Stability of Discrete-time Iterative Learning Control with Random Data Dropouts and Delayed Controlled Signals in Networked Control Systems

Hyo-Sung Ahn‡, Kevin L. Moore‡, and YangQuan Chen§

Abstract—This paper studies the robust stability of discrete-time iterative learning control (ILC) systems in a networked control system setting (NCS). First we consider the problem where data sent from a remote plant to the ILC algorithms is subject to random data dropouts. For this scenario we establish mean-square stability, taking into account the convergence of the covariance of the error along the iteration domain. Next, using this result, we derive a robust stability condition for a more general NCS-based ILC problem with data delays and dropouts in the control signals transmitted to the remote plant as well as data delays in output signals returned from the plant.

Index Terms—Iterative Learning Control, Data Delays, Data Dropouts, Mean-square stability

I. INTRODUCTION

In the field of iterative learning control (ILC) [1] approaches to robustness have included contraction mapping techniques for uncertain nonlinear systems [2], studies of initial state errors [3], ILC without knowledge of control direction [4], ways to handle time-varying uncertainties [5], stochastic ILC [6], and parametric interval variations [7]. Recently we have considered the problem of robustness with respect to intermittent output measurements. [8], [9], [7]. In [7] we showed that it can be useful to do Kalman filtering along the iteration domain. In [8] we considered the problem where data dropouts from a remote plant occurred with the same random variable applied to each component in the multivariable output vector of the plant. In [9] we considered the case where each component in the multivariable output vector of the plant is subject to an independent dropout. Key conclusions in each of these works were that the ILC system would still converge in the face of output data dropout as long as the dropout rate is not unity, that is, 100%.

In this paper we expand on the previous results presented in [8], [9], [7]. Our primary contribution here is to address robust mean-square stability of ILC systems when considering both delays and data dropouts. These results are more general and comprehensive than the results of [8], [9], [7] and thus can handle many general engineering problems such as networked control systems [10] and teleoperation control problems [11].

The paper is organized as follows. In the next section, we give two results. First we consider the problem where data sent from a remote plant to the ILC algorithms is subject to random data dropouts. For this scenario we establish mean-square stability, taking into account the convergence of the covariance of the error along the iteration domain. Next, using this result, we derive a robust stability condition for a more general NCS-based ILC problem with data delays and dropouts in the control signals transmitted to the remote plant as well as data delays in output signals returned from the plant. Conclusions will be given in Section III.

II. STABILITIES OF ILC WITH RANDOM DELAYS AND DATA DROPOUTS

In this section we present robust stability conditions for iterative learning control systems in a NCS scenario with random delays and data dropouts. In the following subsection, we first consider only the dropouts in output signals as they are transmitted from the remote plant via the network. For the stability analysis, we use the notion of mean-square stability, which is formally defined as follows:

Definition 2.1: If state vector $x(k)$ is convergent in the sense of $\|E(x(k)) - q\| \rightarrow 0$ as $k \rightarrow \infty$ and $\|E(x(k)x(k)^T) - Q\| \rightarrow 0$ as $k \rightarrow \infty$, where $q$ and $Q$ are fixed constant vector and matrix, respectively, then the system is said to be mean-square stable.

Throughout the paper, it is supposed that data packets are transferred as a data stream in a computer network or in tele-operated network environment, giving rise to the NCS ILC framework introduced in [8], [9], [7].

We assume that there is a remote discrete-time plant described by $y_k(t) = H(z)u_k(t)$ where $u_k(t)$ is the input and $y_k(t)$ is the output. $t$ is the discrete-time axis and $k$ is the iteration axis. It is supposed that we wish to operate the system in an ILC fashion, but with the update equation $u_{k+1}(t) = u_k(t) + \psi e_k(t+1)$ (assuming a relative degree one plant). Following standard ILC terminology, we can write this system using lifted supervectors as $Y_k = HU_k$ and $U_{k+1} = U_k + \Phi E_k$. Note that because we must transmit the signals $U_k$ and $Y_k$ over a network, the possibility exists that there will be dropped and/or delayed data during the transmission. While the delay effect is not critical, because in ILC we can wait in between iterations for all the data to arrive, the dropout gives more cause for concern.

As a final note about terminology, in the sequel we refer to data dropouts in the “error signals.” In fact, the data dropout occurs in the output signal when it is sent to the ILC algorithm from the remote plant. However, as assumed in [8], [9], [7], we suppose that the output signals (i.e., $y(t)$) are encoded so it can be known whether the output signals have been delivered or lost during the
data flow. If it is known that the output signal is lost or delayed during transfer, we can ascribe that same loss or delay to the error signal as well, because there is no need to explicitly consider an error when there is no output available. That is, suppose that the delivered output can be described by \( y_k^{\text{delivered}}(t) = \eta y_k(t) \), where \( \eta \) is a random variable satisfying \( \eta \in \{0, 1\} \). Then we refer to the error as \( e_k(t) = \eta(y_d(t) - y_k(t)) \) rather than \( e_k(t) = y_d(t) - y_k^{\text{delivered}}(t) = y_d(t) - \eta y_k(t) \).

### A. Random Data Dropouts in Error Signals

Using super-vector framework, we formulate the ILC update rule as:

\[
U_{k+1} = U_k + \Psi \mathcal{N}_k E_k \\
Y_k = H U_k
\]

(1)

(2)

where \( \Psi \) is the control gain matrix and \( \mathcal{N}_k \in \mathbb{R}^{N \times N} \) is a matrix representing the effects of data dropouts during the network transfer. For example, if there are data dropouts in \( e_k(t) \) given by \( e_k(t) = \eta(t) e_k(t) \), where \( \eta(t) \in \{0, 1\} \), then \( \mathcal{N}_k \) can be chosen as:

\[
\mathcal{N}_k = \begin{pmatrix}
\eta(1) & 0 & \cdots & 0 \\
0 & \eta(2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \eta(N)
\end{pmatrix}
\]

(3)

The matrix \( \mathcal{N}_k \) can represent data dropouts of error signals, but it can also represent other phenomena such as data jitters, data delays, and nonuniform sampling, etc. by adding off-diagonal elements into \( \mathcal{N}_k \). For instance, if \( \mathcal{N}_k(2, 1) = \eta(1) \), then it can represent data delays. For example, let us say that \( \eta(2) = 0 \); then \( e_k(2) = \eta(1)e_k(1) = e_k(1) \) if \( \eta(1) = 1 \); thus there exists one sampling delay. Thus the matrix \( \mathcal{N}_k \) is able to represent random delays as well as random data dropouts if \( \eta(i) \) are random variables. They can be represent of more general problems and thus we consider a general random matrix \( \mathcal{N}_k \) throughout the remainder of the paper.

For the sake of derivation, it is assumed that the control gain matrix \( \Psi = \Upsilon \Gamma \), with \( \Gamma = \text{diag}(\gamma_i) \); thus the random matrix \( \mathcal{N}_k \) is commutative in the following way: \( \Psi \mathcal{N}_k = \Upsilon \Gamma \mathcal{N}_k = \mathcal{N}_k \Gamma \) where \( \Psi = \Upsilon \Gamma \Gamma \).\(^1\) Thus we can change (1) and (2) to be:

\[
U_{k+1} = U_k + \Upsilon \mathcal{N}_k \Gamma E_k \\
Y_k = H U_k
\]

(4)

(5)

Now, with no loss of generality, set \( H = H^{-1} \). We are able to obtain:

\[
E_{k+1} = (I + \mathcal{N}_k \Gamma) E_k
\]

(6)

where \( E_k = Y_k - Y_d \). This last expression will be the basis for the development to follow. We note that in the following we assume that \( \Gamma = [g_{ij}] \) is fully-populated in the interest of generality. This will not be the case in a typical ILC problem, where \( \Gamma \) will in fact be diagonal. But, the results of the development to follow are more general than just for ILC and can be readily reduced to the special case of \( \Gamma \) diagonal.

To continue, write \( \mathcal{N}_k \) such as \( \mathcal{N}_k = \overline{\mathcal{N}} + \tilde{\mathcal{N}}_k \) where \( \overline{\mathcal{N}} \) is a deterministic part and \( \tilde{\mathcal{N}}_k \) is a zero-mean random part. The expectation of \( E_{k+1} \) is then evaluated by:

\[
\mathbf{E}[E_{k+1} E_k^T] = \mathbf{E}[(I + \mathcal{N}_k \Gamma) \mathbf{E}_k \mathbf{E}_k^T (I + \mathcal{N}_k \Gamma)^T]
\]

\[
= \mathbf{E}[(I + \Upsilon \Gamma + \tilde{\mathcal{N}}_k \Gamma) \mathbf{E}_k \mathbf{E}_k^T (I + \Upsilon \Gamma + \tilde{\mathcal{N}}_k \Gamma)^T]
\]

\[
= \mathbf{E}[(I + \Upsilon \Gamma) \mathbf{E}_k \mathbf{E}_k^T (I + \Upsilon \Gamma)^T] + \mathbf{E}[(\tilde{\mathcal{N}}_k \Gamma) \mathbf{E}_k \mathbf{E}_k^T (\tilde{\mathcal{N}}_k \Gamma)^T].
\]

(7)

Throughout the paper we use the notations \( \Gamma = [g_{ij}], \mathcal{N}_k = [n_{ij}], \) and \( \mathbf{E}_k = [e_1, e_2, \ldots, e_n]^T \). Since

\[
\Gamma \mathbf{E}_k = \begin{pmatrix}
g_{11}e_1 + g_{12}e_2 + g_{13}e_3 + \cdots + g_{1n}e_n \\
g_{n1}e_1 + g_{n2}e_2 + g_{n3}e_3 + \cdots + g_{nn}e_n \\
\vdots \\
\sum_{j=1}^n g_{ij}e_j
\end{pmatrix}
\]

(8)

if we define \( (\Gamma \mathbf{E}_k)_i = \sum_{j=1}^n g_{ij}e_j \), we are able to have (see Appendix-1 for the detailed derivation of the following equation)

\[
\mathbf{E}[(\mathcal{N}_k \Gamma) \mathbf{E}_k \mathbf{E}_k^T (\mathcal{N}_k \Gamma)^T]_{uu} = \mathbf{E} \left[ \sum_{k=1}^n \sum_{j=1}^n \sum_{k'=1}^n \sum_{j'=1}^n n_{uk}g_{kj}g_{kj'}e_je_{j'} \right].
\]

(9)

Noticing \( n_{ij} \) are random parameters, when \( n_{uk} \neq n_{ek}, \mathbf{E}[n_{uk}n_{ek}] = 0. \) Thus we conclude that \( \mathbf{E}[n_{uk}n_{ek}] \neq 0 \) only if \( u = v \) and \( k = k' \). Therefore \( \mathbf{E}[(\mathcal{N}_k \Gamma) \mathbf{E}_k \mathbf{E}_k^T (\mathcal{N}_k \Gamma)^T]_{uu} = 0 \) if \( u \neq v \). When \( u = v \) and \( k = k' \), we change

\[
\sum_{k=1}^n \sum_{j=1}^n \sum_{k'=1}^n \sum_{j'=1}^n n_{uk}g_{kj}g_{kj'}e_je_{j'}
\]

such as \( \sum_{k'=1}^n \sum_{j'=1}^n \sum_{j=1}^n n_{uk}g_{kj}g_{kj'}e_je_{j'} = \sum_{k=1}^n \sum_{j'=1}^n \sum_{j=1}^n n_{uk}g_{kj}g_{kj'}e_je_{j'} \).

Thus

\[
\mathbf{E}[(\mathcal{N}_k \Gamma) \mathbf{E}_k \mathbf{E}_k^T (\mathcal{N}_k \Gamma)^T]_{uu} = \mathbf{E} \left[ \sum_{k=1}^n \sum_{j=1}^n \sum_{j'=1}^n (n_{uk})^2 g_{kj}g_{kj'}e_je_{j'} \right].
\]

(10)

Also since \( n_{ij} \) and \( e_i \) are independent variables, we are able to change the right-hand side of (10) such as:

\[
\mathbf{E} \left[ \sum_{k=1}^n \sum_{j=1}^n \sum_{j'=1}^n (n_{uk})^2 g_{kj}g_{kj'}e_je_{j'} \right] = \sum_{k=1}^n \sum_{j=1}^n \sum_{j'=1}^n \mathbf{E}[n_{uk}^2] g_{kj}g_{kj'} \mathbf{E}[e_je_{j'}].
\]

(11)
Defining covariance matrix of $E_k$ as $P_{jj'} = E[e_je_{j'}]$, when $u = i$, we change the right-hand side of (11) as follows:

$$
\sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{j'=1}^{n} E[(n_{uk})^2]g_{kj}g_{kj'}E[e_je_{j'}] \\
= E[n_{uk}^2][g_{11}g_{11}P_{11} + g_{11}g_{12}P_{12} + \cdots + g_{1j}g_{1j'}P_{jj'} \\
+ \cdots + g_{1n}g_{1n}P_{nn}] \\
+ E[n_{uk}^2][g_{21}g_{21}P_{11} + g_{21}g_{22}P_{12} + \cdots + g_{2j}g_{2j'}P_{jj'} \\
+ \cdots + g_{2n}g_{2n}P_{nn}] \\
+ E[n_{uk}^2][g_{31}g_{31}P_{11} + g_{31}g_{32}P_{12} + \cdots + g_{3j}g_{3j'}P_{jj'} \\
+ \cdots + g_{3n}g_{3n}P_{nn}] \\
+ \cdots \\
+ E[n_{uk}^2][g_{n1}g_{n1}P_{11} + g_{n1}g_{n2}P_{12} + \cdots \\
+ g_{nj}g_{nj'}P_{jj'} + \cdots + g_{nn}g_{nn}P_{nn}] \\
(12)
$$

If we define $\overline{g}_n = [g_{11}, g_{22}, \cdots, g_{nn}]^T$ and use $P = [P_{ij}]$, then we can further change the above equality such as:

$$
\sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{j'=1}^{n} E[(n_{uk})^2]g_{kj}g_{kj'}E[e_je_{j'}] \\
= E[n_{uk}^2][\overline{g}_n^T P \overline{g} + E[n_{uk}^2][\overline{g}_n^T P \overline{g}] \\
+ \cdots + E[n_{uk}^2][\overline{g}_n^T P \overline{g}] + E[n_{uk}^2][\overline{g}_n^T P \overline{g}] \\
(13)
$$

Therefore we can obtain (14) given in the next page. For simplicity of presentation, we use the following notations:

$$
W_i = \begin{bmatrix}
E[n_{uk}^2] & 0 & \cdots & 0 \\
0 & E[n_{uk}^2] & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & E[n_{uk}^2]
\end{bmatrix}
$$

$$
V_i = [0_{n \times 1} \cdots 0_{n \times 1} \overline{g}_j 0_{n \times 1} \cdots 0_{n \times 1}]
$$

where $V_i$ is the matrix whose $i$-th column is $\overline{g}_j$, but other elements are all zero. Then we are able to represent $E[(\overline{N}_k E_k E_k^T (\overline{N}_k E_k)^T)$ such as (15) given on the next page. Since $P = E[E_k E_k^T]$, using $P_k = E[E_k E_k^T]$ and $P_{k+1} = E[E_{k+1} E_{k+1}^T]$, now we can change (7) as follows:

$$
P_{k+1} = (I + \overline{N} \Gamma) P_k (I + \overline{N} \Gamma)^T + \\
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} [\sqrt{W_j} V_i V_i^T \sqrt{W_j}] P_k V_i V_i^T \sqrt{W_j} \right)
$$

(16)

Now using properties of Kronecker product and using Remark 4 of [12], we can state the following result:

**Proposition 2.1:** If the following inequality holds:

$$
\rho \left( (I + \overline{N} \Gamma) \otimes (I + \overline{N} \Gamma) \right) + \\
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} [\sqrt{W_j} V_i V_i^T \otimes V_i V_i^T \sqrt{W_j}] \right) < 1
$$

where $\rho$ means spectral radius and $\otimes$ is the Kronecker product, then the iterative learning control systems governed by (4) and (5) are mean-square stable, and the steady-state covariance of $P_{\infty}$, which is defined as $P_{\infty} = \lim_{k \to \infty} P_k = \lim_{k \to \infty} E(E_k E_k^T)$, exists and satisfies the following equality:

$$
P_{\infty} = (I + \overline{N} \Gamma) P_{\infty} (I + \overline{N} \Gamma)^T + \\
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} [\sqrt{W_j} V_i V_i^T \otimes V_i V_i^T \sqrt{W_j}] \right)
$$

(18)

**Remark 2.1:** If

$$
\left\| (I + \overline{N} \Gamma) \otimes (I + \overline{N} \Gamma) \\
+ \sum_{j=1}^{n} \left( \sum_{i=1}^{n} [\sqrt{W_j} V_i V_i^T \otimes V_i V_i^T \sqrt{W_j}] \right) \right\| < 1
$$

then the ILC systems will be monotonically convergent in the sense of $\left\| P_{k+1} \right\| < \left\| P_k \right\|$, where $\| P \|_s = \sqrt{\sum_{i,j=1}^{n} (P_{ij}^2)}$.

**B. Randomly Delayed Data Dropouts in Control Signals and Dropouts in Output Measurements**

Next we consider the case where there can be delays and dropouts in both the error signals and control signals, described by:

$$
U_{k+1} = U_k + \mathcal{G} \mathcal{N}_k \Gamma E_k \\
Y_k = H \mathcal{M}_k U_k
$$

(19)

(20)

where $\mathcal{M}_k$ is a random matrix representing control signal loss and delays during the network transfer from the controller to the plant. Figure 1 illustrates data delays and dropouts in the forward channel and the feedback channel. The systems considered in (19) and (20) represent a very general framework as noted in the following remark:

**Remark 2.2:** Let us consider the following linear systems:

$$
x_{k}(t+1) = Ax_k(t) + Bu_k(t) \\
y_k(t) = Cx_k(t)
$$

(21)

(22)

In the plant and/or in the network, the control signal $u_k(t)$ commonly can be delayed according to [13]:

$$
x_{k}(t+1) = Ax_k(t) + Bu_k(t - \tau_0) + Bu_k(t - \tau_1) + \cdots \\
y_k(t) = Cx_k(t)
$$

(23)

In our new framework outlined by (19) and (20), we can consider not only delays but also data dropouts such as:

$$
x_{k}(t+1) = Ax_k(t) + \sum_{i=0}^{m} B \eta_i u_k(t - \tau_i) \\
y_k(t) = Cx_k(t)
$$

(24)

(25)
\[
\mathbf{E}\left[(\bar{\mathbf{N}}_k \Gamma) E_k E_k^T (\bar{\mathbf{N}}_k \Gamma)^T\right] = \begin{bmatrix}
E[n_{11}^2] & 0 & \cdots & 0 \\
0 & E[n_{21}^2] & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & E[n_{n1}^2]
\end{bmatrix} + \begin{bmatrix}
g_1^T P g_1 \\
g_2^T P g_2 \\
\vdots \\
g_n^T P g_n
\end{bmatrix}
\]

\[
\mathbf{E}\left[(\bar{\mathbf{N}}_k \Gamma) E_k E_k^T (\bar{\mathbf{N}}_k \Gamma)^T\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sqrt{W_i(v_i^1)^T P V_i^1 \sqrt{W_i}} + \sqrt{W_1(v_2^1)^T P V_2^1 \sqrt{W_1}} + \cdots + \sqrt{W_1(v_n^1)^T P V_n^1 \sqrt{W_n}}\right]
\]

\[
\mathbf{E}\left[(\bar{\mathbf{N}}_k \Gamma) E_k E_k^T (\bar{\mathbf{N}}_k \Gamma)^T\right] = \sum_{j=1}^{n} \left[\sqrt{W_j(v_j^1)^T P V_j^1 \sqrt{W_j}} + \cdots + \sqrt{W_j(v_n^1)^T P V_n^1 \sqrt{W_n}}\right]
\]

where \(\eta_i \in \{0,1\}\) and \(\tau_i \in \mathbb{N}^+\).

Using \(Y_d = HU_d\), and defining \(E_k = Y_k - Y_d\), we obtain:

\[
E_{k+1} = H(M_{k+1} U_{k+1} - U_d) \quad (26)
\]

\[
E_k = H(M_k U_k - U_d) \quad (27)
\]

which yields

\[
E_{k+1} = E_k + H(M_{k+1} U_{k+1} - M_k U_k)
\]

\[
= E_k + H(M_{k+1} U_{k+1} - \Upsilon N_k \Gamma E_k) - M_k U_k
\]

\[
= (I + H M_{k+1} \Upsilon N_k \Gamma) E_k + H(M_{k+1} - M_k) U_k. \quad (28)
\]

Let \(\Delta U_k = U_d - U_k\). Then we can change the above equation as:

\[
E_{k+1} = (I + H M_{k+1} \Upsilon N_k \Gamma) E_k + H(M_{k+1} - M_k)(U_d - \Delta U_k) \quad (29)
\]

and from \(U_d - U_{k+1} = U_d - U_k - \Upsilon_N \Gamma E_k\), we obtain

\[
\Delta U_{k+1} = \Delta U_k - \Upsilon_N \Gamma E_k \quad (30)
\]

It is supposed that \((I + H M_{k+1} \Upsilon N_k \Gamma) E_k\) and \(H(M_{k+1} - M_k) U_k\) are independent each other because \(M_{k+1} - M_k\) is a random matrix with zero-mean elements (see Appendix-2). We can now present the following lemma.

**Lemma 2.1:** If \(\mathbf{E}[E_0] = 0\) and \(\mathbf{E}[\Delta U_0] = 0\), then \(\mathbf{E}[\Delta U_d \Delta U_d^T S^T] = \mathbf{E}[\Delta U_d \Delta U_d^T S^T] = 0\), where \(S = M_{k+1} - M_k\).

**Proof:** From (28), we have \(\mathbf{E}(E_{k+1}) = \mathbf{E}(I + H M_{k+1} \Upsilon N_k \Gamma) E_k + H(E(M_{k+1} - M_k) U_k\). Since

\[
\mathbf{E}(M_{k+1} - M_k) = 0, \quad \mathbf{E}(E_k) = 0\),

for all \(k\). Now from (30) \(\mathbf{E}(\Delta U_{k+1}) = \mathbf{E}(\Delta U_k) - \mathbf{E}(\Upsilon N_k \Gamma) E_k\), thus since \(\mathbf{E}(E_k) = 0\) for all \(k\), \(\mathbf{E}(\Delta U_k) = 0\) for all \(k\). Thus we can conclude that \(\mathbf{E}[\Delta U_d \Delta U_d^T S^T] = \mathbf{E}[\Delta U_d \Delta U_d^T S^T] = 0\).

If \(\Upsilon\) is not singular, then there exists a matrix \(Y\) such that \(M_{k+1} \Upsilon = YT\) (in fact \(Y\) can be uniquely
calculated as $Y = T^{-1}M_{k+1}Y$. In such case, since $I + H M_{k+1} Y N_k \Gamma = I + H T Y N_k \Gamma$, if $Y = T^{-1}$, we can change $I + H M_{k+1} Y N_k \Gamma = I + T^{-1} M_{k+1} Y N_k \Gamma = I + H M_{k+1} H^{-1} N_k \Gamma$. Now considering $H M_{k+1} H^{-1} N_k \Gamma$ as a random matrix such as $\mathcal{W} = H M_{k+1} H^{-1} N_k \Gamma$, we can separate $\mathcal{W}$ into a deterministic part and a zero-mean random part as $\mathcal{W} = \mathcal{W} + \mathcal{W}$ (see Appendix 3). Thus we can rewrite (28) as:

$$E_{k+1} = (I + \mathcal{W} \Gamma) E_k + \mathcal{W} \Gamma E_k + H (M_{k+1} - M_k) (U_0 - \Delta U_k) \quad (31)$$

Since the terms on the right-hand side of (31) are independent, we have

$$\mathbf{E}[E_{k+1} E_k^T] = (I + \mathcal{W} \Gamma) \mathbf{E}[E_k E_k^T] (I + \mathcal{W} \Gamma)^T$$

$$+ H \mathbf{E}[E_k (M_{k+1} - M_k) (U_0 - \Delta U_k)]$$

$$\times \mathbf{E}[(U_0 - \Delta U_k)^T (M_{k+1} - M_k)^T] H^T \quad (32)$$

Let us denote $\mathcal{S} = [s_{ij}]$ and $\bar{W}_k = [w_{ij}]$ and define the following matrix

$$Z_i = \begin{bmatrix} \mathbf{E}[w_{i1}^2] & 0 & \cdots & 0 \\ 0 & \mathbf{E}[w_{i2}^2] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{E}[w_{in}^2] \end{bmatrix} \quad (33)$$

Then we can use (34) as shown on the next page. To continue, we change $\mathbf{E}[\mathcal{S} \Delta U_k \Delta U_k^T S^T]_{uv}$ such as:

$$\mathbf{E}[\mathcal{S} \Delta U_k \Delta U_k^T S^T]_{uv} = \mathbf{E} \left[ \sum_{k=1}^{n} \sum_{k'=1}^{n} s_{uk} s_{uk'} \delta_{uk} \delta_{uk'} \right] \quad (35)$$

where $\Delta U_k = [\delta_1, \delta_2, \ldots, \delta_n]^T$. Since $\mathbf{E}[s_{uk} s_{uk'}] = 0$ if $s_{uk} \neq s_{uk'}$, we can further simplify the above as follows:

$$\mathbf{E}[\mathcal{S} \Delta U_k \Delta U_k^T S^T]_{uu} = \mathbf{E} \left[ \sum_{k=1}^{n} (s_{uk})^2 \delta_k \right]$$

$$\mathbf{E}[\mathcal{S} \Delta U_k \Delta U_k^T S^T]_{uv} = 0 \quad (u \neq v) \quad (36)$$

Thus when $u = i$, we have

$$\mathbf{E}[\mathcal{S} \Delta U_k \Delta U_k^T S^T]_{ii} = \mathbf{E}[s_{i1}]^2 \mathbf{E}[(\delta_1)^2] + \mathbf{E}[s_{i2}]^2 \mathbf{E}[(\delta_2)^2] + \cdots + \mathbf{E}[s_{in}]^2 \mathbf{E}[(\delta_n)^2] \quad (37)$$

Now denoting $X = \mathbf{E}[\Delta U_k \Delta U_k^T]$, we can generate the equality (38). In (38), $\sigma_i$ is the length $n$ column vector with 1 only at the $i$-th element (but other elements are zeros).

Likewise as above, defining

$$T_i = \begin{bmatrix} \mathbf{E}[s_{i1}^2] & 0 & \cdots & 0 \\ 0 & \mathbf{E}[s_{i2}^2] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{E}[s_{in}^2] \end{bmatrix}$$

$$O_i = \begin{bmatrix} 0_{n \times 1} & \cdots & 0_{n \times 1} \end{bmatrix}$$

we finally obtain:

$$\mathbf{E}[\mathcal{S} \Delta U_k \Delta U_k^T S^T] = \sum_{j=1}^{n} \left[ \sum_{i=0}^{n} \sqrt{\mathbf{T}_i} (O_i^j)^T X O_i^j \right] \mathbf{T}_j$$

which leads

$$P_{k+1} = (I + \mathcal{W} \Gamma) P_k (I + \mathcal{W} \Gamma)^T$$

$$+ \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \sqrt{Z_j} (V_i^j)^T P_k V_i^j \sqrt{Z_j} \right]$$

$$+ H \left( \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \sqrt{T_j} (O_i^j)^T X_k O_i^j \sqrt{T_j} \right] \right) H^T + K \quad (40)$$

where we used $X_k = \mathbf{E}[\Delta U_k \Delta U_k^T]$ and $K = H \mathbf{E} [SU_k U_k^T S^T] H^T$. Next let us change (30) such as:

$$\Delta U_{k+1} = \Delta U_k - H^{-1} \mathcal{N}_k \mathcal{G} E_k - H^{-1} \mathcal{N}_k \mathcal{G} E_k (41)$$

Then using the above equation, we arrive at the following lemma.

**Lemma 2.2:** If $\mathbf{E}(0) = 0$ and $\mathbf{E}(U_0) = 0$, then $\mathbf{E}(\Delta U_k E_k^T) = -X_k H^T$ and $\mathbf{E}(E_k E_k^T) = -H X_k$.

**Proof:** From $E_k = H (M_k U_k - U_0)$, we have $E_k = H (M_k (U_0 - \Delta U_k) - U_0) = H (M_k - I) U_0 - H \Delta U_k$.

Thus $\Delta U_k E_k^T = \Delta U_k (H (M_k - I) U_0 - H \Delta U_k)^T = \Delta U_k U_k^T (M_k - I)^T H^T - \Delta U_k U_k^T H^T$. Therefore we obtain $\mathbf{E}(\Delta U_k E_k^T) = \mathbf{E}(\Delta U_k U_k^T (M_k - I)^T H^T - \Delta U_k U_k^T H^T) = -X_k H^T$.

Now using the above lemma, we can obtain the following relationship:

$$X_{k+1} = X_k + X_k H^T H^{-1} \mathcal{N}_k \Gamma + H^{-1} \mathcal{N}_k \Gamma H X_k$$

$$- H^{-1} \mathcal{N}_k \Gamma P_k (H^{-1} \mathcal{N}_k \Gamma)^T$$

$$- H^{-1} \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \sqrt{W_j} (V_i^j)^T P_k V_i^j \sqrt{W_j} \right] (H^{-1})^T \quad (42)$$

The problem we are seeking at this point is to analyze convergence of the systems governed by (40) and (42), which is summarized in the following proposition.

**Proposition 2.2:** If the inequality given in (43) is true, then $\bar{P}_k \rightarrow P^*$ and $X_k \rightarrow X^*$ as $k \rightarrow \infty$. Furthermore, $P^*$ and $X^*$ are determined by $P_0$, $X_0$, and $K$.

**Proof:** Let us define a notation $\bar{f}(M) = [M_1^T, M_2^T, \cdots, M_n^T]^T$ where $M_i$ are column vectors of square matrix $M$. Then from (40) and (42), using the property of Kronecker product we have

$$\begin{bmatrix} \bar{f}(P_{k+1}) \\ \bar{f}(X_{k+1}) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{f}(P_k) \\ \bar{f}(X_k) \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{f}(K) \\ 0_{N \times 1} \end{bmatrix} \quad (44)$$
\[ P_{k+1} = (I + \mathbb{W} \mathbb{T}) P_k (I + \mathbb{W} \mathbb{T})^T + \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \sqrt{Z_j(V^j_i)^T} P_k V^j_i \sqrt{Z_j} \right] + H E \left[ S(U_d - \Delta U_k)(U_d - \Delta U_k)^T S^T \right] H^T \]

\[ = (I + \mathbb{W} \mathbb{T}) P_k (I + \mathbb{W} \mathbb{T})^T + \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \sqrt{Z_j(V^j_i)^T} P_k V^j_i \sqrt{Z_j} \right] + H E \left[ S(U_d U_d^T S) \right] H^T \]

\[ - H E \left[ S\Delta U_k U_k^T S \right] H^T - H E \left[ S\Delta U_k U_k^T S \right] H^T + H E \left[ S\Delta U_k U_k^T S \right] H^T \]

\[ = (I + \mathbb{W} \mathbb{T}) P_k (I + \mathbb{W} \mathbb{T})^T + \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \sqrt{Z_j(V^j_i)^T} P_k V^j_i \sqrt{Z_j} \right] + H E \left[ S(U_d U_d^T S) \right] H^T \]

\[ + H E \left[ S\Delta U_k U_k^T S \right] H^T \] (34)

\[ \mathbb{E}[S\Delta U_k U_k^T S] = \begin{bmatrix}
    \mathbb{E}[s_{11}^2] & 0 & \cdots & 0 \\
    0 & \mathbb{E}[s_{21}^2] & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \mathbb{E}[s_{n1}^2]
\end{bmatrix} + \begin{bmatrix}
    \mathbb{E}[s_{12}^2] & 0 & \cdots & 0 \\
    0 & \mathbb{E}[s_{22}^2] & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \mathbb{E}[s_{n2}^2]
\end{bmatrix} + \begin{bmatrix}
    \mathbb{E}[s_{1n}^2] & 0 & \cdots & 0 \\
    0 & \mathbb{E}[s_{2n}^2] & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \mathbb{E}[s_{nn}^2]
\end{bmatrix} \] (38)

\[ \rho \left[ \frac{(I + \mathbb{W} \mathbb{T}) \otimes (I + \mathbb{W} \mathbb{T}) + \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \sqrt{Z_j(V^j_i)^T \otimes V^j_i \sqrt{Z_j}} \right] - H^{-1} N_k \Gamma \otimes H^{-1} N_k \Gamma - \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} [H^{-1} \sqrt{T_j O^j_i)^T \otimes H^{-1} \sqrt{T_j O^j_i)^T}] \right] \frac{\left( \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} [H \sqrt{T_j O^j_i)^T \otimes H \sqrt{T_j O^j_i)^T}] \right] \right)} {I \otimes I + (H^T H^{-1} N_k \Gamma \otimes I) + I \otimes (H^{-1} N_k \Gamma H) \right] \] < 1 \] (43)

\[ A_{11} = (I + \mathbb{W} \mathbb{T}) \otimes (I + \mathbb{W} \mathbb{T}) + \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \sqrt{Z_j(V^j_i)^T \otimes V^j_i \sqrt{Z_j}} \right], \] (45)

\[ A_{12} = \left( \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} [H \sqrt{T_j O^j_i)^T \otimes O^j_i \sqrt{T_j H^T}] \right] \right) \], (46)

\[ A_{21} = -H^{-1} N_k \Gamma \otimes (H^{-1} N_k \Gamma) - \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} [H^{-1} \sqrt{T_j O^j_i)^T \otimes V^j_i \sqrt{T_j H^T}] \right], \] (47)

\[ A_{22} = I \otimes I + (H^T H^{-1} N_k \Gamma \otimes I) + I \otimes (H^{-1} N_k \Gamma H) \] (48)
where $A_{11}, A_{12}, A_{21},$ and $A_{22}$ are given in (45)-(48). Now since
\[
\begin{bmatrix}
\overline{f}(P_k)
\end{bmatrix} = A_k \begin{bmatrix}
\overline{f}(P_0)
\end{bmatrix} + [A_k^{k-1} + A_k^{k-2} + \cdots + A + I] \begin{bmatrix}
\overline{f}(K)
\end{bmatrix} \quad (49)
\]
where
\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]
if $\rho(A) < 1$, then $\overline{f}(P_k)$ and $\overline{f}(X_k)$ are convergent as $k \to \infty$, which completes the proof.

Now from the above result, we are able to estimate $P^*$ and $X^*$. Let us denote $\sum^A = \sum_{i=0}^{\infty}A^i = \begin{bmatrix}
\sum_{i=1}^{A_{11}} & \sum_{i=1}^{A_{12}} \\
\sum_{i=1}^{A_{21}} & \sum_{i=2}^{A_{22}}
\end{bmatrix}$. Then $\overline{f}(P_k) \to \overline{f}(P_k)^*$ and $\overline{f}(X_k) \to \overline{f}(X_k)^*$ as $k \to \infty$, which are calculated as:
\[
\overline{f}(P_k)^* = \begin{bmatrix}
[\sum^A_{i=1} & [\sum^A_{i=2} \\
[\sum^A_{j=1} & [\sum^A_{i=2}]
\end{bmatrix} \begin{bmatrix}
\overline{f}(K) \\
0_{N^2 \times 1}
\end{bmatrix}
\]
(51)
\[
\overline{f}(X_k)^* = \begin{bmatrix}
[\sum^A_{i=1} & [\sum^A_{i=2} \\
[\sum^A_{j=1} & [\sum^A_{i=2}]
\end{bmatrix} \begin{bmatrix}
\overline{f}(K) \\
0_{N^2 \times 1}
\end{bmatrix}
\]
(52)

Therefore, finally we can obtain $P^*$ and $X^*$ such as:
\[
P^* = \overline{f}^{-1} \begin{bmatrix}
[\sum^A_{i=1} & [\sum^A_{i=2} \\
[\sum^A_{j=1} & [\sum^A_{i=2}]
\end{bmatrix} \begin{bmatrix}
\overline{f}(K) \\
0_{N^2 \times 1}
\end{bmatrix}
\]
(53)
\[
X^* = \overline{f}^{-1} \begin{bmatrix}
[\sum^A_{i=1} & [\sum^A_{i=2} \\
[\sum^A_{j=1} & [\sum^A_{i=2}]
\end{bmatrix} \begin{bmatrix}
\overline{f}(K) \\
0_{N^2 \times 1}
\end{bmatrix}
\]
(54)

Next to estimate exact value of $P^*$ and $X^*$, let us consider the case of $U_d = H^{-1}Y_d$. From $K = H E[SU_d U_d^T S^T]$, $H^T = H E[S H^{-1} Y_d Y_d^T (H^{-1})^T S^T] H^T$. Using the similar procedure as done to obtain (39), we are able to have:
\[
K = H \sum_{j=1}^{n} \left[ \sqrt{T_j} (O_j^T H^{-1} Y_d \right.
\]
(55)
\[
x Y_d^T (H^{-1})^T O_j^T \sqrt{T_j} \left. H^T \right)
\]

Therefore by inserting (55) into (53) and (54), we can calculate $P^*$ and $X^*$, which completes the proof.

III. CONCLUDING REMARKS

Due to page limits, our simulation results cannot be included in this paper. However, those simulation results will be reported in our future publications.

In this paper we have established robust stability conditions for discrete-time iterative learning control (ILC) systems with random delays within the plant and random data dropouts in the communication channels. To analyze the robust stability condition, we utilized mean-square stability, which characterizes convergence of mean-square of state vector. The framework produced in this paper covers broad ranges of stochastic uncertain characteristics of iterative learning control systems when operating in a networked control setting. As outlined in this paper, random delays and random data dropouts can be considered in a unified frame. Furthermore, non-causality of signals and jamming between successive signals can be considered. We stress this point in the following remark:

**Remark 3.1**: In this paper we have considered random delays and random data dropouts. However the framework established in this paper is able to cover various uncertainties. For example, if the random matrix $N_k$ is given such as:
\[
N_k = \begin{bmatrix}
\eta_{11} & \eta_{12} & \cdots & \eta_{1n} \\
\eta_{21} & \eta_{22} & \cdots & \eta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\eta_{n1} & \eta_{n2} & \cdots & \eta_{nn}
\end{bmatrix}
\]
(56)

then $e_k(2) = \eta_{11} e_k(1) + \eta_{12} e_k(2) + \cdots + \eta_{nn} e_k(n)$ and $e_k(3) = \eta_{21} e_k(1) + \eta_{22} e_k(2) + \cdots + \eta_{nn} e_k(n)$. Thus $e_k(2)$ is a combination of signals $e_k(1)$ and $e_k(2)$, which may be due to a jamming in network communication. In the case of $e_k(3)$ it is affected by non-causal signal $e_k(4)$ if $\eta_{2n} \neq 0$. Thus the framework we introduced in this paper can take account of not only time delays but also data forwarding, which may be due to existence of non-causality of ILC systems. Thus the framework presented here can be considered as a general unified framework for network-based ILC systems.

IV. APPENDIX

A. Appendix-1

In this appendix, we derive (9). From (59), since $(NTE) = \sum_{k=1}^{n} n_{uk} \sum_{j=1}^{n} g_{kj} e_j$, we have (60). Notice that $\eta_{ij}$ implies
\[
E \left[ (N_k \Gamma) E_k E_k^T (N_k \Gamma)^T \right]_{uv} = E \left[ \sum_{k=1}^{n} n_{uk} \sum_{j=1}^{n} g_{kj} e_j \sum_{k'=1}^{n} n_{vk} g_{kj} e_j \right]
\]
(61)

B. Appendix-2

Let us show that $E[|M_{k+1} - M_k|] = 0$ when $M_{k+1}$ and $M_k$ are random matrices, but with a fixed probability such as:
\[
M_{k+1} = M_k = [m_{ij}]
\]
(62)

where $m_{ij}$ could be zero or one; i.e., $m_{ij} \in \{0,1\}$ with probability of $P_1(m_{ij} = 0) = p_{0}^0$ and $P_1(m_{ij} = 0) = p_{0}^0 = 1 - p_{1}^0$, it is immediate to conclude that $E[|M_{k+1} - M_k|] = E[(M_{k+1} - M_k)] = 0$.

C. Appendix-3

\[
W = H M_{k+1} H^{-1} N_k
\]
(63)
\[
\tilde{N}_k \Gamma E_k = \begin{pmatrix}
  n_{11}(\Gamma E_k) + n_{12}(\Gamma E_k) + n_{13}(\Gamma E_k) + \cdots + n_{1n}(\Gamma E_k) \\
  n_{11}(\Gamma E_k) + n_{22}(\Gamma E_k) + n_{13}(\Gamma E_k) + \cdots + n_{nn}(\Gamma E_k) \\
  \vdots \\
  n_{11}(\Gamma E_k) + n_{22}(\Gamma E_k) + n_{33}(\Gamma E_k) + \cdots + n_{nn}(\Gamma E_k)
\end{pmatrix}
\]

(57)

\[
= \sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right] \\
= \sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right] \\
\sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right] \\
\sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right]
\]

(58)

\[
\sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right] = \sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right] = \sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right] = \sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right]
\]

(59)

\[
(\tilde{N}_k \Gamma) E_k E_k^T (\tilde{N}_k \Gamma)^T = \left( \sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right] \right) \left( \sum_{k=1}^{n} \left[ n_{1k}(\Gamma E_k) \right] \right)
\]

(60)

where \( \tilde{W}_{ij} \) is computed as:

\[
\tilde{W}_{ij} = \sum_{k=1}^{n} \sum_{k=1}^{n} \left[ H_{ik_1} M_{k_1 k_2} H_{k_2 k_3}^{-1} (\tilde{N}_k)_{k_3 j} \right] + H_{ik_1} (\tilde{M}_{k+1})_{k_1 k_2} H_{k_2 k_3}^{-1} \tilde{N}_k k_3 j + H_{ik_1} (\tilde{M}_{k+1})_{k_1 k_2} H_{k_2 k_3}^{-1} \tilde{N}_k k_3 j
\]

(64)

Since \((\tilde{N}_k)_{ij}\) and \((\tilde{M}_{k+1})_{ij}\) are independent each other, we are able to obtain:

\[
E[\tilde{W}_{ij}^2] = \sum_{k=1}^{n} \sum_{k=1}^{n} \sum_{k=1}^{n} \left[ (H_{ik_1} M_{k_1 k_2} H_{k_2 k_3}^{-1})^2 E[\tilde{N}_k k_3 j] \right] + \sum_{k=1}^{n} \sum_{k=1}^{n} \left[ \tilde{N}_k k_3 j \right]^2 E[(\tilde{M}_{k+1})_{k_1 k_2} (\tilde{M}_{k+1})_{k_1 k_2} \tilde{N}_k k_3 j]
\]

(65)

\[
E[M_{k_1 k_2}^2] = M_{k_1 k_2} (1 - M_{k_1 k_2}) \quad \text{and} \quad E[\tilde{N}_k k_3 j] = \tilde{N}_k k_3 j (1 - \tilde{N}_k k_3 j)
\]

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