{e}^{-(\tau_{n-1}+\alpha_{n-1})} \mathrm{e}^{\mathrm{i}(\Theta_j-\Theta_n)}$$

$$\times \begin{vmatrix} \lambda_{11l_1} & \cdots & \lambda_{1nl_1} \\ \vdots & & \vdots \\ \lambda_{j-11l_{j-1}} & \cdots & \lambda_{j-1nl_{j-1}} \\ 1 & \cdots & 1 \\ \lambda_{j+11l_j} & \cdots & \lambda_{j+1nl_j} \\ \vdots & & \vdots \\ \lambda_{n1l_{n-1}} & \cdots & \lambda_{nnl_{n-1}} \end{vmatrix} \mathrm{e}^{-\mathrm{i}\Theta_j} \boldsymbol{u}_j,$$

while those in the denominator (4.2) are

$$\left(\prod_{k=n}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{\substack{\{l_1,\ldots,l_{n-1}\} \\ =\{1,\ldots,n-1\}}} \mathrm{e}^{-(\tau_1+\alpha_1)} \cdots \mathrm{e}^{-(\tau_{n-1}+\alpha_{n-1})} \begin{vmatrix} \lambda_{11l_1} & \cdots & \lambda_{1n-1l_1} \\ \vdots & \ddots & \vdots \\ \lambda_{n-11l_{n-1}} & \cdots & \lambda_{n-1n-1l_{n-1}} \end{vmatrix}$$

$$+ \left(\prod_{k=n+1}^{N} \frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k} \right) \sum_{\substack{\{l_1,\ldots,l_n\} \\ =\{1,\ldots,n\}}} \mathrm{e}^{-(\tau_1+\alpha_1)} \cdots \mathrm{e}^{-(\tau_n+\alpha_n)} \begin{vmatrix} \lambda_{11l_1} & \cdots & \lambda_{1nl_1} \\ \vdots & \ddots & \vdots \\ \lambda_{n1l_n} & \cdots & \lambda_{nnl_n} \end{vmatrix}.$$

As a natural extension of (3.6), we define $\phi_{i_1 i_2 \dots i_p}$ for distinct positive integers $i_1, i_2, \dots, i_p$ by

$$\mathrm{e}^{-2\phi_{i_1 i_2 \dots i_p}} \equiv \frac{1}{\lambda_{i_1 i_1 i_1} \cdots \lambda_{i_p i_p i_p}} \sum_{\substack{\{l_1,\dots,l_p\} \\ =\{i_1,\dots,i_p\}}} \begin{vmatrix} \lambda_{i_1 i_1 l_1} & \cdots & \lambda_{i_1 i_p l_1} \\ \vdots & \ddots & \vdots \\ \lambda_{i_p i_1 l_p} & \cdots & \lambda_{i_p i_p l_p} \end{vmatrix}. \tag{4.4}$$

We should note that $\phi_{i_1 i_2 \dots i_p}$ is symmetric with respect to permutations of the subscripts $i_1, i_2, \dots, i_p$. We will prove later that $\phi_{i_1 i_2 \dots i_p}$ is always taken as real. In terms of $\phi_{i_1 i_2 \dots i_p}$, we can express the asymptotic form of $\boldsymbol{q}$ in this region as

$$\begin{aligned} \boldsymbol{q} \simeq\ & 2\eta_n \operatorname{sech}(\tau_n + \alpha_n + \phi_{12\dots n} - \phi_{12\dots n-1}) \mathrm{e}^{-\mathrm{i}\Theta_n} \mathrm{e}^{\phi_{12\dots n} + \phi_{12\dots n-1}} \\ & \times \frac{1}{\lambda_{111} \cdots \lambda_{n-1\,n-1\,n-1}} \sum_{j=1}^{n} \sum_{\substack{\{l_1,\dots,l_{n-1}\} \\ =\{1,\dots,n-1\}}} \begin{vmatrix} \lambda_{11l_1} & \cdots & \lambda_{1nl_1} \\ \vdots & & \vdots \\ \lambda_{j-1\,1\,l_{j-1}} & \cdots & \lambda_{j-1\,n\,l_{j-1}} \\ 1 & \cdots & 1 \\ \lambda_{j+1\,1\,l_j} & \cdots & \lambda_{j+1\,n\,l_j} \\ \vdots & & \vdots \\ \lambda_{n1l_{n-1}} & \cdots & \lambda_{nnl_{n-1}} \end{vmatrix} \boldsymbol{u}_j. \end{aligned} \tag{4.5}$$

(N^-) $\tau_1, \dots, \tau_{N-1} (\to -\infty) \ll \tau_N (\sim \text{finite})$

The dominant terms are those which contain the factor $\mathrm{e}^{-\tau_1 - \cdots - \tau_{N-1}}$. With calculations similar to the case (n^-), we obtain the asymptotic form of $\boldsymbol{q}$ given by (4.5) with $n = N$.

In the limit $t \to +\infty$, we have

$$\frac{\tau_1}{\eta_1} \gg \frac{\tau_2}{\eta_2} \gg \cdots \gg \frac{\tau_N}{\eta_N}.$$

In this case, we have to consider the following N regions (N^+)–(1^+) separately. It is easily seen that $\boldsymbol{q} \simeq \boldsymbol{0}$ in the other regions.

(N^+) $\tau_1, \dots, \tau_{N-1} (\to +\infty) \gg \tau_N (\sim \text{finite})$

The dominant terms are those which contain the factor $\mathrm{e}^{\tau_1 + \cdots + \tau_{N-1}}$. Using the relation $\lambda_{NNN} = 1/(2\eta_N)$, we obtain

$$\begin{aligned} \boldsymbol{q} &\simeq \frac{2 \left(\prod_{k=1}^{N-1} \frac{\mathrm{e}^{\tau_k + \alpha_k}}{2\eta_k} \right) \mathrm{e}^{-\mathrm{i}\Theta_N} \boldsymbol{u}_N}{\prod_{k=1}^{N} \frac{\mathrm{e}^{\tau_k + \alpha_k}}{2\eta_k} + \left(\prod_{k=1}^{N-1} \frac{\mathrm{e}^{\tau_k + \alpha_k}}{2\eta_k} \right) \mathrm{e}^{-(\tau_N + \alpha_N)} \lambda_{NNN}} \\ &= 2\eta_N \operatorname{sech}(\tau_N + \alpha_N) \mathrm{e}^{-\mathrm{i}\Theta_N} \boldsymbol{u}_N. \end{aligned} \tag{4.6}$$

(n^+) $\tau_1, \ldots, \tau_{n-1}(\to +\infty) \gg \tau_n(\sim \text{finite}) \gg \tau_{n+1}, \ldots, \tau_N(\to -\infty), \qquad n = 2, \ldots, N-1$

The dominant terms are those which contain the factor $\mathrm{e}^{\tau_1+\cdots+\tau_{n-1}-\tau_{n+1}-\cdots-\tau_N}$. Then, those in the numerator of (4.1) are

$$2\sum_{j=n}^{N}\left(\prod_{k=1}^{n-1}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\sum_{\substack{\{l_1,\ldots,l_{N-n}\}\\=\{n+1,\ldots,N\}}}\mathrm{e}^{-(\tau_{n+1}+\alpha_{n+1})}\cdots\mathrm{e}^{-(\tau_N+\alpha_N)}\mathrm{e}^{\mathrm{i}(\Theta_j-\Theta_n)}$$
$$\times\begin{vmatrix}\lambda_{nnl_1} & \cdots & \lambda_{nNl_1}\\ \vdots & & \vdots\\ \lambda_{j-1\,n\,l_{j-n}} & \cdots & \lambda_{j-1\,N\,l_{j-n}}\\ 1 & \cdots & 1\\ \lambda_{j+1\,n\,l_{j-n+1}} & \cdots & \lambda_{j+1\,N\,l_{j-n+1}}\\ \vdots & & \vdots\\ \lambda_{Nnl_{N-n}} & \cdots & \lambda_{NNl_{N-n}}\end{vmatrix}\mathrm{e}^{-\mathrm{i}\Theta_j}\boldsymbol{u}_j,$$

while those in the denominator (4.2) are

$$\left(\prod_{k=1}^{n}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\sum_{\substack{\{l_1,\ldots,l_{N-n}\}\\=\{n+1,\ldots,N\}}}\mathrm{e}^{-(\tau_{n+1}+\alpha_{n+1})}\cdots\mathrm{e}^{-(\tau_N+\alpha_N)}\begin{vmatrix}\lambda_{n+1\,n+1\,l_1} & \cdots & \lambda_{n+1\,N\,l_1}\\ \vdots & \ddots & \vdots\\ \lambda_{N\,n+1\,l_{N-n}} & \cdots & \lambda_{NNl_{N-n}}\end{vmatrix}$$
$$+\left(\prod_{k=1}^{n-1}\frac{\mathrm{e}^{\tau_k+\alpha_k}}{2\eta_k}\right)\sum_{\substack{\{l_1,\ldots,l_{N-n+1}\}\\=\{n,\ldots,N\}}}\mathrm{e}^{-(\tau_n+\alpha_n)}\cdots\mathrm{e}^{-(\tau_N+\alpha_N)}\begin{vmatrix}\lambda_{nnl_1} & \cdots & \lambda_{nNl_1}\\ \vdots & \ddots & \vdots\\ \lambda_{Nnl_{N-n+1}} & \cdots & \lambda_{NNl_{N-n+1}}\end{vmatrix}.$$

In terms of $\phi_{i_1 i_2 \ldots i_p}$ defined by (4.4), we can express the asymptotic form of $\boldsymbol{q}$ in this region as

$$\boldsymbol{q} \simeq 2\eta_n\,\mathrm{sech}(\tau_n+\alpha_n+\phi_{nn+1\ldots N}-\phi_{n+1\,n+2\ldots N})\mathrm{e}^{-\mathrm{i}\Theta_n}\mathrm{e}^{\phi_{n\,n+1\ldots N}+\phi_{n+1\,n+2\ldots N}}$$
$$\times\frac{1}{\lambda_{n+1\,n+1\,n+1}\cdots\lambda_{NNN}}\sum_{j=n}^{N}\sum_{\substack{\{l_1,\ldots,l_{N-n}\}\\=\{n+1,\ldots,N\}}}\begin{vmatrix}\lambda_{nnl_1} & \cdots & \lambda_{nNl_1}\\ \vdots & & \vdots\\ \lambda_{j-1\,n\,l_{j-n}} & \cdots & \lambda_{j-1\,N\,l_{j-n}}\\ 1 & \cdots & 1\\ \lambda_{j+1\,n\,l_{j-n+1}} & \cdots & \lambda_{j+1\,N\,l_{j-n+1}}\\ \vdots & & \vdots\\ \lambda_{Nnl_{N-n}} & \cdots & \lambda_{NNl_{N-n}}\end{vmatrix}\boldsymbol{u}_j. \tag{4.7}$$

(1$^+$) $\tau_1(\sim \text{finite}) \gg \tau_2, \ldots, \tau_N(\to -\infty)$

The dominant terms are those which contain the factor $\mathrm{e}^{-\tau_2-\cdots-\tau_N}$. With calculations similar to the case (n^+), we obtain the asymptotic form of $\boldsymbol{q}$ given by (4.7) with $n = 1$.

We shall simplify the above asymptotic forms using the following abbreviations:

$$c_{jk} \equiv \frac{\mathrm{i}}{\zeta_k - \zeta_j^*}, \qquad d_{jk} \equiv \frac{\mathrm{i}}{\zeta_k - \zeta_j^*} \left(\boldsymbol{u}_j \cdot \boldsymbol{u}_k^\dagger \right). \tag{4.8}$$

According to the definition of λ_{jkl} (2.11), we have $\lambda_{jkl} = 2\eta_l c_{kl} d_{jl}$. Then we can rewrite the definition of $\phi_{i_1 i_2 \ldots i_n}$ ((4.4) with $p \to n$) as a factorized form (cf. the paragraph below (4.2)):

$$\begin{aligned}
\mathrm{e}^{-2\phi_{i_1 i_2 \ldots i_n}} &= \prod_{l=1}^{n} (2\eta_{i_l}) \times \sum_{l_1 = i_1, \ldots, i_n} \cdots \sum_{l_n = i_1, \ldots, i_n} \begin{vmatrix} 2\eta_{l_1} c_{i_1 l_1} d_{i_1 l_1} & \cdots & 2\eta_{l_1} c_{i_n l_1} d_{i_1 l_1} \\ \vdots & \ddots & \vdots \\ 2\eta_{l_n} c_{i_1 l_n} d_{i_n l_n} & \cdots & 2\eta_{l_n} c_{i_n l_n} d_{i_n l_n} \end{vmatrix} \\
&= \prod_{l=1}^{n} (2\eta_{i_l})^2 \times \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_n} \\ \vdots & \ddots & \vdots \\ d_{i_n i_1} & \cdots & d_{i_n i_n} \end{vmatrix} \times \begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 i_n} & \cdots & c_{i_n i_n} \end{vmatrix}.
\end{aligned} \tag{4.9}$$

Here $i_1, i_2, \ldots, i_n$ are distinct positive integers. We have for any nonzero vector $(y_{i_1}, \ldots, y_{i_n})$ that

$$\begin{aligned}
(y_{i_1}, \ldots, y_{i_n}) \underline{\begin{pmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_n} \\ \vdots & \ddots & \vdots \\ d_{i_n i_1} & \cdots & d_{i_n i_n} \end{pmatrix}} \begin{pmatrix} y_{i_1}^* \\ \vdots \\ y_{i_n}^* \end{pmatrix} &= \sum_{j,k = i_1, \ldots, i_n} \frac{\mathrm{i}}{\zeta_k - \zeta_j^*} \left(\boldsymbol{u}_j \cdot \boldsymbol{u}_k^\dagger \right) y_j y_k^* \\
&= \sum_{j,k = i_1, \ldots, i_n} \int_0^\infty \mathrm{e}^{\mathrm{i}(\zeta_k - \zeta_j^*) z} \left(\boldsymbol{u}_j \cdot \boldsymbol{u}_k^\dagger \right) y_j y_k^* \mathrm{d}z \\
&= \int_0^\infty \left\| \sum_{j = i_1, \ldots, i_n} \mathrm{e}^{-\mathrm{i}\zeta_j^* z} y_j \boldsymbol{u}_j \right\|^2 \mathrm{d}z \\
&> 0.
\end{aligned}$$

Thus the eigenvalues of the underlined Hermitian matrix are all positive. This proves that the second term on the right-hand side of (4.9) is positive. Considering the special case where all the vectors $\boldsymbol{u}_{i_1}, \ldots, \boldsymbol{u}_{i_n}$ are identical, we can prove the same fact for the third term in (4.9). Therefore, the right-hand side of (4.9) is positive and $\phi_{i_1 i_2 \ldots i_n}$ is always taken as real. In the same way as that in (4.9), we can rewrite the second lines of (4.5) or (4.7) as a factorized form:

$$
\frac{1}{\lambda_{i_1 i_1 i_1}\cdots\lambda_{i_{n-1} i_{n-1} i_{n-1}}}\sum_{j=1}^{n}\sum_{\substack{\{l_1,\dots,l_{n-1}\}\\=\{i_1,\dots,i_{n-1}\}}}\begin{vmatrix}
\lambda_{i_1 i_1 l_1} & \cdots & \lambda_{i_1 i_n l_1}\\
\vdots & & \vdots\\
\lambda_{i_{j-1} i_1 l_{j-1}} & \cdots & \lambda_{i_{j-1} i_n l_{j-1}}\\
1 & \cdots & 1\\
\lambda_{i_{j+1} i_1 l_j} & \cdots & \lambda_{i_{j+1} i_n l_j}\\
\vdots & & \vdots\\
\lambda_{i_n i_1 l_{n-1}} & \cdots & \lambda_{i_n i_n l_{n-1}}
\end{vmatrix}\boldsymbol{u}_{i_j}
$$

$$
=\prod_{l=1}^{n-1}(2\eta_{i_l})\times\sum_{j=1}^{n}\sum_{l_1=i_1,\dots,i_{n-1}}\cdots\sum_{l_{n-1}=i_1,\dots,i_{n-1}}\begin{vmatrix}
2\eta_{l_1}c_{i_1 l_1}d_{i_1 l_1} & \cdots & 2\eta_{l_1}c_{i_n l_1}d_{i_1 l_1}\\
\vdots & & \vdots\\
2\eta_{l_{j-1}}c_{i_1 l_{j-1}}d_{i_{j-1} l_{j-1}} & \cdots & 2\eta_{l_{j-1}}c_{i_n l_{j-1}}d_{i_{j-1} l_{j-1}}\\
1 & \cdots & 1\\
2\eta_{l_j}c_{i_1 l_j}d_{i_{j+1} l_j} & \cdots & 2\eta_{l_j}c_{i_n l_j}d_{i_{j+1} l_j}\\
\vdots & & \vdots\\
2\eta_{l_{n-1}}c_{i_1 l_{n-1}}d_{i_n l_{n-1}} & \cdots & 2\eta_{l_{n-1}}c_{i_n l_{n-1}}d_{i_n l_{n-1}}
\end{vmatrix}\boldsymbol{u}_{i_j}
$$

$$
=\prod_{l=1}^{n-1}(2\eta_{i_l})^2\times\sum_{j=1}^{n}\begin{vmatrix}
d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & 0\\
\vdots & & \vdots & \vdots\\
d_{i_{j-1} i_1} & \cdots & d_{i_{j-1} i_{n-1}} & 0\\
0 & \cdots & 0 & 1\\
d_{i_{j+1} i_1} & \cdots & d_{i_{j+1} i_{n-1}} & 0\\
\vdots & & \vdots & \vdots\\
d_{i_n i_1} & \cdots & d_{i_n i_{n-1}} & 0
\end{vmatrix}\times\begin{vmatrix}
c_{i_1 i_1} & \cdots & c_{i_n i_1}\\
\vdots & & \vdots\\
c_{i_1 i_{n-1}} & \cdots & c_{i_n i_{n-1}}\\
1 & \cdots & 1
\end{vmatrix}\boldsymbol{u}_{i_j}
$$

$$
=\prod_{l=1}^{n-1}(2\eta_{i_l})^2\times\begin{vmatrix}
c_{i_1 i_1} & \cdots & c_{i_n i_1}\\
\vdots & & \vdots\\
c_{i_1 i_{n-1}} & \cdots & c_{i_n i_{n-1}}\\
1 & \cdots & 1
\end{vmatrix}\times\begin{vmatrix}
d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1}\\
\vdots & & \vdots & \vdots\\
d_{i_n i_1} & \cdots & d_{i_n i_{n-1}} & \boldsymbol{u}_{i_n}
\end{vmatrix}. \tag{4.10}
$$

The determinant which contains vectors in the last column represents a vector defined by the Laplace expansion with respect to the last column.

We can simplify the asymptotic forms further by noting some relations for the conventional determinants in (4.9) or (4.10). We have the following lemma:

Lemma 4.1. *The following equalities for determinants hold:*

$$\begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_n i_{n-1}} \\ 1 & \cdots & 1 \end{vmatrix} = \prod_{l=1}^{n-1} \frac{\zeta_{i_l}^* - \zeta_{i_n}^*}{\zeta_{i_l} - \zeta_{i_n}^*} \times \begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_{n-1} i_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_{n-1} i_{n-1}} \end{vmatrix}, \tag{4.11}$$

$$\begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 i_n} & \cdots & c_{i_n i_n} \end{vmatrix} = \frac{\mathrm{i} \prod_{l=1}^{n-1} (\zeta_{i_n} - \zeta_{i_l})}{\prod_{l=1}^{n} (\zeta_{i_n} - \zeta_{i_l}^*)} \times \begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_n i_{n-1}} \\ 1 & \cdots & 1 \end{vmatrix}, \tag{4.12}$$

$$\begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_n i_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 i_n} & \cdots & c_{i_n i_n} \end{vmatrix} = \frac{\mathrm{i}}{\zeta_{i_n} - \zeta_{i_n}^*} \prod_{l=1}^{n-1} \left| \frac{\zeta_{i_l} - \zeta_{i_n}}{\zeta_{i_l} - \zeta_{i_n}^*} \right|^2 \times \begin{vmatrix} c_{i_1 i_1} & \cdots & c_{i_{n-1} i_1} \\ \vdots & \ddots & \vdots \\ c_{i_1 i_{n-1}} & \cdots & c_{i_{n-1} i_{n-1}} \end{vmatrix}. \tag{4.13}$$

Proof. We can prove (4.11) by subtracting on the left-hand side the last column from the other columns and using the relation,

$$c_{jk} - c_{nk} = \frac{\zeta_j^* - \zeta_n^*}{\zeta_k - \zeta_n^*} c_{jk}.$$

Similarly, (4.12) is proved by subtracting on the left-hand side the last row from the other rows and using the relation,

$$c_{jk} - c_{jn} = \frac{\zeta_n - \zeta_k}{\zeta_n - \zeta_j^*} c_{jk}.$$

(4.13) is a direct consequence of (4.12) and (4.11). □

Summing up (4.3) and (4.5) ($n = 2, \ldots, N$), or (4.6) and (4.7) ($n = 1, \ldots, N-1$) with the help of (4.9), (4.10) and Lemma 4.1, we finally arrive at the following proposition.

Proposition 4.2. *The asymptotic forms of the N-soliton solution of the Manakov model (1.1) are as follows.*
As $t \to -\infty$,

$$\boldsymbol{q} \simeq \sum_{n=1}^{N} 2\eta_n \operatorname{sech}(\tau_n + \alpha_n + \phi_{\{1,\ldots,n-1\},n}) \mathrm{e}^{-\mathrm{i}\Theta_n} \boldsymbol{u}_{\{1,\ldots,n-1\},n}.$$

As $t \to +\infty$,

$$\boldsymbol{q} \simeq \sum_{n=1}^{N} 2\eta_n \operatorname{sech}(\tau_n + \alpha_n + \phi_{\{n+1,\dots,N\},n}) \mathrm{e}^{-\mathrm{i}\Theta_n} \boldsymbol{u}_{\{n+1,\dots,N\},n}.$$

Here $\phi_{\{i_1,\dots,i_{n-1}\},i_n}$ *and* $\boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},i_n}$ *are defined for distinct positive integers* $i_1, \dots, i_{n-1}, i_n$ *by*

$$\begin{aligned} \mathrm{e}^{-2\phi_{\{i_1,\dots,i_{n-1}\},i_n}} &\equiv \mathrm{e}^{-2(\phi_{i_1\dots i_{n-1}i_n} - \phi_{i_1\dots i_{n-1}})} \\ &= \prod_{l=1}^{n-1} \left| \frac{\zeta_{i_l} - \zeta_{i_n}}{\zeta_{i_l} - \zeta_{i_n}^*} \right|^2 \times \frac{\begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_n} \\ \vdots & \ddots & \vdots \\ d_{i_n i_1} & \cdots & d_{i_n i_n} \end{vmatrix}}{d_{i_n i_n} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} \end{vmatrix}} \ (> 0), \end{aligned} \qquad (4.14)$$

and

$$\begin{aligned} \boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},i_n} &\equiv \frac{\mathrm{e}^{\phi_{i_1\dots i_{n-1}i_n} + \phi_{i_1\dots i_{n-1}}}}{\lambda_{i_1 i_1 i_1} \cdots \lambda_{i_{n-1} i_{n-1} i_{n-1}}} \sum_{j=1}^{n} \sum_{\substack{\{l_1,\dots,l_{n-1}\} \\ =\{i_1,\dots,i_{n-1}\}}} \begin{vmatrix} \lambda_{i_1 i_1 l_1} & \cdots & \lambda_{i_1 i_n l_1} \\ \vdots & & \vdots \\ \lambda_{i_{j-1} i_1 l_{j-1}} & \cdots & \lambda_{i_{j-1} i_n l_{j-1}} \\ 1 & \cdots & 1 \\ \lambda_{i_{j+1} i_1 l_j} & \cdots & \lambda_{i_{j+1} i_n l_j} \\ \vdots & & \vdots \\ \lambda_{i_n i_1 l_{n-1}} & \cdots & \lambda_{i_n i_n l_{n-1}} \end{vmatrix} \boldsymbol{u}_{i_j} \\ &= \mathrm{e}^{\phi_{\{i_1,\dots,i_{n-1}\},i_n}} \prod_{l=1}^{n-1} \frac{\zeta_{i_l}^* - \zeta_{i_n}^*}{\zeta_{i_l} - \zeta_{i_n}^*} \times \frac{\begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & & \vdots & \vdots \\ d_{i_n i_1} & \cdots & d_{i_n i_{n-1}} & \boldsymbol{u}_{i_n} \end{vmatrix}}{\begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} \end{vmatrix}}, \end{aligned} \qquad (4.15)$$

respectively. When the set $\{i_1, \dots, i_{n-1}\}$ *is empty, the definitions (4.14) and (4.15) should read as* $\mathrm{e}^{-2\phi_{\{\ \},i}} \equiv 1$ *and* $\boldsymbol{u}_{\{\ \},i} \equiv \boldsymbol{u}_i$.

We will prove in the next section that $\boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},i_n}$ is always a unit vector, i.e. $\left\| \boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},i_n} \right\| = 1$. This ensures that an N-soliton collision does not change the amplitudes of solitons. The vector $\boldsymbol{u}_{\{1,\ldots,n-1\},n}$ gives the polarization vector of soliton-n before an N-soliton collision, while $\boldsymbol{u}_{\{n+1,\ldots,N\},n}$ gives that after the collision. Using the abbreviation,

$$\boldsymbol{q}_{\{i_1,\ldots,i_{n-1}\},i_n} \equiv 2\eta_{i_n}\,\mathrm{sech}(\tau_{i_n} + \alpha_{i_n} + \phi_{\{i_1,\ldots,i_{n-1}\},i_n})\mathrm{e}^{-\mathrm{i}\Theta_{i_n}}\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},i_n}, \tag{4.16}$$

we can diagram the asymptotic behavior of the N-soliton solution in a simplest way (see Fig. 2). We should note that $\boldsymbol{q}_{\{i_1,\ldots,i_{n-1}\},i_n}$ is symmetric with respect to permutations of $i_1,\ldots,i_{n-1}$. The last subscript i_n denotes a soliton number which is of course time-independent. The significance of the other subscripts $i_1,\ldots,i_{n-1}$ in $\{\ \}$ will be seen in the next section.

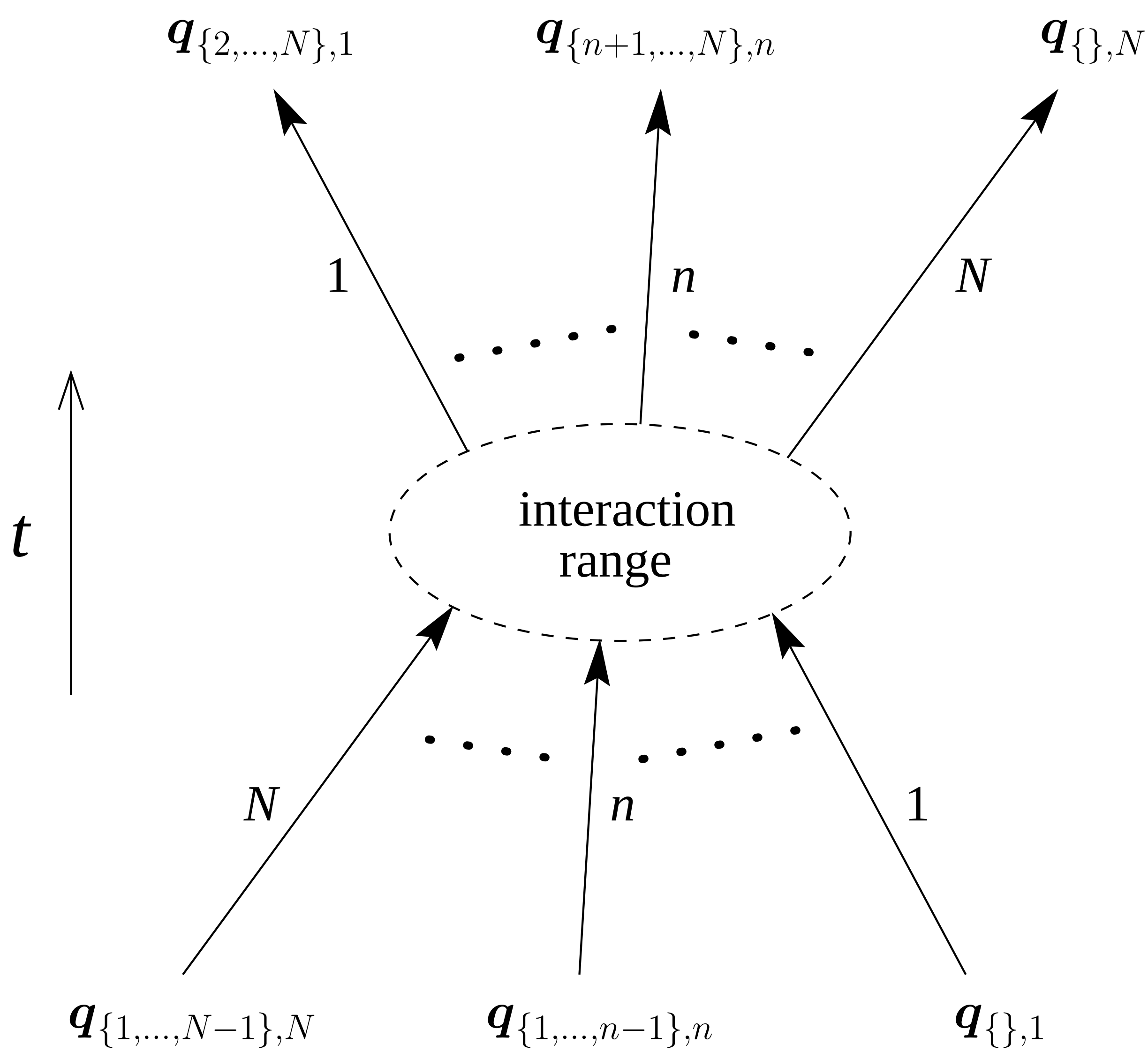

Figure 2: Asymptotic behavior of the N-soliton solution

5 Factorization of an N-soliton collision into superposition of pair collisions

In this section, based on the collision laws of two solitons presented in section 3, we examine the asymptotic behavior of the N-soliton solution obtained in section 4. We conclude that an N-soliton collision in the Manakov model (1.1) is factorized into nonlinear superposition of pair collisions in arbitrary order.

We first prove a lemma needed later to compute the Hermitian product between $\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},j}$ and $\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},k}$.

Lemma 5.1. *For any set of unit vectors $\boldsymbol{u}_{i_1},\ldots,\boldsymbol{u}_{i_{n-1}},\boldsymbol{u}_j,\boldsymbol{u}_k$ and d_{il} defined by (4.8), the following equality holds:*

$$\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{ji_1} & \cdots & d_{ji_{n-1}} & \boldsymbol{u}_j \end{vmatrix} \cdot \begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{ki_1} & \cdots & d_{ki_{n-1}} & \boldsymbol{u}_k \end{vmatrix}^\dagger$$
$$= \frac{\zeta_k - \zeta_j^*}{\mathrm{i}} \times \begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} \end{vmatrix} \times \begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} & d_{i_1k} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} & d_{i_{n-1}k} \\ d_{ji_1} & \cdots & d_{ji_{n-1}} & d_{jk} \end{vmatrix}. \tag{5.1}$$

Proof. In the proof of this lemma, we use D to denote the last determinant in (5.1):

$$D \equiv \begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} & d_{i_1k} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} & d_{i_{n-1}k} \\ d_{ji_1} & \cdots & d_{ji_{n-1}} & d_{jk} \end{vmatrix}.$$

We express minor determinants obtained by deleting one row and one column from D as

$$D\begin{bmatrix} i_l \\ k \end{bmatrix} = \begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} \\ \vdots & & \vdots \\ d_{i_{l-1}i_1} & \cdots & d_{i_{l-1}i_{n-1}} \\ d_{i_{l+1}i_1} & \cdots & d_{i_{l+1}i_{n-1}} \\ \vdots & & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} \\ d_{ji_1} & \cdots & d_{ji_{n-1}} \end{vmatrix},$$

$$D\left[\begin{array}{c} j \\ i_l \end{array}\right] = \left|\begin{array}{cccccc} d_{i_1 i_1} & \cdots & d_{i_1 i_{l-1}} & d_{i_1 i_{l+1}} & \cdots & d_{i_1 i_{n-1}} & d_{i_1 k} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{l-1}} & d_{i_{n-1} i_{l+1}} & \cdots & d_{i_{n-1} i_{n-1}} & d_{i_{n-1} k} \end{array}\right|,$$

$$D\left[\begin{array}{c} j \\ k \end{array}\right] = \left|\begin{array}{ccc} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} \end{array}\right|.$$

Using these abbreviations and the Laplace expansion of determinants, we can rewrite the left-hand side of (5.1) as

$$\begin{aligned}
\text{l.h.s.} &= \left\{ \sum_{p=1}^{n-1} (-1)^{n+p} D\left[\begin{array}{c} i_p \\ k \end{array}\right] \boldsymbol{u}_{i_p} + D\left[\begin{array}{c} j \\ k \end{array}\right] \boldsymbol{u}_j \right\} \cdot \left\{ \sum_{q=1}^{n-1} (-1)^{n+q} D\left[\begin{array}{c} j \\ i_q \end{array}\right] \boldsymbol{u}_{i_q}^\dagger + D\left[\begin{array}{c} j \\ k \end{array}\right] \boldsymbol{u}_k^\dagger \right\} \\
&= \sum_{p=1}^{n-1} \sum_{q=1}^{n-1} (-1)^{p+q} D\left[\begin{array}{c} i_p \\ k \end{array}\right] D\left[\begin{array}{c} j \\ i_q \end{array}\right] \times \left\{ \frac{(\zeta_{i_q} - \zeta_j^*) + (\zeta_j^* - \zeta_{i_p}^*)}{\mathrm{i}} \right\} d_{i_p i_q} \\
&\quad + \sum_{q=1}^{n-1} (-1)^{n+q} D\left[\begin{array}{c} j \\ k \end{array}\right] D\left[\begin{array}{c} j \\ i_q \end{array}\right] \times \frac{(\zeta_{i_q} - \zeta_j^*)}{\mathrm{i}} d_{j i_q} \\
&\quad + \sum_{p=1}^{n-1} (-1)^{n+p} D\left[\begin{array}{c} i_p \\ k \end{array}\right] D\left[\begin{array}{c} j \\ k \end{array}\right] \times \left\{ \frac{(\zeta_k - \zeta_j^*) + (\zeta_j^* - \zeta_{i_p}^*)}{\mathrm{i}} \right\} d_{i_p k} \\
&\quad + D\left[\begin{array}{c} j \\ k \end{array}\right] D\left[\begin{array}{c} j \\ k \end{array}\right] \times \frac{(\zeta_k - \zeta_j^*)}{\mathrm{i}} d_{jk} \\
&= \sum_{q=1}^{n-1} (-1)^{n+q} D\left[\begin{array}{c} j \\ i_q \end{array}\right] \frac{(\zeta_{i_q} - \zeta_j^*)}{\mathrm{i}} \left\{ \sum_{p=1}^{n-1} (-1)^{n+p} D\left[\begin{array}{c} i_p \\ k \end{array}\right] d_{i_p i_q} + D\left[\begin{array}{c} j \\ k \end{array}\right] d_{j i_q} \right\} \\
&\quad + \sum_{p=1}^{n-1} (-1)^{n+p} D\left[\begin{array}{c} i_p \\ k \end{array}\right] \frac{(\zeta_j^* - \zeta_{i_p}^*)}{\mathrm{i}} \left\{ \sum_{q=1}^{n-1} (-1)^{n+q} D\left[\begin{array}{c} j \\ i_q \end{array}\right] d_{i_p i_q} + D\left[\begin{array}{c} j \\ k \end{array}\right] d_{i_p k} \right\} \\
&\quad + D\left[\begin{array}{c} j \\ k \end{array}\right] \frac{(\zeta_k - \zeta_j^*)}{\mathrm{i}} \left\{ \sum_{p=1}^{n-1} (-1)^{n+p} D\left[\begin{array}{c} i_p \\ k \end{array}\right] d_{i_p k} + D\left[\begin{array}{c} j \\ k \end{array}\right] d_{jk} \right\} \\
&= \sum_{q=1}^{n-1} (-1)^{n+q} D\left[\begin{array}{c} j \\ i_q \end{array}\right] \frac{(\zeta_{i_q} - \zeta_j^*)}{\mathrm{i}} \left|\begin{array}{cccc} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & d_{i_1 i_q} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & d_{i_{n-1} i_q} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & d_{j i_q} \end{array}\right|
\end{aligned}$$

$$
+\sum_{p=1}^{n-1}(-1)^{n+p}D\left[\begin{array}{c} i_p \\ k \end{array}\right]\frac{(\zeta_j^*-\zeta_{i_p}^*)}{\mathrm{i}}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} & d_{i_1k} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} & d_{i_{n-1}k} \\ d_{i_pi_1} & \cdots & d_{i_pi_{n-1}} & d_{i_pk} \end{vmatrix}
$$

$$
+D\left[\begin{array}{c} j \\ k \end{array}\right]\frac{(\zeta_k-\zeta_j^*)}{\mathrm{i}}D.
$$

Is is easily seen that in the last expression only the last term remains, which is the right-hand side of (5.1). □

Corollary 5.2. *The vector* $\boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},i_n}$ *defined for distinct positive integers* $i_1,\dots,i_{n-1},i_n$ *by (4.15) with (4.14) is a unit vector, i.e.*

$$
\left\|\boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},i_n}\right\| = 1.
$$

Proof. Using Lemma 5.1 in the special case $j=k(\equiv i_n)$, we have

$$
\begin{aligned}
\left\|\boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},i_n}\right\|^2 &= \boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},i_n}\cdot\boldsymbol{u}^{\dagger}_{\{i_1,\dots,i_{n-1}\},i_n} \\
&= \prod_{l=1}^{n-1}\left|\frac{\zeta_{i_l}-\zeta_{i_n}^*}{\zeta_{i_l}-\zeta_{i_n}}\right|^2 \times \frac{d_{i_ni_n}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} \end{vmatrix}}{\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_n} \\ \vdots & \ddots & \vdots \\ d_{i_ni_1} & \cdots & d_{i_ni_n} \end{vmatrix}} \\
&\quad\times\prod_{l=1}^{n-1}\left|\frac{\zeta_{i_l}^*-\zeta_{i_n}^*}{\zeta_{i_l}-\zeta_{i_n}^*}\right|^2 \times \frac{\dfrac{\zeta_{i_n}-\zeta_{i_n}^*}{\mathrm{i}}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} \end{vmatrix}\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_n} \\ \vdots & \ddots & \vdots \\ d_{i_ni_1} & \cdots & d_{i_ni_n} \end{vmatrix}}{\begin{vmatrix} d_{i_1i_1} & \cdots & d_{i_1i_{n-1}} \\ \vdots & \ddots & \vdots \\ d_{i_{n-1}i_1} & \cdots & d_{i_{n-1}i_{n-1}} \end{vmatrix}^2} \\
&= 1.
\end{aligned}
$$

□

We are now able to apply the collision laws defined by Theorem 3.1 to the two-soliton collision that soliton $\boldsymbol{q}_{\{i_1,\dots,i_{n-1},j\},k}$ overtakes soliton $\boldsymbol{q}_{\{i_1,\dots,i_{n-1}\},j}$ (cf. the definition (4.16)). Here $i_1,\dots,i_{n-1},j,k$ are distinct positive integers.

Proposition 5.3. *The two-soliton collision that* $\boldsymbol{q}_{\{i_1,\ldots,i_{n-1},j\},k}$ *overtakes* $\boldsymbol{q}_{\{i_1,\ldots,i_{n-1}\},j}$ *changes these solitons to* $\boldsymbol{q}_{\{i_1,\ldots,i_{n-1}\},k}$ *and* $\boldsymbol{q}_{\{i_1,\ldots,i_{n-1},k\},j}$ *as shown in Fig. 3. According to Theorem 3.1, this is equivalent to the following set of equalities:*

$$\mathrm{e}^{-2\left(\phi_{\{i_1,\ldots,i_{n-1},j\},k}-\phi_{\{i_1,\ldots,i_{n-1}\},k}\right)} = \mathrm{e}^{-2\left(\phi_{\{i_1,\ldots,i_{n-1},k\},j}-\phi_{\{i_1,\ldots,i_{n-1}\},j}\right)} \tag{5.2a}$$

$$= \left|\frac{\zeta_j-\zeta_k}{\zeta_j-\zeta_k^*}\right|^2 \left\{1+\frac{(\zeta_j-\zeta_j^*)(\zeta_k-\zeta_k^*)}{|\zeta_j-\zeta_k^*|^2}\left|\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},j}\cdot\boldsymbol{u}^{\dagger}_{\{i_1,\ldots,i_{n-1}\},k}\right|^2\right\}, \tag{5.2b}$$

$$\begin{aligned}\boldsymbol{u}_{\{i_1,\ldots,i_{n-1},j\},k} =\ & \mathrm{e}^{\phi_{\{i_1,\ldots,i_{n-1},j\},k}-\phi_{\{i_1,\ldots,i_{n-1}\},k}}\frac{\zeta_j^*-\zeta_k^*}{\zeta_j-\zeta_k^*}\Big\{\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},k}\\ & -\frac{\zeta_j-\zeta_j^*}{\zeta_j-\zeta_k^*}\left(\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},k}\cdot\boldsymbol{u}^{\dagger}_{\{i_1,\ldots,i_{n-1}\},j}\right)\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},j}\Big\},\end{aligned} \tag{5.3}$$

$$\begin{aligned}\boldsymbol{u}_{\{i_1,\ldots,i_{n-1},k\},j} =\ & \mathrm{e}^{\phi_{\{i_1,\ldots,i_{n-1},k\},j}-\phi_{\{i_1,\ldots,i_{n-1}\},j}}\frac{\zeta_k^*-\zeta_j^*}{\zeta_k-\zeta_j^*}\Big\{\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},j}\\ & -\frac{\zeta_k-\zeta_k^*}{\zeta_k-\zeta_j^*}\left(\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},j}\cdot\boldsymbol{u}^{\dagger}_{\{i_1,\ldots,i_{n-1}\},k}\right)\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},k}\Big\}.\end{aligned} \tag{5.4}$$

Proof. Throughout the proof, we employ the following notation to write determinants compactly:

$$d\begin{pmatrix} j_1, j_2, \ldots, j_l \\ k_1, k_2, \ldots, k_l \end{pmatrix} \equiv \begin{vmatrix} d_{j_1k_1} & d_{j_1k_2} & \cdots & d_{j_1k_l} \\ d_{j_2k_1} & d_{j_2k_2} & \cdots & d_{j_2k_l} \\ \vdots & \vdots & \ddots & \vdots \\ d_{j_lk_1} & d_{j_lk_2} & \cdots & d_{j_lk_l} \end{vmatrix}.$$

To prove (5.2), we first rewrite the left-hand side of (5.2a) as

$$\mathrm{e}^{-2\left(\phi_{\{i_1,\ldots,i_{n-1},j\},k}-\phi_{\{i_1,\ldots,i_{n-1}\},k}\right)} = \left|\frac{\zeta_j-\zeta_k}{\zeta_j-\zeta_k^*}\right|^2 \times \frac{d\begin{pmatrix} i_1,\ldots,i_{n-1},j,k \\ i_1,\ldots,i_{n-1},j,k\end{pmatrix} d\begin{pmatrix} i_1,\ldots,i_{n-1} \\ i_1,\ldots,i_{n-1}\end{pmatrix}}{d\begin{pmatrix} i_1,\ldots,i_{n-1},j \\ i_1,\ldots,i_{n-1},j\end{pmatrix} d\begin{pmatrix} i_1,\ldots,i_{n-1},k \\ i_1,\ldots,i_{n-1},k\end{pmatrix}}, \tag{5.5}$$

by using the definition of $\phi_{\{i_1,\ldots,i_{n-1}\},i_n}$ (4.14). Obviously, the right-hand side of (5.5) is symmetric with respect to interchange of j and k. It follows that equality (5.2a) holds.

Next, we prove equality (5.2b). Using the definition of $\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},i_n}$ (4.15) and Lemma 5.1, we obtain

$$\boldsymbol{u}_{\{i_1,\ldots,i_{n-1}\},j}\cdot\boldsymbol{u}^{\dagger}_{\{i_1,\ldots,i_{n-1}\},k}$$

$$= \mathrm{e}^{\phi_{\{i_1,\dots,i_{n-1}\},j}+\phi_{\{i_1,\dots,i_{n-1}\},k}} \times \frac{\zeta_k - \zeta_j^*}{\mathrm{i}} \times \prod_{l=1}^{n-1} \frac{(\zeta_{i_l}^* - \zeta_j^*)(\zeta_{i_l} - \zeta_k)}{(\zeta_{i_l} - \zeta_j^*)(\zeta_{i_l}^* - \zeta_k)} \times \frac{d\begin{pmatrix} i_1,\dots,i_{n-1},j \\ i_1,\dots,i_{n-1},k \end{pmatrix}}{d\begin{pmatrix} i_1,\dots,i_{n-1} \\ i_1,\dots,i_{n-1} \end{pmatrix}}. \tag{5.6}$$

Multiplying (5.6) by its complex conjugate on each side, we obtain with the help of (4.14) that

$$\left|\boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},j} \cdot \boldsymbol{u}^\dagger_{\{i_1,\dots,i_{n-1}\},k}\right|^2 = -\frac{\left|\zeta_k - \zeta_j^*\right|^2}{(\zeta_j - \zeta_j^*)(\zeta_k - \zeta_k^*)} \times \frac{d\begin{pmatrix} i_1,\dots,i_{n-1},j \\ i_1,\dots,i_{n-1},k \end{pmatrix} d\begin{pmatrix} i_1,\dots,i_{n-1},k \\ i_1,\dots,i_{n-1},j \end{pmatrix}}{d\begin{pmatrix} i_1,\dots,i_{n-1},j \\ i_1,\dots,i_{n-1},j \end{pmatrix} d\begin{pmatrix} i_1,\dots,i_{n-1},k \\ i_1,\dots,i_{n-1},k \end{pmatrix}}.$$

Then, we can rewrite the right-hand side of (5.2b) as

$$\begin{aligned} &\left|\frac{\zeta_j - \zeta_k}{\zeta_j - \zeta_k^*}\right|^2 \left\{1 + \frac{(\zeta_j - \zeta_j^*)(\zeta_k - \zeta_k^*)}{|\zeta_j - \zeta_k^*|^2} \left|\boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},j} \cdot \boldsymbol{u}^\dagger_{\{i_1,\dots,i_{n-1}\},k}\right|^2\right\} \\ &= \left|\frac{\zeta_j - \zeta_k}{\zeta_j - \zeta_k^*}\right|^2 \left\{1 - \frac{d\begin{pmatrix} i_1,\dots,i_{n-1},j \\ i_1,\dots,i_{n-1},k \end{pmatrix} d\begin{pmatrix} i_1,\dots,i_{n-1},k \\ i_1,\dots,i_{n-1},j \end{pmatrix}}{d\begin{pmatrix} i_1,\dots,i_{n-1},j \\ i_1,\dots,i_{n-1},j \end{pmatrix} d\begin{pmatrix} i_1,\dots,i_{n-1},k \\ i_1,\dots,i_{n-1},k \end{pmatrix}}\right\}. \end{aligned} \tag{5.7}$$

Here, thanks to the Jacobi formula for determinants, we have

$$\begin{aligned} &d\begin{pmatrix} i_1,\dots,i_{n-1},j \\ i_1,\dots,i_{n-1},j \end{pmatrix} d\begin{pmatrix} i_1,\dots,i_{n-1},k \\ i_1,\dots,i_{n-1},k \end{pmatrix} - d\begin{pmatrix} i_1,\dots,i_{n-1},j \\ i_1,\dots,i_{n-1},k \end{pmatrix} d\begin{pmatrix} i_1,\dots,i_{n-1},k \\ i_1,\dots,i_{n-1},j \end{pmatrix} \\ &= d\begin{pmatrix} i_1,\dots,i_{n-1} \\ i_1,\dots,i_{n-1} \end{pmatrix} d\begin{pmatrix} i_1,\dots,i_{n-1},j,k \\ i_1,\dots,i_{n-1},j,k \end{pmatrix}. \end{aligned} \tag{5.8}$$

Thus, (5.7) equals (5.5). This completes the proof of equality (5.2b).

To prove equality (5.3), we need to extend the Jacobi formula (5.8). We remark that, although the matrix elements d_{il} here are given by (4.8), both sides of (5.8) are equal as a polynomial for general elements d_{il}. Therefore, keeping the validity of (5.8), we can replace the columns with column index k by columns consisting of vectors $\boldsymbol{u}_{i_1},\dots,\boldsymbol{u}_{i_{n-1}},\boldsymbol{u}_j$ or $\boldsymbol{u}_k$:

$$d\begin{pmatrix} i_1,\dots,i_{n-1},j \\ i_1,\dots,i_{n-1},j \end{pmatrix} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{k i_1} & \cdots & d_{k i_{n-1}} & \boldsymbol{u}_k \end{vmatrix} - d\begin{pmatrix} i_1,\dots,i_{n-1},k \\ i_1,\dots,i_{n-1},j \end{pmatrix}$$

$$
\times \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & \boldsymbol{u}_j \end{vmatrix} = d \begin{pmatrix} i_1, \dots, i_{n-1} \\ i_1, \dots, i_{n-1} \end{pmatrix} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & d_{i_1 j} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & d_{i_{n-1} j} & \boldsymbol{u}_{i_{n-1}} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & d_{jj} & \boldsymbol{u}_j \\ d_{k i_1} & \cdots & d_{k i_{n-1}} & d_{kj} & \boldsymbol{u}_k \end{vmatrix}. \tag{5.9}
$$

We rewrite the right-hand side of (5.3) using (4.14), (4.15) and (5.6) (with $j \leftrightarrow k$) as

$$
\begin{aligned}
& \mathrm{e}^{\phi_{\{i_1,\dots,i_{n-1},j\},k} - \phi_{\{i_1,\dots,i_{n-1}\},k}} \frac{\zeta_j^* - \zeta_k^*}{\zeta_j - \zeta_k^*} \Big\{ \boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},k} - \frac{\zeta_j - \zeta_j^*}{\zeta_j - \zeta_k^*} \Big(\boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},k} \cdot \boldsymbol{u}^\dagger_{\{i_1,\dots,i_{n-1}\},j} \Big) \\
& \qquad\qquad \times \boldsymbol{u}_{\{i_1,\dots,i_{n-1}\},j} \Big\} \\
&= \mathrm{e}^{\phi_{\{i_1,\dots,i_{n-1},j\},k}} \times \frac{\zeta_j^* - \zeta_k^*}{\zeta_j - \zeta_k^*} \prod_{l=1}^{n-1} \frac{\zeta_{i_l}^* - \zeta_k^*}{\zeta_{i_l} - \zeta_k^*} \times \frac{1}{d \begin{pmatrix} i_1, \dots, i_{n-1}, j \\ i_1, \dots, i_{n-1}, j \end{pmatrix} d \begin{pmatrix} i_1, \dots, i_{n-1} \\ i_1, \dots, i_{n-1} \end{pmatrix}} \\
& \quad \times \left\{ d \begin{pmatrix} i_1, \dots, i_{n-1}, j \\ i_1, \dots, i_{n-1}, j \end{pmatrix} \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{k i_1} & \cdots & d_{k i_{n-1}} & \boldsymbol{u}_k \end{vmatrix} - d \begin{pmatrix} i_1, \dots, i_{n-1}, k \\ i_1, \dots, i_{n-1}, j \end{pmatrix} \right. \\
& \qquad \left. \times \begin{vmatrix} d_{i_1 i_1} & \cdots & d_{i_1 i_{n-1}} & \boldsymbol{u}_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ d_{i_{n-1} i_1} & \cdots & d_{i_{n-1} i_{n-1}} & \boldsymbol{u}_{i_{n-1}} \\ d_{j i_1} & \cdots & d_{j i_{n-1}} & \boldsymbol{u}_j \end{vmatrix} \right\}.
\end{aligned}
$$

Owing to the extended Jacobi formula (5.9), this equals $\boldsymbol{u}_{\{i_1,\dots,i_{n-1},j\},k}$ (cf. the definition (4.15)). Now the proof of equality (5.3) is complete. The proof of equality (5.4) is accomplished by interchanging j and k in the proof of (5.3). $\square$

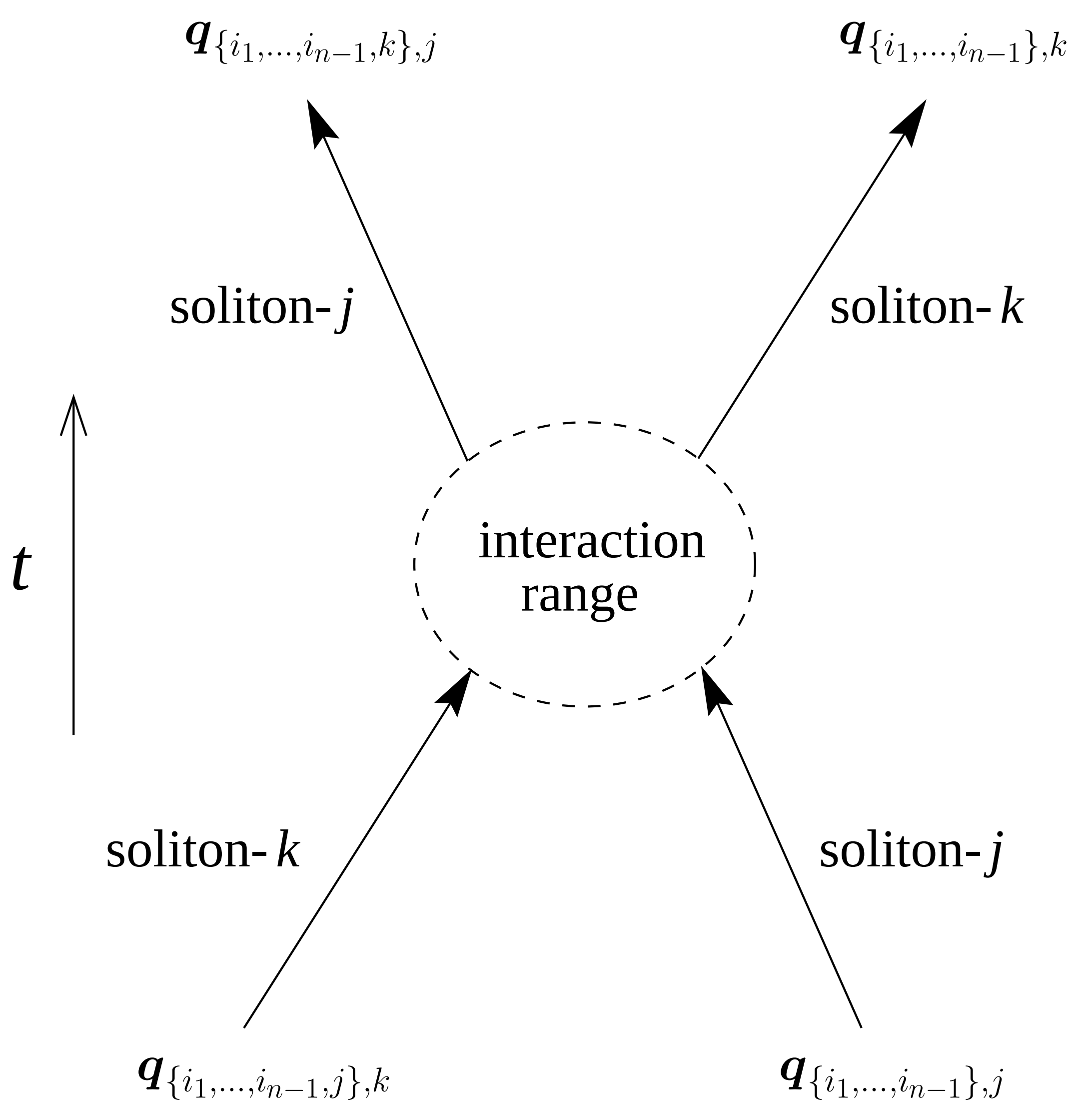

Figure 3: Two-soliton collision in the presence of other solitons

Proposition 5.3 is applicable to two-soliton collisions satisfying the following condition:

- Before the collision, the subscript set in the left-hand soliton's { } equals the whole subscript set of the right-hand soliton: $\{i_1, \ldots, i_{n-1}, j\} = \{i_1, \ldots, i_{n-1}\} \cup \{j\}$.

We note the following properties (cf. Fig. 3):

- After the collision, the subscript set in the left-hand soliton's { } still equals the whole subscript set of the right-hand soliton: $\{i_1, \ldots, i_{n-1}, k\} = \{i_1, \ldots, i_{n-1}\} \cup \{k\}$.
- The whole subscript set of the left-hand soliton is unchanged: $\{i_1, \ldots, i_{n-1}, j\} \cup \{k\} = \{i_1, \ldots, i_{n-1}, k\} \cup \{j\}$.
- The subscript set in the right-hand soliton's { } is unchanged: $\{i_1, \ldots, i_{n-1}\} = \{i_1, \ldots, i_{n-1}\}$.
- The overtaken soliton's number is removed from the overtaking soliton's { }, while the overtaking soliton's number is added to the overtaken soliton's { }.

We are now able to obtain the main result of this paper.

Theorem 5.4. *An N-soliton collision in the Manakov model (1.1) is factorized into nonlinear superposition of $\binom{N}{2}$ pair collisions in arbitrary order.*

Proof. According to the asymptotic behavior of the N-soliton solution as $t \to -\infty$ (see Fig. 2), solitons-N, ..., 1 are initially distributed along the x-axis as

$$\boldsymbol{q}_{\{1,\ldots,N-1\},N}, \ldots, \boldsymbol{q}_{\{1,\ldots,n-1\},n}, \ldots, \boldsymbol{q}_{\{\ \},1}.$$

We take this initial state as a point of departure and assume that the solitons collide pairwise in a given order. Then, a pair collision takes place

$$\binom{N}{2} = \frac{N(N-1)}{2} \text{ times.}$$

What will the final state be under this assumption? We note the following two points:

- The subscript set in each soliton's { } always equals the whole subscript set of the next soliton to the right. This ensures that Proposition 5.3 is applicable to every pair collision.
- Soliton-n will overtake solitons-1, ..., $n-1$ and will be overtaken by solitons-$n+1$, ..., N.

We see that, regardless of the given order of pair collisions, solitons-1, ..., N are finally distributed along the x-axis as

$$\boldsymbol{q}_{\{2,\ldots,N\},1}, \ldots, \boldsymbol{q}_{\{n+1,\ldots,N\},n}, \ldots, \boldsymbol{q}_{\{\,\},N}.$$

This final state is exactly the same as the asymptotic behavior of the N-soliton solution as $t \to +\infty$ (see Fig. 2).

Q.E.D.

6 Concluding remarks

In this paper, we have investigated soliton collisions in the Manakov model with general m components (1.1) by a straightforward approach. We first derived the general N-soliton solution of the Manakov model from that of the matrix NLS equation (2.2) through a simple reduction [12]. We considered the limits $t \to \mp\infty$ for the $N = 2$ case and obtained the collision laws of two solitons in the Manakov model. Next, we considered the same limits for the general N case and obtained the asymptotic behavior of the N-soliton solution. We could diagram the asymptotic behavior in a simplest way in terms of the abbreviation $\boldsymbol{q}_{\{i_1,\ldots,i_{n-1}\},i_n}$ defined by (4.16) (see Fig. 2). Taking advantage of this, we proved with a simple combinatorial discussion that an N-soliton collision in the Manakov model is factorized into nonlinear superposition of $\binom{N}{2}$ pair collisions in arbitrary order.

This result is far from being trivial in the $m \geq 2$ case. In the $m = 1$ case (scalar NLS), all the soliton parameters which play an essential role in the collision laws (in the notation of this paper, $\zeta_1, \zeta_2, \ldots, \zeta_N$) are invariant in time. A pair collision results only in a displacement of the soliton centers and a shift of the phases, which will not change the effects of future pair collisions. Thus, superposition of $\binom{N}{2}$ pair collisions gives the same results for every order of pair collisions. It is not difficult to prove in this case that an N-soliton collision reduces to a pair collision [8, 9, 10]. In contrast, in the $m \geq 2$ case, a pair collision results in a change of the polarization vectors, which will change the effects of future pair collisions completely. Therefore, it was not obvious until this work that nonlinear superposition of $\binom{N}{2}$ pair collisions gives the same results for every order of pair collisions or that it exactly coincides with an N-soliton collision. The key to proving these is a highly nontrivial relation among determinants and extended determinants, Lemma 5.1. This implies a possibility that some new relations similar to Lemma 5.1 are obtained through investigating soliton collisions in multi-component integrable systems.

Finally, we make some comments on the literature. Multi-soliton solutions of the Manakov model have already been obtained through the Hirota method [14, 15, 16] (see also [17, 18] for results by another method). In this respect, although it is very useful, the explicit formula for the N-soliton solution (2.9) with (2.10) and (2.11) may not be essentially new. The main contribution of this work lies in an elucidation of the pairwise nature of soliton collisions in the Manakov model. This removes the longtime misunderstanding that multi-particle effects exist in the Manakov model. The results of this paper were obtained by the author in the summer of 2000 and presented at the autumn meeting of the Physical Society of Japan in the year. Very recently, he encountered some papers [19, 20, 21] the authors of which posed the same problem (but did not solve it completely).

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