

Research Article

Solving Abel's Type Integral Equation with Mikusinski's Operator of Fractional Order

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This paper gives a novel explanation of the integral equation of Abel's type from the point of view of Mikusinski's operational calculus. The concept of the inverse of Mikusinski's operator of fractional order is introduced for constructing a representation of the solution to the integral equation of Abel's type. The proof of the existence of the inverse of the fractional Mikusinski operator is presented, providing an alternative method of treating the integral equation of Abel's type.

1. Introduction

Abel studied a physical problem regarding the relationship between kinetic and potential energies for falling bodies [1]. One of his integrals stated in [1] is expressed in the form

$$f(t) = \int_a^t \frac{g(u)}{\sqrt{t-u}} du, \quad a > 0, \quad (1)$$

where $f(t)$ is known, but $g(t)$ is unknown. The previous expression is in the literature nowadays called Abel's integral equation [2]. In addition to (1), Abel also worked on the integral equation in [1] in the following form:

$$f(t) = \int_a^t \frac{g(u)}{(t-u)^\lambda} du, \quad a > 0, \quad 0 < \lambda < 1, \quad a \leq t \leq b, \quad (2)$$

which is usually termed the integral equation of Abel's type [3] or the generalized Abel integral equation [4]. The function $(t-u)^{-\lambda}$ may be called Abel's kernel. It is seen that (1) is a special case of (2) for $\lambda = 1/2$. This paper is in the aspect of (2). Without generality losing, for the purpose of facilitating the discussions, we multiply the left side of (1) with the constant $1/\Gamma(\lambda)$, and let $a = 0$. That is, we rewrite (2) by

$$f(t) = \frac{1}{\Gamma(\lambda)} \int_0^t \frac{g(u)}{(t-u)^\lambda} du, \quad 0 < \lambda < 1, \quad 0 \leq t \leq b. \quad (3)$$

The integral equation of Abel's type attracts the interests of mathematicians and physicists. In mathematics, for example, for solving the integral equation of Abel's type, [5] discusses a transformation technique, [6] gives a method of orthogonal polynomials, [7] adopts the method of integral operators, [8, 9] utilize the fractional calculus, [10] is with the Bessel functions, [11, 12] study the wavelet methods, [13, 14] describe the methods based on semigroups, [15] uses the almost Bernstein operational matrix, and so forth [16, 17], just to mention a few. Reference [18] represents a nice description of the importance of Abel's integral equations in mathematics as well as engineering, citing [19–23] for the various applications of Abel's integral equations.

The above stands for a sign that the theory of Abel's integral equations is developing. New methods for solving such a type of equations are demanded in this field. This paper presents a new method to describe the integral equation of Abel's type from the point of view of the Mikusinski operator of fractional order. In addition, we will give a solution to the integral equation of Abel's type by using the inverse of the Mikusinski operator of fractional order.

The remainder of this article is organized as follows. In Section 2, we shall express the integral equation of the Abel's type using the Mikusinski operator of fractional order and give the solution to that type of equation in the constructive way based on the inverse of the fractional-order Mikusinski

operator. Section 3 consists of two parts. One is the proof of the existence of the inverse of the fractional-order Mikusinski operator. The other is the computation of the solution to Abel's type integral equation. Finally, Section 4 concludes the paper.

2. Constructive Solution Based on Fractional-Order Mikusinski Operator

Denote the operation of Mikusinski's convolution by \otimes . Let \oplus be the operation of its inverse. Then, for $a(t), b(t) \in C(0, \infty)$, one has

$$a(t) \otimes b(t) = \int_0^t a(t-\tau) b(\tau) d\tau = c(t). \quad (4)$$

The inverse of the previous expression is the deconvolution, which is denoted by (see [24–26])

$$c(t) \oplus a(t) = b(t), \quad c(t) \oplus b(t) = a(t). \quad (5)$$

In (4) and (5), the constraint $a(t), b(t) \in C(0, \infty)$ may be released. More precisely, we assume that $a(t)$ and $b(t)$ may be generalized functions. Therefore, the Diract- δ function in the following is the identity in this convolution system. That is,

$$a(t) \otimes \delta(t) = \delta(t) \otimes a(t) = a(t). \quad (6)$$

Consequently,

$$a(t) \oplus a(t) = \delta(t). \quad (7)$$

Let l be an operator that corresponds to the function $1(t)$ such that

$$la(t) = 1(t) \otimes a(t) = \int_0^t a(\tau) d\tau. \quad (8)$$

Therefore, the operator l^2 implies

$$l^2 \iff 1(t) \otimes 1(t) = \int_0^t d\tau = \frac{t}{1}. \quad (9)$$

For $n = 1, \dots$, consequently, we have

$$l^n \iff \frac{t^{n-1}}{(n-1)!}, \quad (10)$$

where $0! = 1$.

The Cauchy integral formula may be expressed by using l^n , so that

$$l^n g(t) = \frac{t^{n-1}}{(n-1)!} \otimes g(t) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} g(\tau) d\tau. \quad (11)$$

Generalizing l^n to l^λ in (12) for $\lambda > 0$ yields the Mikusinski operator of fractional order given by

$$l^\lambda \iff \frac{t^{\lambda-1}}{(\lambda-1)!} = \frac{t^{\lambda-1}}{\Gamma(\lambda)}. \quad (12)$$

Thus, taking into account (12), we may represent the integral equation of Abel's type by

$$l^\lambda g(t) = \frac{t^{\lambda-1}}{\Gamma(\lambda)} \otimes g(t) = \int_0^t \frac{(t-\tau)^{\lambda-1}}{\Gamma(\lambda)} g(\tau) d\tau = f(t). \quad (13)$$

Rewrite the above by

$$l^\lambda g(t) = f(t). \quad (14)$$

Then, the solution to Abel's type integral equation (3) may be represented by

$$g(t) = l^{-\lambda} f(t), \quad (15)$$

where $l^{-\lambda}$ is the inverse of l^λ .

There are two questions in the constructive solution expressed by (15). One is whether $l^{-\lambda}$ exists. The other is how to represent its computation. We shall discuss the answers next section.

3. Results

3.1. Existence of the Inverse of Mikusinski's Operator of Order λ . Let \mathbf{G} and \mathbf{F} be two normed spaces for $g(t) \in \mathbf{G}$ and $f(t) \in \mathbf{F}$, respectively. Then, the operator l^λ regarding Abel's type integral equation (13) may be expressed by

$$l^\lambda : \mathbf{G} \longrightarrow \mathbf{F}. \quad (16)$$

The operator l^λ is obviously linear. Note that (3) is convergent [1]. Thus, one may assume that

$$m \leq \int_0^b \left| \frac{(t-\tau)^{\lambda-1}}{\Gamma(\lambda)} g(\tau) \right| d\tau \leq M, \quad (17)$$

where

$$m \geq 0, \quad M \geq 0. \quad (18)$$

Define the norm of $f(t)$ by

$$\|f(t)\| = \max_{0 < t < b} f(t). \quad (19)$$

Then, we have

$$\|l^\lambda g(t)\| \leq M \|f(t)\|. \quad (20)$$

The above implies that l^λ is bounded. Accordingly, it is continuous [27, 28].

Since

$$\|l^\lambda g(t)\| \geq m \|f(t)\|, \quad (21)$$

$l^{-\lambda}$ exists. Moreover, the inverse of l^λ is continuous and bounded according to the inverse operator theorem of Banach [27, 28]. This completes the proof of (15).

3.2. *Computation Formula.* According to the previous analysis, $I^{-\lambda}$ exists. It actually corresponds to the differential of order λ . Thus,

$$g(t) = I^{-\lambda} f(t) = \frac{d^\lambda f(t)}{dt^\lambda}. \tag{22}$$

In (13), we write $\int_0^t ((t-\tau)^{\lambda-1}/\Gamma(\lambda))g(\tau)d\tau = f(t)$ by

$$\int_0^t (t-\tau)^{\lambda-1} g(\tau) d\tau = \Gamma(\lambda) f(t). \tag{23}$$

Following [29, p. 13, p. 527], [30], therefore,

$$\begin{aligned} g(t) = I^{-\lambda} f(t) &= \frac{\sin(\pi\lambda)}{\pi} \frac{d}{dt} \int_0^t \frac{\Gamma(\lambda) f(u)}{(t-u)^{1-\lambda}} du \\ &= \frac{\Gamma(\lambda) \sin(\pi\lambda)}{\pi} \left[\frac{f(0)}{t^{1-\lambda}} + \int_0^t \frac{f'(t) dt}{(t-u)^{1-\lambda}} \right]. \end{aligned} \tag{24}$$

Since

$$\frac{\sin(\pi\lambda)}{\pi} = \frac{1}{\Gamma(\lambda)\Gamma(1-\lambda)}, \tag{25}$$

we write (24) by

$$g(t) = \frac{1}{\Gamma(1-\lambda)} \left[\frac{f(0)}{t^{1-\lambda}} + \int_0^t \frac{f'(t) dt}{(t-u)^{1-\lambda}} \right]. \tag{26}$$

In the solution (26), if $f(0) = 0$, one has

$$g(t) = \frac{1}{\Gamma(1-\lambda)} \int_0^t \frac{f'(t) dt}{(t-u)^{1-\lambda}}, \tag{27}$$

which is a result described by Gelfand and Vilenkin in [9, Section 5.5].

Note that Mikusinski’s operational calculus is a tool usually used for solving linear differential equations [24–26], but we use it in this research for the integral equation of the Abel’s type from a view of fractional calculus. In addition, we suggest that the idea in this paper may be applied to studying other types of equations, for instance, those in [31–50], to make the possible applications of Mikusinski’s operational calculus a step further.

4. Conclusions

We have presented the integral equation of Abel’s type using the method of the Mikusinski operational calculus. The constructive representation of the solution to Abel’s type integral equation has been given with the Mikusinski operator of fractionally negative order, giving a novel interpretation of the solution to Abel’s type integral equation.

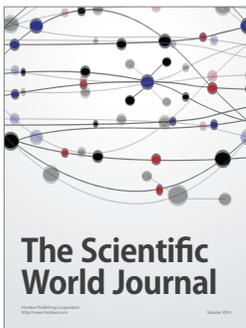
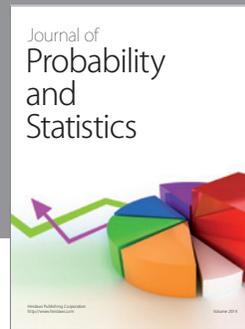
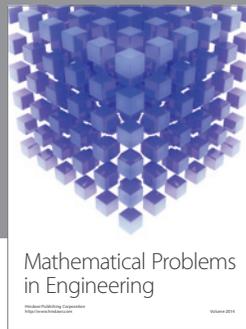
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