

# Spherical-Wave Source-Scattering Matrix Analysis of Coupled Antennas; A General System Two-Port Solution

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**Abstract**—Expressions are given for the coupling between two antennas in terms of each antenna's spherical-wave source-scattering matrix. A comparison with the "classical" scattering matrix representation is given in sufficient detail to permit conversion back and forth between the source-scattering matrix and the classical scattering matrix. Expressions for the transmission formulas, showing two different expressions corresponding to reversing the direction of propagation are given. However, if both antennas are reciprocal with equal characteristic waveguide impedances, then the two-port scattering matrix is a symmetric matrix.

EXPRESSIONS FOR THE coupling between two antennas when each antenna is described by a spherical-wave source-scattering matrix representation are presented. Our expressions are derived using matrix algebra, thus avoiding the cumbersome modal-subscript and summation notation otherwise required. We explicitly account for multiple reflections between the antennas, and exhibit formal differences between transmission formulas when the propagation directions are reversed. Thus, we present a complete analytical picture from which we extract the simplified coupling equation commonly used as a starting point for analyzing spherical near-field scanning. Previously, Jensen [1] expressed the transmission from an arbitrary antenna to a probe, using the Lorentz reciprocity relation, in order to formulate a spherical near-field to far-field transformation. Subsequently, Wacker [2] reexpressed Jensen's transmission formula from a scattering matrix approach, neglecting multiple reflections. In the general context of describing antenna coupling using scattering-matrix analysis, the earliest work was the plane-wave scattering-matrix formulation of Kerns and Dayhoff [3]. Wasylkiwskyj and Kahn [4] used spherical-wave scattering matrices to express the coupling between minimum-scattering antennas, which by the definition of minimum scattering ignores multiple reflections between the antennas. Yaghjian [5] gave a complete cylindrical-wave source-scattering matrix analysis of two-antenna coupling presenting the first-order multiple-reflection term. Yaghjian also coined the term, source-scattering matrix, to distinguish a scattering matrix formulation, using cylindrical (or spherical) waves, in which the exciting spatial modes contain Bessel functions of the first kind as the radial-distance function. This particular radial-

variable function enters into the formulation in expressing the effect, on the cylindrical or spherical modes, of radially translating the coordinate origin. A source-scattering matrix formalism is also introduced by Appel-Hansen [6, ch. 8], in which he reexpresses Yaghjian's cylindrical analysis [5] and then goes on to reobtain Wacker's result [2]. Finally, a summary version of the present analysis is given in the 1981 Antennas and Propagation Society Symposium Digest [7].

The spherical-wave source-scattering matrix representation of an antenna may be expressed as

$$\begin{bmatrix} b_0 \\ Q \end{bmatrix} = \begin{bmatrix} \Gamma & R \\ T & S \end{bmatrix} \begin{bmatrix} a_0 \\ P \end{bmatrix}. \quad (1)$$

This constitutes the source-scattering matrix representation for the "exterior" region, which consists of the region exterior to a spherical surface enclosing the antenna. Here,  $b_0$  and  $a_0$  represent the emergent and incident mode coefficients at the waveguide leads to the antenna, while  $Q$  and  $P$  are infinite column matrices representing the emergent and exciting spatial-mode coefficients at the hypothetical spherical boundary enclosing the antenna. For simplicity, only a single waveguide feed mode is assumed to propagate. The normalization on the modal coefficient  $a_0$  is such that  $1/2|a_0|^2/Z_0$  represents the incident power at a hypothetical terminal surface in the waveguide feed, where  $Z_0$  is the characteristic impedance of the waveguide. The elements of the  $P$  matrix are coefficients of vector spherical wave functions whose product sums up to equal the electric field incident on the enclosing spherical boundary. Also,  $\Gamma$  is the waveguide reflection coefficient of the antenna,  $T$  is an infinite column matrix representing the antenna's transmission properties,  $R$  is an infinite row matrix representing the antenna's receiving properties, while  $S$  is an infinite square matrix representing the antenna's scattering properties. The spatial modes associated with the  $Q$  matrix each contain a spherical Hankel function of the first kind, which represents an outgoing-wave mode when an  $e^{-i\omega t}$  time dependence is assumed. The spatial modes associated with the  $P$  matrix, on the other hand, each contain a spherical Bessel function as the radial coordinate function.

We can contrast the source-scattering matrix (1) with the classical spherical-wave scattering matrix [8], for which the incident or exciting spatial-wave modes contain spherical Hankel functions of the second kind. Thus, the classical scattering matrix for the "exterior" region may be expressed,

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in terms of the matrix elements in (1), as [9]

$$\begin{bmatrix} b_0 \\ U \end{bmatrix} = \begin{bmatrix} \Gamma & 2R \\ T & I+2S \end{bmatrix} \begin{bmatrix} a_0 \\ V \end{bmatrix},$$

where  $I$  is the identity matrix,  $U=1/2P+Q$ , and  $V=1/2P$ . The attraction of this representation is that each mode clearly exhibits the characteristics of an incoming or outgoing wave.

In Fig. 1 we show two antennas; the antenna on the left is associated with the unprimed coordinate system, while the antenna on the right is associated with the doubly primed coordinate system. In Fig. 1, the Euler angles  $\phi$ ,  $\theta$ ,  $\chi$  describe the orientation of the singly primed coordinate system with respect to the unprimed coordinate system. Here,  $z'$  and  $z''$  are collinear while the remaining singly primed coordinates are, respectively, parallel to their doubly primed coordinate system counterparts. Also,  $r$  is the separation distance between the unprimed and doubly primed coordinate systems. Equation (1) gives the source-scattering matrix for the antenna on the left, while the exterior-region source-scattering matrix for the antenna on the right is given by

$$\begin{bmatrix} b_0'' \\ Q'' \end{bmatrix} = \begin{bmatrix} \hat{\Gamma}'' & \hat{R}'' \\ \hat{T}'' & \hat{S}'' \end{bmatrix} \begin{bmatrix} a_0'' \\ P'' \end{bmatrix}. \quad (2)$$

Here, the carets on the scattering matrix elements designate the right-hand antenna, while the primes designate the coordinate system. Similarly, the absence of carets on the scattering matrix elements in (1) designates that the left-hand antenna is represented, while the absence of primes designates that the scattering matrix elements are expressed with respect to the unprimed coordinate system.

We now wish to determine the effect, on the scattering matrix elements  $\hat{\Gamma}''$ ,  $\hat{R}''$ ,  $\hat{T}''$ , and  $\hat{S}''$ , of transforming the doubly primed coordinate system into the unprimed coordinate system. That is, although the two antennas shown in Fig. 1 remain fixed in space, we can employ a coordinate system transformation to obtain a scattering-matrix representation, referenced to the unprimed coordinate system, for the antenna on the right in Fig. 1. Then, with the scattering matrices of both antennas referenced to the same coordinate system, the mutual coupling equations can readily be obtained. The resulting "interior-region" source-scattering matrix representation for the antenna on the right, referenced to the unprimed coordinate system, would relate the same spatial-mode coefficient matrices,  $P$  and  $Q$ , that are related by (1). Thus, we seek the transformed source-scattering matrix elements in the relation

$$\begin{bmatrix} b_0'' \\ P \end{bmatrix} = \begin{bmatrix} \hat{\Gamma} & \hat{R} \\ \hat{T} & \hat{S} \end{bmatrix} \begin{bmatrix} a_0'' \\ Q \end{bmatrix}. \quad (3)$$

This "interior-region" representation is valid for the region interior to a sphere, centered about the unprimed coordinate system origin, whose volume just excludes the antenna represented, (i.e., the antenna on the right in Fig. 1). Thus we have a common region, between the regions of validity of (1) and (3), where the same modal-coefficient matrices,  $P$  and  $Q$ , are used in the common representation of the electromagnetic

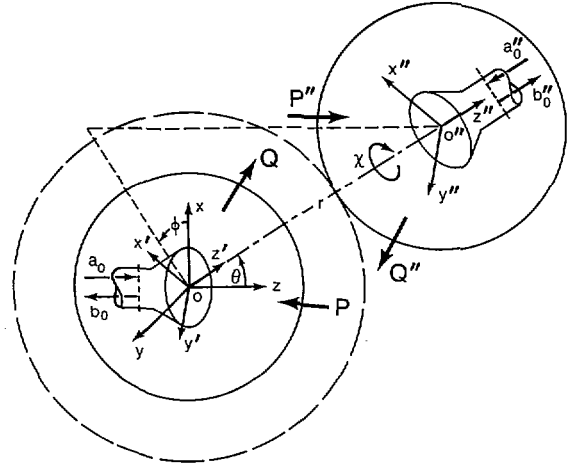


Fig. 1. Problem geometry, showing emergent ( $Q$ ) modes associated with spherical Hankel functions and exciting ( $P$ ) modes associated with spherical Bessel functions.

field. In Fig. 1, the solid-line circles each denote the boundary of the circumscribed antenna's "exterior" region representation, while the dotted-line circle denotes the boundary of the right-hand antenna's "interior" region representation. The common region is represented as the space between the two concentric circles.

We still require a determination of the scattering matrix elements of (3) in terms of the known scattering matrix elements of (2). These results, upon applying the coordinate-system transformation operations to the matrix elements in (2),

$$\begin{aligned} \hat{\Gamma} &= \hat{\Gamma}'' & \hat{R} &= \hat{R}'' CD^\dagger \\ \hat{T} &= D\alpha\hat{T}'' & \hat{S} &= D\alpha\hat{S}'' CD^\dagger. \end{aligned} \quad (4)$$

Here,  $D$  is a known [10] coordinate-system rotation matrix corresponding to a rotation through the Euler angles  $\phi$ ,  $\theta$ ,  $\chi$ .  $D^\dagger$  denotes the transposed complex conjugate of the  $D$  matrix, which means that it corresponds to the inverse rotation. Also,  $C$  is a coordinate-system translation matrix corresponding to a rigid coordinate-system translation a distance  $r$  along the  $z'$  axis [11]. The  $\alpha$  matrix corresponds to a coordinate-system translation in the opposite direction. The construction of the  $\alpha$  and  $C$  matrices is presented in the Appendices, where it is shown that the elements of the  $\alpha$  and  $C$  are related by  $\alpha_{ij} = (-)^{i+j}C_{ij}$ . The source-scattering matrix transformations (4) are derived in Appendix I.

The coupling between the two antennas shown in Fig. 1 is determined by solving (1) and (3) to obtain

$$\begin{aligned} P &= (I - \hat{S}\hat{S})^{-1}(\hat{S}Ta_0 + \hat{T}a_0'') \\ Q &= (I - \hat{S}\hat{S})^{-1}(Ta_0 + \hat{S}\hat{T}a_0'') \end{aligned} \quad (5)$$

and

$$\begin{aligned} b_0'' &= \hat{\Gamma}a_0'' + \hat{R}Q \\ b_0 &= \Gamma a_0 + RP. \end{aligned} \quad (6)$$

Upon substituting (5) into (6) we are able to evaluate the equivalent system-two-port scattering matrix, which is defined

as (cf. [12])

$$\begin{bmatrix} b_0 \\ b_0'' \end{bmatrix} = \begin{bmatrix} M_{00} & M_{00}'' \\ M_{0''0} & M_{0''0}'' \end{bmatrix} \begin{bmatrix} a_0 \\ a_0'' \end{bmatrix},$$

and which is formed by considering just the waveguide leads to the two antennas. There results

$$\begin{bmatrix} b_0 \\ b_0'' \end{bmatrix} = \begin{bmatrix} \Gamma + R\hat{S}(I - S\hat{S})^{-1}T & R(I - S\hat{S})^{-1}\hat{T} \\ \hat{R}(I - S\hat{S})^{-1}T & \hat{T} + \hat{R}S(I - S\hat{S})^{-1}\hat{T} \end{bmatrix} \begin{bmatrix} a_0 \\ a_0'' \end{bmatrix}. \quad (7)$$

It may be noted that this result is similar to that obtained by Kerns [12] using plane-wave scattering-matrix analysis. Here, we have used the matrix identity,  $(I - S\hat{S})^{-1}\hat{S} = \hat{S}(I - S\hat{S})^{-1}$ .

As special cases of (7), we have the transmission formulas, neglecting multiple reflections,

$$M_{0''0} = \hat{R}'' CD^\dagger T \quad (8)$$

and

$$M_{00}'' = RD\alpha\hat{T}'' \quad (9)$$

The principal diagonal elements of (7) just reduce to the waveguide reflection coefficients when multiple reflections are neglected.

At this point, we present expressions showing the angular dependence of the two oppositely directed transmission formulas. The spherical angles  $\theta$  and  $\phi$  are defined in Fig. 1, while the rotation of the right-hand antenna about the  $z''$ -axis is characterized by the angle  $\chi$ . Let us define the translated receiving and transmitting characteristic matrices  $\hat{R}' = \hat{R}''C$  and  $\hat{T}' = \alpha\hat{T}''$ . Then (8) and (9) can be written, respectively, as

$$M_{0''0} = \sum_{s=1}^2 \sum_{n=1}^{\infty} \sum_{m=-n}^n \sum_{\mu=-n}^n \hat{R}'_{s\mu n} D_{m\mu}^{n*}(\phi, \theta, \chi) T^{smn} \quad (10)$$

and

$$M_{00}'' = \sum_{s=1}^2 \sum_{n=1}^{\infty} \sum_{\mu=-n}^n \sum_{m=-n}^n R_{smn} D_{m\mu}^n(\phi, \theta, \chi) \hat{T}'_{s\mu n}. \quad (11)$$

Similar expressions have also been obtained by Larsen [13]. In the above, the index  $s$  characterizes either transverse electric (TE) or transverse magnetic (TM) modes, while  $m$  and  $n$  characterize the order and degree of the spherical wave functions. The respective indexes  $\mu$  and  $m$  serve as functions. Also in the above,  $R_{smn}$  and  $T^{smn}$  are receiving and transmitting matrix elements from (1), while  $\hat{R}'_{s\mu n}$  and  $\hat{T}'_{s\mu n}$  are translated receiving and transmitting matrix elements characterizing the right-hand antenna. The asterisk in (10) denotes the complex conjugate, while the elements of the  $D$

matrix are defined [10, ch. 4] as

$$D_{m\mu}^n(\phi, \theta, \chi) = e^{-im\phi} d_{m\mu}^n(\theta) e^{-i\mu\chi}. \quad (12)$$

The  $d_{m\mu}^n(\theta)$  functions are closely related [10] to the Jacobi polynomials.

As special cases of (10), when the antenna on the right in Fig. 1 is an ideal  $x$ -directed dipole, it has been verified for  $\chi = 0$  and  $\chi = \pi/2$  that our expression for  $M_{0''0}$  is proportional to the  $\theta$  and  $\phi$  components, respectively, of the vector-spherical-wave-function expansion of the electric field radiated by the antenna on the left.

We conclude by noting the reciprocity relation that exists between (10) and (11) when both antennas in Fig. 1 are reciprocal. For this case, it is shown in Appendix II that the preceding expressions are related according to

$$M_{0''0} = \frac{\hat{Z}_0''}{Z_0} M_{00}'' \quad (13)$$

where  $Z_0$ ,  $\hat{Z}_0''$  are the characteristic impedances of the waveguide leads to the two antennas.

#### APPENDIX I

##### COORDINATE-SYSTEM TRANSLATION AND ROTATION TRANSFORMATIONS OF THE SOURCE-SCATTERING MATRIX ELEMENTS

Following Stratton [14], we introduce vector spherical wave functions  $\mathbf{M}_{mn}$  and  $\mathbf{N}_{mn}$  which satisfy the vector Helmholtz equation,  $\nabla^2 \mathbf{F} + k^2 \mathbf{F} = 0$ ;  $k = 2\pi/\lambda$  where  $\lambda$  is the wavelength. Here,  $\mathbf{N}_{mn} = 1/k \nabla \times \mathbf{M}_{mn}$  and  $\mathbf{M}_{mn} = -\hat{\mathbf{r}} \times \nabla \psi_{mn}$ , where  $\hat{\mathbf{r}}$  is the radial vector and  $\psi_{mn}$  satisfies the scalar wave equation  $(\nabla^2 + k^2)\psi_{mn} = 0$ . In particular,  $\psi_{mn}$  contains the associated Legendre function of degree  $n$  and order  $m$  as well as the spherical Bessel function of the first kind of order  $n$ . We also introduce the vector spherical wave functions  $\mathbf{N}_{mn}^{(1)} = 1/k \nabla \times \mathbf{M}_{mn}^{(1)}$  and  $\mathbf{M}_{mn}^{(1)} = -\hat{\mathbf{r}} \times \nabla \psi_{mn}^{(1)}$ , where  $\psi_{mn}^{(1)}$  contains a spherical Hankel function of the first kind as the function of the radial coordinate. Thus, the first set of vector basis functions would be associated with the coefficients in the  $P$  matrix, while the second set of vector basis functions would be associated with the coefficients in the  $Q$  matrix, where the  $P$  and  $Q$  matrices are introduced by (1). We have yet to specify the normalization and the functional dependence of the equatorial angle  $\hat{\phi}$  for the scalar spherical wave functions  $\psi_{mn}$  and  $\psi_{mn}^{(1)}$ , as constraints on these choices are imposed by the rotation of coordinates transformation.

Rather than carry two separate notations for the vector spherical wave functions, we define

$$\mathbf{F}_{1mn} = \mathbf{M}_{mn}, \quad \mathbf{F}_{2mn} = \mathbf{N}_{mn}$$

with similar definitions for  $\mathbf{F}_{smn}^{(1)}$ ;  $s = 1, 2$ . We now complete the determination of  $\mathbf{F}_{smn}$  and  $\mathbf{F}_{smn}^{(1)}$  by requiring that the normalization and choice of the angular-variable functions be such that under a rotation of coordinates transformation the vector spherical wave functions transform according to

$$\mathbf{F}'_{s\mu n} = \sum_{m=-n}^n \mathbf{F}_{smn} D_{m\mu}^n(\phi, \theta, \chi), \quad (14)$$

and similarly for  $\mathbf{F}_{smn}^{(1) \prime}$ . Here, the primes on the vector spherical wave functions indicate that these functions are defined with respect to rotated coordinates, where the singly primed coordinate-system's axes are parallel to the axes of the doubly primed coordinate system as shown in Fig. 1. The constraints imposed by (14) on the angular functions used in the vector spherical wave functions are similar to the constraints noted by Griffin [15] for specifying the spherical harmonics. Our resulting expressions for  $\mathbf{M}_{mn}$ ,  $\mathbf{N}_{mn}$  and  $\mathbf{M}_{mn}^{(1)}$ ,  $\mathbf{N}_{mn}^{(1)}$  will correspond with those definitions of the vector spherical wave functions given in [16], except that our expressions will require the additional factor  $\sqrt{(n-m)!/(n+m)!}$ . Noting that  $d_{m0}^n(\hat{\theta})$  is equal to this square-root factor multiplied by the associated Legendre function  $P_n^m(\cos \hat{\theta})$ , we can express the scalar spherical wave functions as

$$\psi_{mn} = j_n(k\hat{r})d_{m0}^n(\hat{\theta})e^{im\hat{\phi}}, \quad \psi_{mn}^{(1)} = h_n^{(1)}(k\hat{r})d_{m0}^n(\hat{\theta})e^{im\hat{\phi}}$$

where  $\hat{r}$ ,  $\hat{\theta}$ ,  $\hat{\phi}$  denote the spherical coordinates of the observation point,  $j_n(x)$  is the spherical Bessel function of the first kind, of order  $n$  and argument  $x$ , while  $h_n^{(1)}(x)$  is the spherical Hankel function of the first kind [17].

The translation of coordinates transformation, from the singly primed to the doubly primed coordinate system, can be expressed as the addition theorem [16]

$$\mathbf{M}_{\mu n}^{(1) \prime} = \sum_{\nu=(1,|\mu|)}^{\infty} (A_{\mu\nu}^{\mu n} \mathbf{M}_{\mu\nu}^{\prime\prime} + B_{\mu\nu}^{\mu n} \mathbf{N}_{\mu\nu}^{\prime\prime}) \quad (15)$$

and

$$\mathbf{N}_{\mu n}^{(1) \prime} = \sum_{\nu=(1,|\mu|)}^{\infty} (A_{\mu\nu}^{\mu n} \mathbf{N}_{\mu\nu}^{\prime\prime} + B_{\mu\nu}^{\mu n} \mathbf{M}_{\mu\nu}^{\prime\prime})$$

where the notation  $(1, |\mu|)$  denotes the larger of 1 or  $|\mu|$ . The fact that an addition-theorem expansion of this type must exist is self evident, since  $\mathbf{M}_{\mu\nu}^{\prime\prime}$  and  $\mathbf{N}_{\mu\nu}^{\prime\prime}$  constitute a complete set of basis functions for describing fields (or field components) at and in the vicinity of the origin  $0''$ . The above expansion applies to a rigid coordinate-system translation from  $0 = 0'$  to  $0''$  a distance  $r$  along the  $z''$ -axis (see Fig. 1). As stated in [16], when translating from  $0''$  to  $0'$  the coefficients  $A_{\mu\nu}^{\mu n}$  and  $B_{\mu\nu}^{\mu n}$  are preceded by the factors  $(-)^{n+\nu}$  and  $(-)^{n+\nu+1}$ , respectively. The translation coefficients corresponding to this latter case are given in [16], from which we define  $A_{\mu\nu}^{\mu n}$  and  $B_{\mu\nu}^{\mu n}$ , except that our definitions include the additional factor  $\sqrt{(n-\mu)!(\nu+\mu)!/(n+\mu)!(\nu-\mu)!}$  to account for replacing  $P_n^m(\cos \hat{\theta})$  with  $d_{m0}^n(\hat{\theta})$ .

Let us now form an infinite row matrix of the vector spherical wave functions  $\mathbf{F}_{smn}^{(1)}$ , which we shall call  $\mathbf{F}^{(1)}$ . For our purposes, the matrix element  $\mathbf{F}_{smn}^{(1)}$  is located at position  $I(s, m, n)$  in  $\mathbf{F}^{(1)}$ , where  $I(s, m, n) = 2[n^2 + (s-1)n - 1] + s + m + n$ . That is, for a given value of  $n$ , all of the  $s = 1$  elements would be grouped together, starting with  $m = -n$  and running through  $m = +n$ ; then these in turn would be followed by all of the  $s = 2$  elements and then by elements corresponding to the next value of  $n$ . We also define the row

matrices  $\mathbf{M}^{(1)}$  and  $\mathbf{N}^{(1)}$  with suitably interspaced zero elements such that

$$\mathbf{F}^{(1)} = \mathbf{M}^{(1)} + \mathbf{N}^{(1)}.$$

It should be obvious, for corresponding matrix elements in  $\mathbf{M}^{(1)}$  and  $\mathbf{N}^{(1)}$ , that one or the other of these matrix elements will be equal to zero.

We can now write (15) as the matrix equation

$$\mathbf{F}^{(1) \prime} = \mathbf{F}'' \mathbf{C} \quad (16)$$

where  $\mathbf{C}$  is the infinite square matrix

$$\mathbf{C} = \mathbf{A} + \mathbf{B}. \quad (17)$$

In (17) the elements of the  $\mathbf{A}$  and  $\mathbf{B}$  matrices are  $A_{\mu\nu}^{\mu n}$  and  $B_{\mu\nu}^{\mu n}$ , respectively, with suitably interspaced zero elements. For translation in the opposite direction we have the matrix equation

$$\mathbf{F}^{(1) \prime\prime} = \mathbf{F}' \alpha. \quad (18)$$

From (31) of Appendix II, it is apparent that the elements of the  $\alpha$  and  $\mathbf{C}$  matrices are related by  $\alpha_{ij} = (-)^{i+j} C_{ij}$ . Next, we can rewrite (14) in the form

$$\mathbf{F}' = \mathbf{F} \mathbf{D}. \quad (19)$$

Moreover, since  $\mathbf{D}$  is a unitary matrix,  $\mathbf{D}^{-1} = \mathbf{D}^\dagger$  so that

$$\mathbf{F}^{(1)} = \mathbf{F}^{(1) \prime} \mathbf{D}^\dagger. \quad (20)$$

With the vector-spherical-wave-function row matrices defined, we can express the electric field vector  $\mathcal{E}$ , with respect to the unprimed coordinate system, as

$$\mathcal{E} = \mathbf{F}^{(1)} \mathbf{Q} + \mathbf{F} \mathbf{P}. \quad (21)$$

Similarly, the electric field vector can be expressed with respect to the doubly primed coordinate system as

$$\mathcal{E} = \mathbf{F}'' \mathbf{P}'' + \mathbf{F}^{(1) \prime\prime} \mathbf{Q}''. \quad (22)$$

Let us consider the first term on the right in (21). From (16) and (20) we see that

$$\mathbf{F}^{(1)} \mathbf{Q} = \mathbf{F}'' \mathbf{C} \mathbf{D}^\dagger \mathbf{Q}. \quad (23)$$

Now the right-hand side of (23) represents the exciting electric field with respect to the doubly primed coordinate system. Consequently, upon comparing (23) with (22) we see that

$$\mathbf{P}'' = \mathbf{C} \mathbf{D}^\dagger \mathbf{Q}. \quad (24)$$

Similarly, upon applying (18) and (19) to the second term on the right in (22) and comparing that result with (21) we see that

$$\mathbf{P} = \mathbf{D} \alpha \mathbf{Q}''. \quad (25)$$

Now from (2) we have  $\mathbf{Q}'' = \hat{\mathbf{T}}'' a_0'' + \hat{\mathbf{S}}'' \mathbf{P}''$ . If we substitute this expression into (25) and use (24) for  $\mathbf{P}''$  we obtain

$$\mathbf{P} = \mathbf{D} \alpha \hat{\mathbf{T}}'' a_0'' + \mathbf{D} \alpha \hat{\mathbf{S}}'' \mathbf{C} \mathbf{D}^\dagger \mathbf{Q}. \quad (26)$$

Also from (2) and (24) we obtain

$$b_0'' = \hat{\Gamma}'' a_0'' + \hat{R}'' CD^\dagger Q. \quad (27)$$

Consequently, comparing (26) and (27) with (3) we obtain the transformations given in (4).

It may be noted that a very complicated result is obtained if one should seek a "classical" interior-region scattering matrix (i.e., using spherical Hankel functions of the second kind for the emergent modes). Thus, in terms of the transformed scattering matrix elements defined by (3), such a "classical" interior region scattering matrix would take the form [9]

$$\begin{bmatrix} b_0'' \\ V \end{bmatrix} = \begin{bmatrix} \hat{\Gamma} - \hat{R}(2I + \hat{S})^{-1} \hat{T} & 2\hat{R}(2I + \hat{S})^{-1} \\ (2I + \hat{S})^{-1} \hat{T} & (2I + \hat{S})^{-1} \hat{S} \end{bmatrix} \begin{bmatrix} a_0'' \\ U \end{bmatrix}.$$

However, using literally the above interior-region scattering matrix and the previously given "classical" exterior-region scattering matrix, it is a straight forward exercise in matrix algebra to generate (7), the general two-port scattering matrix for coupled antennas.

## APPENDIX II

### RECIPROCIITY RELATIONS FOR COUPLED ANTENNAS

We can write the translation transformations as

$$\hat{R}'_{sun} = \sum_{\sigma=1}^2 \sum_{\nu=(1,|\mu|)}^{\infty} C_{\sigma\mu\nu}^{sn} \hat{R}''_{\sigma\mu\nu} \quad (28)$$

and

$$\hat{T}'_{s\mu n} = \sum_{\sigma=1}^2 \sum_{\nu=(1,|\mu|)}^{\infty} \alpha_{s\mu n}^{\sigma\nu} \hat{T}''_{\sigma\mu\nu}, \quad (29)$$

where

$$C_{\sigma\mu\nu}^{sn} = A_{\mu\nu}^{\sigma n} \frac{1 + (-)^{s+\sigma}}{2} + B_{\mu\nu}^{\sigma n} \frac{1 - (-)^{s+\sigma}}{2} \quad (30)$$

and

$$\alpha_{s\mu n}^{\sigma\nu} = (-)^{n+\nu} (-)^{s+\sigma} C_{s\mu n}^{\sigma\nu}. \quad (31)$$

Now from our expressions for  $A_{\mu\nu}^{\sigma n}$  and  $B_{\mu\nu}^{\sigma n}$ , and by using certain symmetry relations between the so-called Wigner 3-j symbols [18], we are able to obtain the additional relation between the elements of the  $\alpha$  and  $C$  matrices

$$\alpha_{s\mu n}^{\sigma\nu} = \frac{2n+1}{n(n+1)} \frac{\nu(\nu+1)}{2\nu+1} C_{\sigma,-\mu,\nu}^{sn}. \quad (32)$$

Consequently, upon substituting (32) into (29), we obtain

$$\hat{T}'_{s,-\mu,n} = \frac{2n+1}{n(n+1)} \sum_{\sigma=1}^2 \sum_{\nu=(1,|\mu|)}^{\infty} \frac{\nu(\nu+1)}{2\nu+1} C_{\sigma\mu\nu}^{sn} \hat{T}''_{\sigma,-\mu,\nu}. \quad (33)$$

Up to this point we have not made any assumptions concerning reciprocity. From the definition of the scattering

matrix elements (1), we can apply techniques similar to those utilized by Kerns [12] to obtain a reciprocity relation between the elements of the  $\hat{R}''$  and  $\hat{T}''$  matrices. Briefly, we know that the Lorentz reciprocity surface integral, evaluated over a closed surface enclosing a source free isotropic region, is equal to zero. We assume that the antenna on the right in Fig. 1 is reciprocal, so that the theorem applies, and we find that the only nonzero contributions to the surface integral come from integrating over a hypothetical spherical boundary enclosing the antenna and from a hypothetical terminal surface in the waveguide feed. Using the orthogonality relation between the electric and magnetic basis fields for the waveguide feed [12], the orthogonality relations between the vector spherical wavefunctions on the spherical boundary [8], [14], the Wronskian relation between spherical Bessel functions [17], and (2), we obtain the reciprocity relation

$$\hat{R}''_{\sigma\mu\nu} = \frac{2\pi\eta}{k^2} \hat{Z}''_0(-)^{\mu} \frac{\nu(\nu+1)}{2\nu+1} \hat{T}''_{\sigma,-\mu,\nu}, \quad (34)$$

where  $\eta = \sqrt{\epsilon/\mu}$ , the characteristic admittance of free space. Now if we substitute (34) into (33) and make use of (28) we obtain a reciprocity relation between the translated transmitting and receiving coefficients,

$$\hat{Z}''_0 \hat{T}'_{s,-\mu,n} = \frac{k^2}{2\pi\eta} (-)^{\mu} \frac{2n+1}{n(n+1)} \hat{R}'_{s\mu n}. \quad (35)$$

This result is seen to be very similar to (34).

We now note the relation [10]

$$D_{-m,-\mu}^n(\phi, \theta, \chi) = (-)^{m+\mu} D_{m\mu}^{n*}(\phi, \theta, \chi). \quad (36)$$

Consequently, substituting (35) and (36) into (11) results in

$$M_{00}'' = \frac{k^2}{2\pi\eta \hat{Z}''_0} \sum_{snm\mu} (-)^m \frac{2n+1}{n(n+1)}$$

$$R_{s,-m,n} D_{m\mu}^{n*}(\phi, \theta, \chi) \hat{R}'_{s\mu n}. \quad (37)$$

If we make the assumption that both antennas in Fig. 1 are reciprocal, we can substitute the reciprocity relation (cf. (34)) between  $R$  and  $T$  matrix elements into (37) and compare that result with (10), thus obtaining (13).

We conclude by noting (37) implies that we only need one algorithm (e.g.: (10)) for spherical-scanning near-field to far-field transformations, provided that the probe antenna is reciprocal. Reciprocity for the test antenna is not required.

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