On Clock Offset Estimation in Wireless Sensor Networks with Weibull Distributed Network Delays

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Abstract—We consider the problem of Maximum Likelihood (ML) estimation of clock parameters in a two-way timing exchange scenario where the random delays assume a Weibull distribution, which represents a more generalized model. The ML estimate of the clock offset for the case of exponential distribution was obtained earlier. Moreover, it was reported that when the fixed delay is known, MLE is not unique. We determine the uniformly minimum variance unbiased (UMVU) estimators for exponential distribution under such a scenario and produce biased estimators having lower MSE than UMVU for all values of clock offset. We then consider the case when shape parameter is greater than one and reduce the corresponding optimization problems to their equivalent convex forms, thus guaranteeing convergence to a global minimum.

Keywords- Sensor networks, clock synchronization, estimation.

I. INTRODUCTION

A typical wireless sensor network (WSN) consists of a large number of geographically distributed sensors, deployed to observe some phenomenon of interest and report their observations to a distant fusion center. Wireless sensor networks are finding increasing applications in diverse areas such as environmental monitoring, battlefield surveillance, industrial process monitoring, target localization and tracking, etc. [1]. Much of the current research has focussed on power efficient transmission and finding scalable algorithms that can deliver certain performance guarantees while minimizing the overall energy utilization. Applications such as object tracking, efficient duty cycling, etc. require the sensors clocks to be sufficiently synchronized so that node lifetime is boosted by minimizing power utilization during sleep mode. Therefore, efficient and robust time synchronization between nodes becomes critically important.

Clock synchronization has been a topic of interest for the past several years. Chaudhari et.al. [2] presented algorithms for joint ML estimation of clock offset and skew under the exponential delay model in a two-way timing regime. Several estimators were proposed in [3] assuming an exponential distribution for the random link delays and it was stated that MLE of clock offset \( \theta \) is not unique. However, it was reported in [4] that the MLE does exist uniquely for the case when fixed delay \( d \) is unknown irrespective of whether the exponential delay parameter \( \alpha \) is known or unknown. Much of the work produced thus far, has been limited to the case of exponential distribution. The Weibull distribution, that includes several other distributions as special cases, has been found to capture the network delays efficiently and presents an interesting scenario to study clock synchronization.

In this paper, we estimate the clock offset in a two-way timing message exchange mechanism under a Weibull network delay distribution. We assume that the shape parameter \( k \) is known and solve the corresponding ML estimation problem by reducing it to an equivalent convex form. It was reported in [4] that when the fixed delay \( d \) is known, the MLE of \( \theta \) is not unique for an exponential delay distribution \((k=1)\). Under such a setting, we first determine the UMVU estimator among all possible solutions. Since a better performance index of an estimator is its MSE, we obtain biased estimator having lower MSE subject to the given constraints for all values of \( \theta \) using minimax technique. The same estimator also suffices for the case when \( d \) is unknown. Then, for the general case of shape parameter \( k > 1 \), we reduce the ML estimation problem to its analogous convex form, thus guaranteeing convergence to global optimum, irrespective of initialization.

II. SYSTEM MODEL

We consider a two-way message exchange mechanism in a wireless sensor network where the fixed link delay is symmetric and the random delays in each direction assume a Weibull distribution whose shape parameter is known. In such a scenario, the clock offset model can be expressed as [4]

\[
U_j = d + \theta + X_j,
\]

\[
V_j = d - \theta + Y_j, \quad j = 1, ..., N
\]

where \( d \) denotes the fixed link delay and \( \theta \) is the clock offset. \( \{X_j\}_{j=1}^{N} \) and \( \{Y_j\}_{j=1}^{N} \) are i.i.d. Weibull random variables signifying the random delays in the two paths and are distributed as \( f_X(t) = f_Y(t) = \frac{\theta}{\lambda} \left( \frac{t}{\lambda} \right)^{k-1} \exp \left( -\frac{t}{\lambda} \right)^k u(t) \) where \( u(t) \) is the unit step function. The shape parameter \( k > 0 \) is known and the scale parameter \( \lambda > 0 \) is unknown.
Our task is to estimate $d$ and $\theta$ based on the i.i.d observations $\{U_j\}_{j=1}^N$ and $\{V_j\}_{j=1}^N$.

The likelihood function based on the independent data sets $\{U_j\}_{j=1}^N$ and $\{V_j\}_{j=1}^N$ can be written as

$$L(d, \theta, \lambda) = \left(\frac{k}{\lambda}\right)^{2N} \prod_{j=1}^N \frac{(U_j - d - \theta)}{\lambda}^{k-1} \left(\frac{V_j - d + \theta}{\lambda}\right)^{k-1} \exp\left\{-\frac{k}{\lambda} \sum_{j=1}^N (U_j + V_j - 2d)\right\}$$

subject to $U_j - d - \theta \geq 0$ and $V_j - d + \theta \geq 0$ for $j = 1, ..., N$. Notice that the scale parameter $\lambda$ is independent of the constraints and hence, the likelihood can be maximized by determining the maximizing value of $\lambda$ for all fixed values of $d$ and $\theta$. The maximum likelihood estimate, $\lambda_{ML}$, of the scale parameter $\lambda$ is given by $\lambda_{ML} = \frac{N}{N} \sum_{j=1}^N (U_j + V_j - 2d)$.

The reduced likelihood function, thus, can be expressed as

$$L_R = C - 2Nk \log\left(\frac{\sum_{j=1}^N (U_j + V_j - 2d)}{2N}\right) + (k-1) \sum_{j=1}^N \log\left[(U_j - d - \theta)(V_j - d + \theta)\right]$$

subject to $U_{(1)} - d - \theta \geq 0$ and $V_{(1)} - d + \theta \geq 0$. $C = 2 N \log k - 2Nk$ is a known constant and $U_{(1)}$ and $V_{(1)}$ denote the first order statistics of $\{U_j\}_{j=1}^N$ and $\{V_j\}_{j=1}^N$, respectively. We distinguish two cases based on the value of the shape parameter $k$.

A. Exponential Distribution ($k = 1$)

The Weibull distribution reduces to an exponential distribution when $k = 1$. In the exponential case, the ML estimates of $d$ and $\theta$ were reported in [4]. While corroborating those results, we seek estimators having lower MSE than the ML estimate. Two subcases can be distinguished based on whether $d$ is known.

2.1.1. Known $d$: When the scale parameter $k = 1$, the reduced likelihood equation in (2) becomes

$$L_R = -2N - 2N \log\left(\frac{\sum_{j=1}^N (U_j + V_j - 2d)}{2N}\right) \cdot u(U_{(1)} - d - \theta) \cdot u(V_{(1)} - d + \theta)$$

The optimization problem can, hence, be stated as

$$\hat{\theta} = \arg \max_{\theta} L_R,$$

s.t. $U_{(1)} - d - \theta \geq 0$, $V_{(1)} - d + \theta \geq 0$. (4)

Clearly, there are infinite solutions to the MLE problem that satisfy the convex constraints. Firstly, we determine a UMVU estimator, $\hat{\theta}_{UMVU}$, of $\theta$.

Theorem 1: The UMVU estimator of $\theta$ subject to the convex constraints in (4) is $\hat{\theta}_{UMVU} = \frac{U_{(1)} - V_{(1)}}{2}$ with variance $\frac{N^2}{2N^2}$.

Proof: Consider the vector $T \triangleq [U_{(1)}, V_{(1)}]'$. The reduced likelihood function in (3), ignoring the constant, can be expressed as

$$L_R = -2N \log\left(\frac{\sum_{j=1}^N (U_j + V_j - 2d)}{2N}\right) \cdot u(U_{(1)} - d - \theta) \cdot u(V_{(1)} - d + \theta).$$

where $g_1(U_j, V_j)$ is solely a function of data and $g_2(T, \theta)$ and $g_3(T, \theta)$ are functions of $\theta$ that depend on the data only through $T$. Hence using Neyman Fisher Factorization Theorem [5], $T$ is a sufficient statistic for estimating $\theta$. The pdf of $T$ is given as

$$f(T; \theta) = \frac{N^2}{\lambda^2} \cdot e^{-\frac{N}{\lambda}(U_{(1)} + V_{(1)} - 2d)} \cdot u(U_{(1)} - d - \theta) \cdot u(V_{(1)} - d + \theta).$$

It can be shown that there is only one unbiased function of $T$ for estimating $\theta$. Hence, $T$ is a complete sufficient statistic for estimating $\theta$. An unbiased estimator of $\theta$ as a function of $T$ is $\hat{\theta}(T, \theta) = \frac{U_{(1)} - V_{(1)}}{2}$ and using Rao-Blackwell-Lehmann-Scheffe’ theorem, it is the UMVU estimator. We have that $\text{var}(U_{(1)}) = \text{var}(V_{(1)}) = \frac{N^2}{2N^2}$. Therefore, $\text{var}(\hat{\theta}_{UMVU}) = \frac{N^2}{2N^2}$.

Since a better performance index for an estimator is MSE, a problem of interest is to determine an estimator that has lower MSE than the UMVU estimator. Herein, we restrict ourselves to an estimator $\theta'$ obtained as a linear transformation of $\hat{\theta}_{UMVU}$ i.e., $\theta' = (1 + B)\hat{\theta}_{UMVU}$ where $B$ is a constant whose value is to be chosen so that $MSE_{\theta'} < MSE_{\hat{\theta}_{UMVU}}$.

The MSE of an estimator $\theta'$ is given by

$$MSE_{\theta'} = \text{var}(\theta') + b^2(\theta').$$

where $b(\theta') = E[\theta'] - \theta$ is the bias of the estimator. For $\theta' = (1 + B)\hat{\theta}_{UMVU}$, MSE can be expressed as

$$MSE_{\theta'} = (1 + B)^2\text{var}(\hat{\theta}_{UMVU}) + B^2\theta'^2.$$ (6)

For all practical purposes, we can assume here that it is known a-priori that $|\theta| \leq Q$. As our goal is to reduce the MSE as much as possible, we can choose $B$ that produces the most negative value of the difference of MSE’s of estimators $\theta'$ and $\hat{\theta}_{UMVU}$ [6]. In particular we seek a solution to the minimax problem

$$\hat{B} = \arg \min_B \max_{|\theta| \leq Q} (MSE_{\theta'} - MSE_{\hat{\theta}_{UMVU}}).$$ (7)
Theorem 2: The minimax problem (7) has a solution \( \hat{B} = \frac{\sqrt{\gamma}}{Q^2 + \gamma} \), where \( \gamma = \frac{\lambda}{2N} \). Moreover, the estimator \( \hat{\theta}' = (1 + B)\hat{\theta}_{UMVU} \) has a lower MSE than \( \hat{\theta}_{UMVU} \forall \theta \leq Q \).

Proof: Using (6), the minimax problem in (7) can be stated as

\[
\hat{B} = \arg \min_B \max_{|\theta| \leq Q} \left((1 + B)^2 \gamma + B^2 \theta^2 - \gamma\right),
\]

which is monotonically increasing in \( |\theta| \). Hence, (8) is maximized over \( \theta \) by choosing \( |\theta| = Q \). The problem, therefore, reduces to

\[
\hat{B} = \arg \min_B \left((1 + B)^2 \gamma + B^2 Q^2\right),
\]

which being a quadratic function in \( B \) is easily minimized to yield

\[
\hat{B} = \frac{-\gamma}{Q^2 + \gamma}.
\]

The resulting estimator \( \hat{\theta}' = \left(\frac{Q^2}{Q^2 + \gamma}\right)^2 \hat{\theta}_{UMVU} \) has variance \( \left(\frac{Q^2}{Q^2 + \gamma}\right)^2 \gamma \). Hence, it follows that \( \hat{\theta}' \) has a lower MSE than \( \hat{\theta}_{UMVU} \forall \theta \leq Q \). \( \square \)

2.1.2. Unknown \( d \): For the case of unknown \( d \), the ML estimation problem, using (3), can be equivalently stated as

\[
\min_{d, \theta} \sum_{j=1}^{N} (U_j + V_j - 2d) \quad \text{s.t. } U_{(1)} - d - \theta \geq 0, \quad V_{(1)} - d + \theta \geq 0.
\] (9)

Since the objective and the constraint functions are all affine, (9) is a linear program (LP) [7]. The Lagrangian can be expressed as

\[
L = \sum_{j=1}^{N} (U_j + V_j - 2d) + \mu_1 (d + \theta - U_{(1)}) + \mu_2 (d - \theta - V_{(1)}).
\]

The KKT conditions are

\[ -2N + \mu_1 + \mu_2 = 0, \quad \mu_1 - \mu_2 = 0. \]

\[ \mu_1 (d + \theta - U_{(1)}) = 0, \quad \mu_2 (d - \theta - V_{(1)}) = 0. \]

A simple solution to the above system yields

\[
\left[ \hat{d}_{ML}, \hat{\theta}_{ML} \right] = \left[ \frac{U_{(1)} + V_{(1)}}{2}, \frac{U_{(1)} - V_{(1)}}{2} \right].
\]

Using Theorem 1, \( \mathbf{T} = \left[U_{(1)} \ V_{(1)}\right]' \) is again a complete sufficient statistic for estimating \( d \) and \( \theta \). The UMVU estimators are, therefore,

\[
\left[ \hat{d}_{UMVU}, \hat{\theta}_{UMVU} \right] = \left[ \frac{U_{(1)} + V_{(1)}}{2}, \frac{U_{(1)} - V_{(1)}}{2} \right].
\]

Hence, the UMVU estimator of \( \theta \) for the case of unknown \( d \) is given by the same expression as for known \( d \). Therefore, the estimator \( \hat{\theta}' \) obtained through Theorem 2 has a lower MSE than \( \hat{\theta}_{UMVU} \forall \theta \leq Q \).

B. \( k > 1 \)

Herein, we restrict ourselves to the case where the shape parameter \( k > 1 \). The problem of finding the ML estimates of \( d \) and \( \theta \) has remained largely unexplored thus far. We frame an equivalent convex form by distinguishing two cases based on whether \( d \) is known.

2.2.1. Known \( d \): When \( d \) is known, using (2), the ML estimation problem can be equivalently stated as

\[
\hat{\theta}_{ML} = \arg \min_{\theta} -(k - 1) \sum_{j=1}^{N} \log (U_j - d - \theta) - (k - 1) \sum_{j=1}^{N} \log (V_j - d + \theta) \quad \text{s.t. } U_{(1)} - d - \theta \geq 0, \quad V_{(1)} - d + \theta \geq 0.
\] (10)

Theorem 3: The optimization problem (10) is convex.

Proof: Note that the constraints are inactive here. We know that \( \log(x) \) is concave on \( \mathbb{R}_+ \). Since \( k - 1 > 0 \) and summation preserves convexity, the proof follows.

2.2.2. Unknown \( d \): When \( d \) and \( \theta \) are unknown, the ML estimation problem can be equivalently stated using (2) as

\[
\min_{d, \theta} 2Nk \log \left( \frac{\sum_{j=1}^{N} (U_j + V_j - 2d)}{2N} \right) - (k - 1) \sum_{j=1}^{N} \left( \log (U_j - d - \theta) + \log (V_j - d + \theta) \right) \quad \text{s.t. } U_{(1)} - d - \theta \geq 0, \quad V_{(1)} - d + \theta \geq 0.
\] (11)

The optimization problem (11) is, essentially, non convex. The following theorem characterizes an equivalent convex form for (11).

Theorem 4: An equivalent convex form for (11) is expressed as

\[
\min_{x, y} \left( \frac{U + V}{2} + x + y \right)^{2Nk} + \sum_{j=1}^{N} [(U_j + x)(V_j + y)]^{1-k} \quad \text{s.t. } U_{(1)} + x \geq 0, \quad V_{(1)} + y \geq 0.
\] (12)

where \( x \overset{\Delta}{=} -(d + \theta), \ y \overset{\Delta}{=} -(d - \theta), \ U = \sum_{j=1}^{N} U_j \) and \( V = \sum_{j=1}^{N} V_j \).
Proof: Notice that with $x$ and $y$ defined as above, the objective function in (11) can be rewritten as

$$f(x, y) = \log \left( \frac{U + V}{2} + x + y \right)^{2Nk} + \sum_{j=1}^{N} \log \left[ (U_j + x)(V_j + y) \right]^{1-k}.$$ 

which is analogous to minimizing the exponentiated function

$$f'(x, y) = \left( \frac{U + V}{2} + x + y \right)^{2Nk} + \sum_{j=1}^{N} \left[ (U_j + x)(V_j + y) \right]^{1-k}.$$ 

It can be verified that each of the terms in the above expression is a convex function of $(x, y)$. Since summation preserves convexity, the proof immediately follows.

In general, (10) and (12) do not admit closed form solutions. Although, it is typically cumbersome to solve the ML estimation problem numerically, due to complicated search algorithms required for finding local minima, this is not a problem here. The established convexity of the optimization problem guarantees the convergence of Newton algorithm to the global minimum, regardless of initialization.

III. Simulations

We present some simulation results to evaluate the MSE performance of the estimators. Fig. 1 shows a comparison between the MSE's of UMVU estimator and the biased estimator for the case of exponential distribution. Here we take $\lambda = 5$ and $Q = 2$. Other parameters are $d = 2$ and $\theta = -1$. Clearly the resulting MSE is reduced when we form a biased estimator and this improvement is achieved when few observations are available, a challenge often encountered in WSN.

Fig. 2 shows the MSE performance of the ML estimator for the case of Rayleigh distribution ($k = 2$). Due to established convexity of the problem, the descent algorithm converges in few iterations and low MSE is achieved for small number of samples. Quite expectedly, the MSE performance is better when $d$ is known as compared to the case where both $d$ and $\theta$ are unknown.

IV. Conclusion

This paper considers a two-way timing exchange mechanism for estimating the clock offset assuming a Weibull distribution for the random link delays. When the shape parameter $k = 1$, estimators are proposed, using minimax technique, which have lower MSE than the corresponding UMVU estimators for all values of $\theta$ subject to the underlying constraints. For the general case of $k > 1$, equivalent convex forms are proposed, thus guaranteeing convergence to a global optimum.

REFERENCES


