Optimal Control of Crop Irrigation based on the Hamilton-Jacobi-Bellman Equation

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Abstract

Water management in agriculture is a key issue due to the increasing problem of water scarcity worldwide. Based on the recent progress in the dynamic modeling of plant growth in interaction with the water resource, our objective is to study the optimal control problem of crop irrigation. For this purpose, we first describe the LNAS model for sugar beet growth, driving the dynamics of both plant biomass and soil water reserve. We then introduce the utility function corresponding to the farmer’s profit and derive the value function from the Hamilton-Jacobi-Bellman (HJB) equation. Then a backward finite-difference scheme is implemented to solve the HJB equation. It is proved to converge under a proper Courant-Friedrichs-Lewy condition for the discretization step. A few numerical simulations are provided to illustrate the resolution.

1 Introduction

Considering the world’s increasing need of agriculture products, agriculture water management is inevitably a crucial point. The key challenge is not only to address issues related to effective and sustainable use of water in agriculture but also to take into account the water scarcity and to preserve the water resources. Important research efforts in agronomy have been devoted to breed new varieties which are more resistant to water stress (see for example [2] for sunflower) or to develop more refined irrigation techniques. For most irrigation scheduling methods, measured or estimated soil moisture status are regarded as a criterion to determine when stress appears and when irrigation becomes necessary [8]. More recently, with the progress in plant-soil interaction modeling, a few attempts have been made to improve water use based on the implementation of optimization or control optimal techniques. In [15], the gradient of the profit function is computed with adjoint modeling or automatic differentiation, but it proves to be limited from a numerical perspective, since gradient-based methods fail to converge to global optima due to the strong non-convexity of the problem. In [13] or [14], heuristic methods are used to solve the optimization problem resulting from the parameterization of the control function. In this paper, we propose a novel approach based on a new model of crop growth in interaction with the water resource. A criterion corresponding to the farmer’s profit is introduced, taking into account both the price of yield and the cost of the water resource. The numerical resolution of the optimal control problem is achieved through the discretization of the Hamilton-Jacobi-Bellman (HJB) equation.

In Section 2, the LNAS model for sugar beet growth, which drives the dynamics of both plant biomass and soil water reserve, is introduced. In Section 3, the optimal control problem is formulated and the HJB equation is derived for the value function. The numerical scheme for the resolution of the HJB equation is then presented in Section 4 with the investigation of the convergence conditions. In Section 5, the results of a few test cases are presented, and the perspectives of this work are given in Section 6.

2 LNAS model

The LNAS model (Log Normal Accumulation and Senescence) is a continuous-time model that describes the equilibrium between the plant growth (here we will consider sugar beet) and the water level of the soil. This model is rather simple, involving only a few essential parameters and characterized by several important biological processes like plant transpiration, biomass production by photosynthesis, biomass allocation and plant senescence. In this study, we use the LNAS model to describe sugar beet growth in interaction with the water resource. Another time scale, the thermal time at day $t$, $\tau_t$, defined as an accumulated temperature (in °C days) is used to describe plant development and organ expansion:

$$\tau_t := \int_0^t \max(0, T(s) - T_b) ds$$

with $T_b$ a base temperature (here we take $T_b = 0$). Temperature $T(t)$ is given on a daily basis and will
be considered as a piecewise constant function.

Let \( q \geq 0 \) denote the total biomass of the plant, \( q_r \geq 0 \) the accumulated biomass for leaves, \( q_l \geq 0 \) the biomass of green leaves and \( r \geq 0 \) the soil water reserve that is available for plants. The plant-soil interaction is regulated by the following equations detailed in the next subsections:

\[
\begin{aligned}
\dot{r}(t) &= -E(t) - T(t) + P(t) + u(t), \\
\dot{q}(t) &= \mu_S p R(t) (1 - e^{-\xi(t)q(t)}) \min \left( \frac{r(t)}{r_0}, 1 \right), \\
\dot{q}_l(t) &= \gamma(\tau_i) \dot{q}(t) - \dot{q}_r(t), \\
q_g(t) &= (1 - \phi(\tau_i))q_l(t)
\end{aligned}
\]  

\[ (S) \]

\[ (2.2) \]

2.1 Soil:

\[ (2.3) \]

\[ \dot{r}(t) = -E(t) - T(t) + P(t) + u(t). \]

The first equation concerns the balance of soil water level which is a simple version of reservoir models (see for example [12, 9] for more sophisticated versions). The soil is considered as a single layer of constant depth \( D \) (expressed in meter). If \( \theta \) (in mm/m) denotes the volumetric soil water content, we have \( r(t) = D \times (\theta(t) - \theta_{\text{min}}) \), with \( \theta_{\text{min}} \) the minimum volumetric content admissible (also called permanent wilting point). We also consider that drainage keeps the volumetric water content below a maximum \( \theta_{\text{max}} \) (called field capacity). \( r(t) \) is then bounded between 0 and \( r_{\text{max}} \). \( T(t) \) (in mm/day) is the plant transpiration at time \( t \) and \( E(t) \) (in mm/day) is the soil evaporation, \( P(t) \) (in mm/day) is the precipitation. The irrigation \( u(t) \) is thus the control variable.

The evapotranspiration mechanism takes place only if water is available, i.e. if \( r(t) \geq 0 \), then we have:

\[ (2.4) \]

\[ T(t) = ET_0 K(t) C_p(t) f_w(t). \]

\( ET_0 \) (in mm) is a reference evapotranspiration determined by meteorological conditions, \( K(t) = K_0 (1 - e^{-LAI(t)}) \) is a coefficient that depends on the crop and on the growth stage and \( C_p(t) = 1 - e^{-kLAI(t)} \) is the portion of ground covered by crop canopy. The LAI (t) factor (no unit) involved in \( C_p \) and \( K \) is the Leaf Area Index, it is defined by :

\[ (2.5) \]

\[ LAI(t) = d(1 - \phi(t)) \frac{q_l(t)}{e}, \]

with \( d \) (no unit) the field density, \( e \) (in g) the surfacic foliar mass and \( \phi(t) \) the percentage of senescent leaves (see definition below). Finally, the last term \( f_w(t) \) is a factor to account for water stress \( (f_w(t) = 1 \text{ when there is no stress, and } 0 \text{ for a maximal stress}) \) which modulates the effect of the transpiration. We take:

\[ (2.6) \]

\[ f_w(t) = \min \left( \frac{r(t)}{r_0}, 1 \right). \]

\( r_0 \) is the threshold level under which transpiration is limited and we use \( r_0 = 0.4 \times r_{\text{max}} \) (as proposed in [12]).

For the soil evaporation the equation is similar:

\[ (2.7) \]

\[ E(t) = ET_0 (1 - C_p(t)) f_w(t). \]

2.2 Biomass: The second equation describes the neat biomass production by photosynthesis, we have:

\[ (2.8) \]

\[ \dot{q}(t) = \mu S_p R(t) (1 - e^{-\xi(t)q(t)}) f_w(t), \]

with \( \xi(t) = \frac{K}{\theta_{\text{max}}} (1 - \phi(\tau_i)) \).

\( \mu \) denotes the radiation use efficiency, \( R(t) \) (in MJ/day) is the incident photosynthetically active radiation, \( k \) is the Beer-Lambert extinction coefficient and \( S_p \) is a characteristic surface in m². \( 1 - e^{-\xi(t)q(t)} \) is a factor which represents the portion of intercepted radiation. Finally the last term \( f_w(t) \) in (2.8) is the water stress: if the water level is below a threshold \( r_0 \), a “stress” will affect the production by decreasing the radiation use efficiency [7] proportionally to the ratio \( \frac{r(t)}{r_0} \).

2.3 Allocation: The third equation rules the allocation of biomass production between the root and the leaves. A rigorous model for the allocation would require the notion of demand function for each plant organ (as in the GreenLab model [11]), but here we describe the allocation at compartment level with an empirical function which depends on the thermal time \( \gamma(\tau) \). \( \gamma(\tau) \) is precisely the fraction of biomass production given to leaves \( (1 - \gamma \text{ is for root}) \). We use a lognormal cumulative distribution \( F_\gamma \) with parameters \( (\mu_1, \sigma_1) \in \mathbb{R}_+^2 : \gamma(\tau) = \gamma_1 + (\gamma_f - \gamma_1) F_\gamma(\tau) \). We then obtain:

\[ (2.9) \]

\[ \dot{q}_l(t) = \gamma(\tau_i) \dot{q}(t) = \dot{q}(t) - \dot{q}_r(t). \]

2.4 Senescence: The last equation concerns the senescence process. Let \( \phi(\tau) \) be the percentage of senescent leaves, described as above by a lognormal cumulative distribution of parameters \( (\mu_2, \sigma_2) \in \mathbb{R}_+^2 \). \( \tau_{\text{sen}} \) is the thermal time corresponding to the beginning of the senescence process:

\[ \phi(\tau) = \begin{cases} F_\phi(\tau - \tau_{\text{sen}}) & \text{if } \tau > \tau_{\text{sen}}, \\ 0 & \text{otherwise}. \end{cases} \]

The relation between \( q_g \) and \( q_l \) is then:

\[ (2.10) \]

\[ q_g(t) = (1 - \phi(\tau_i))q_l(t). \]
3 Problem formulation

Our objective is to optimize the irrigation strategy under given meteorological conditions, so that the utility (profits) is maximized. For sugar beet, the utility is the difference between the price of the yield (the root mass) and the cost of irrigation water. We first introduce some notations.

As the evolution of $q$ and $q_\tau$ can be directly deduced from that of $q_t$, it is possible to reduce the dynamic system from 4 to 2 state variables. The presentation of the 4 state variables in section 2 was done for the sake of clarity. Let $X = (X_s)_{s \geq 0}$ be a controlled process, $\mathbb{R}^2$-valued such that $\forall s \in [0, T]$, $X_s = (q_t(s), r(s))$ and such that:

$$
\frac{dX_s}{ds} = b(s, X_s, u_s),
$$

$b$ summarizes $q_t$ and $r$ in the LNAS model:

$$
b(s, X_s, u_s) = \left(\gamma(s)R_s(s) \beta \right) (1 - e^{-\xi(s)}q_t(s)) f_w(s) - E(s) - T(s) + P(s) + u_s.
$$

The control $U = (u_s)_{0 \leq s \leq T}$ is a progressively measurable process with values in the control space $A$, a compact interval of $\mathbb{R}_+$. $b$ defined on $[0, T] \times \mathbb{R}^2 \times A$, satisfies the conditions such that a unique strong solution exists for the differential equation (3.11). It is the case when $b(s, x, u)$ is a Lipschitz function on $(s, x)$ uniformly in $u$, and when $U$ is square integrable (which is the case when $A$ is a compact space). Let $\mathcal{A}$ be the set of all admissible control functions $U$. Given the initial conditions $(t, x) \in [0, T] \times \mathbb{R}^2$ and a control $U \in \mathcal{A}$, we note $\{X_s^{t, x}, t \leq s \leq T\}$ the unique strong solution of (3.11) with initial condition $x$ at time $s = t$.

Let $J$ be the utility function and $f$ the instantaneous utility function, we have:

$$
J(t, x, U) = \int_t^T f(s, x_s, u_s)ds = \int_t^T (p q_t(s) - C(u_s))ds.
$$

for all $(t, x) \in [0, T] \times \mathbb{R}^2$, where $p$ is the sugar beet price at harvest time, $q_t$ is the instantaneous production of root and $C$ is the cost production. We choose for $C$ a quadratic form in the first place (we also consider the linear case by setting $\beta = 0$):

$$
C(u_s) = \alpha u_s + \frac{\beta}{2} u_s^2.
$$

$\alpha$ is the unit cost of water used for irrigation and $\beta$ a penalization term. We consider that the unit cost is constant over time. This assumption is realistic as water price is generally set by regulatory rules and does not depend on supply/demand balance. $q_\tau$ and $C$ are Borel functions and are bounded. We introduce the value function which is the maximal achievable utility through all admissible controls:

$$
v(t, x) = \sup_{U \in \mathcal{A}} J(t, x, U).
$$

For an initial state $(t, x)$, we regard $U \in \mathcal{A}$ as an optimal control if $v(t, x) = J(t, x, U)$. It is well-known that using Bellman’s optimality principle (see [6], Chapter 1) we can show that the value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation which gives the local behavior of $v$ on $[0, T] \times \mathbb{R}^2$:

$$
-\frac{\partial v}{\partial t}(t, x) - \sup_{u \in \mathcal{A}} [L^u v(t, x) + f(t, x, u)] = 0,
$$

where $L^u$ is the infinitesimal generator of $X$:

$$
L^u v = b(s, x, u) D_x v,
$$

with $D_x v$ the gradient of $v(s,.)$ (with respect to $x$).

On $[0, T] \times \mathbb{R}^2$, it is frequently stated as:

$$
-\frac{\partial v}{\partial t}(t, x) - H(t, x, D_x v(s, x)) = 0,
$$

where for all $(s, x, p) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$:

$$
H(s, x, p) = \sup_{u \in \mathcal{A}} \left[b(s, x, u)p + f(s, x, u)\right],
$$

$H$ is called the Hamiltonian of the control problem. As the control takes values in a compact, $H$ is finite. There is also a terminal condition to this equation:

$$
v(T, x) = 0 , x \in \mathbb{R}^2,
$$

which results from the definition of $v$.

Finding a solution to the HJB equation is far from easy. Generally, we have what is known as a verification theorem which consists in proving that a smooth solution of the HJB equation (3.16) coincides under suitable sufficient conditions with the value function. The verification theorem provides jointly the optimal control. We can find such a theorem in Fleming and Soner [6]. However, in our case, nothing ensures that the regularity conditions are met by $v$ and notably the classical $C^2$-condition is too strong. However, we can define weak solutions to the HJB equation, weak in the sense that we relax the regularity condition. The suitable class of weak solutions is that of the viscosity solutions. A deep review of this notion can be found in [4]. We only mention here that, in this framework, the value...
function $v$ is the unique continuous solution of HJB. The only regularity condition we expect for $v$ is to be locally bounded, which results from its definition.

But $v$ is rarely given by an analytical formula. In our case, because of the complexity of the dynamic system, it is not possible to find such a formula. Therefore, a numerical scheme is implemented in order to find an approximate solution. We discretize the nonlinear HJB equation with the finite difference method. This scheme has been initially proposed by Crandall and Lions (see [5]) for first order Hamiltonians.

4 Resolution

Before describing the numerical scheme, we focus on the Hamiltonian in order to find the instantaneous optimal control.

4.1 Optimal control: We have for all $(t, x) \in [0, T] \times \mathbb{R}^2$, with $x = (q, r)$:

$$H(t, x, D_x v(t, x)) = \sup_{u \in A} \left[ \frac{\partial v}{\partial q}(t, x) \mu_S R(t) \left(1 - e^{-\xi(t)q(t)} \right) \min \left( \frac{r(t)}{\theta_0}, 1 \right) + \frac{\partial v}{\partial r}(t, x) \{-E(t) - T(t) + P(t) + u\} + pq(t) - C(u) \right]$$

By only considering the terms involving $u$, we have to find:

$$\sup_{u \in A} \left[ -\beta u^2 + u \left( \frac{\partial v}{\partial r}(t, x) - \alpha \right) \right].$$

The control space is $A = [0, u_{\max}]$ (which means that we set a maximum irrigation quantity $u_{\max}$). Therefore, if we take a linear cost for the irrigation with $\beta = 0$ and $\alpha = p_w$, where $p_w$ is the unitary cost of water, then the instantaneous optimal irrigation is:

$$u^*(t, x) = \begin{cases} 0 & \text{if } \partial_r v(t, x) < p_w \\ u_{\max} & \text{if } \partial_r v(t, x) \geq p_w. \end{cases}$$

In this on-off control system, the criterion to trigger irrigation is the following: if the marginal rise of the value function obtained by changing the water level, $v$, is above the unitary cost of water, then the irrigation is triggered, which makes sense. The drawback of this control is that we do not choose the quantity of irrigated water, given that the maximal output $u_{\max}$ is fixed initially. Thus irrigation could be decided even though the marginal gain, i.e., $\partial_r v - p_w$, is low.

If we take a quadratic cost with $\beta > 0$ the optimal irrigation is given by:

$$u^*(t, x) = \left[ \frac{\partial_r v(t, x) - p_w}{\beta} \right]_+.$$
of \( v \) on the mesh at time step \( n+1 \).

The initial condition of this backward scheme is therefore:

\[
(4.25) \quad v^{N}_{j,k} = 0, \quad \forall (j, k) \in [0, J] \times [0, K]
\]

**Remark 4.1.** We also have to approximate the optimal control \( u^*(t, x) \) at point \( x_{j,k}^{n+1} \), the approximate value will be denoted \( u^*_{j,k} \) (because it is the control to apply between time step \( n \) and \( n+1 \) when we are at the node \((j, k)\)). We will use, when it is possible, a second order estimation for \( \partial_x v(t, x) \):

\[
(4.26) \quad \partial_x v = \frac{v^{n+1}_{j,k+1} - v^{n+1}_{j,k-1}}{2\Delta x}
\]

for \( k \neq 0, k \neq K \) and \( n \neq N \).

An important point to note is that when we are at the node \( x_{j,k}^{N} \), there is no control possible. We will hence fix \( u^*_{j,k} = 0 \).

**Proposition 4.1.** The numerical scheme leads to the following algorithm:

1. If \( n = N \), then \( \forall (j, k) \in [0, J] \times [0, K] \), \( u^*_{j,k} = 0 \) and \( v^{N}_{j,k} = 0 \).

2. If \( n \in [0, N-1] \), suppose that we know all the values on the mesh at time step \( n+1 \), i.e. the set of \( v^{n+1}_{j,k} \), then for all \((j, k)\) we first compute the approximate value of the optimal control \( u^*_{j,k} \), afterwards we compute \( v^n_{j,k} \) by (4.24).

Implementing this algorithm gives us at each current point, a 3-uplet \((x^n_{j,k}, u^{n+1}_{j,k}, v^n_{j,k})\) that we store in a global mapping. \( x^n_{j,k} \) is the current point of the algorithm, \( u^*_{j,k} \) is the control to be applied at \( x^n_{j,k} \) and \( v^n_{j,k} \) is the approximate value of the value function at this same point.

**Proposition 4.2.** Once \((x^n_{j,k}, u^{n+1}_{j,k}, v^n_{j,k})\) is computed \( \forall (j, k) \in [0, J] \times [0, K] \) and for all \( n \in [0, N] \), the following algorithm is performed in order to retrieve the optimal control when the starting point is \( X_0 = (q_0, r_0) \):

1. First step - We determine \( \tilde{X}_0 \) corresponding to the node \((j_0, k_0)\) which is the closest to \( X_0 \) on the mesh. We then take the associated control \( u^0_{j_0,k_0} \) which is denoted by \( u_0 \). An Euler explicit scheme on (3.11) with time step \( \Delta t \) is implemented in order to make the dynamic system evolve:

\[
(4.27) \quad X_1 = X_0 + b(0, X_0, u_0) \Delta t.
\]

2. General step - Suppose that until \( n \in [1, N-1] \) a real dynamic system trajectory \((X_0, X_1, ..., X_n)\), an approximate trajectory \((\tilde{X}_0, ..., \tilde{X}_n)\) and a control vector \((u_0, ..., u_n)\) are built. The Euler scheme is implemented to make the dynamic system evolve:

\[
(4.28) \quad X_{n+1} = X_n + b(t^n, X_n, u_n) \Delta t.
\]

Then we determine \( \tilde{X}_{n+1} \), the value of the mesh which is the closest to \( X_{n+1} \) and the corresponding node \((j_{n+1}, k_{n+1})\). The associated control \( u^*_{j_{n+1},k_{n+1}} \) is then stored in the global mapping which is denoted by \( u_{n+1} \) for clarity.

3. Final step - For \( n = N \), once \( X_N \) is computed, we set \( u_N = 0 \).

This algorithm allows the step by step computation of a piecewise constant optimal control \( \tilde{U} = (u_0, u_1, ..., u_N) \) of the problem for the initial state \( X_0 \).

**Remark 4.2.** The control that we apply at the point \( X_n \) is actually the optimal control corresponding to the point \( \tilde{X}_n \). In order to assess the computation error, we use the equality : \( v(0, X_0) = J(0, X_0, \tilde{U}) \). We will compare \( v_0 := v^0_{j_0,k_0} \), an approximation of \( v(0, X_0) \) given by the numerical scheme, with \( \tilde{J}_N \), the approximation of the utility function (3.13), given when the dynamic system evolves with the control \( \tilde{U} \):

\[
(4.29) \quad \tilde{J}_N := \sum_{j=0}^{N} (p_{f}(j \Delta t) - p_{w} u_j).
\]

We observed that the difference between these two values tended to 0 when the space step sizes tended to 0, as expected.

**4.3 Convergence of the numerical scheme:** In order to have an estimation of the convergence rate between \( v^n \) and \( v \), the scheme has to be consistent, stable and monotone (see [5] for details). The consistency is straightforward, since when the time and space step sizes tend to 0 in (4.24) the scheme gives back the HJB equation. The monotonicity refers to the fact that \( S \) has to be an increasing function of each coordinate of the vector \( V^{n+1} \). It is obvious that the scheme is increasing for the variables \( v^{n+1}_{j,k}, v^{n+1}_{j-1,k}, v^{n+1}_{j,k+1} \) and \( v^{n+1}_{j,k-1} \), because \( b^+_{x} \) and \( b^+_{r} \) are positive functions.

The scheme is increasing for \( v^{n+1}_{j,k} \) if and only if the following CFL condition is satisfied:

\[
(4.30) \quad 1 - \frac{\Delta t}{\Delta x} (b^+_{x}(q^i, r^k, u^*) d_{j\neq j} + b^+_{r}(q^i, r^k, u^*) d_{j=0})
\]

\[
- \frac{\Delta t}{\Delta y} (b^+_{y}(q^i, r^k, u^*) d_{k\neq k} + b^+_{y}(q^i, r^k, u^*) d_{k=0}) \geq 0.
\]
Thus we have to choose $\Delta t$, $\Delta q$ and $\Delta r$ such that the previous inequality is satisfied. In this case, the main result of Crandall and Lions in [5] can be applied and we obtain:

**Proposition 4.3.** Using the proposed finite difference scheme (4.24) and if the CFL condition is satisfied, then there exists a constant $c$ such that:

\[
|v(t^n, q^j, r^k) - v_{i,j,k}^n| \leq c \sqrt{\Delta t}
\]

for $0 \leq n \leq N$ and for all $(j, k) \in [0, J] \times [0, K]$.

As a consequence, the optimal control found is a piecewise constant approximation of the real control which allows us to approach the maximum of the value function with an error of order $\sqrt{\Delta t}$. Notice that obtaining a piecewise constant control is relevant as the irrigation is usually handled on a daily basis.

5 Numerical results

The LNAS model and the numerical algorithms were implemented in the C++ platform PYGMALION, which is developed to facilitate the simulation, analysis, identification and optimization of plant growth models [3]. The test case corresponds to the optimal irrigation of sugar beet crop, with model parameterization and data coming from the study presented in [11] with an experimental site in Loiret (North of France).

We use daily climatic data (photosynthetically active radiation, average daily temperature, and potential evapotranspiration) provided by the French meteorological advisory services (Météo France) from March to June. However, we set the precipitation $P(t)$ to 0 in order to observe how our model reacts in a drought period.

Given the climate data, the following time and space step sizes which verify the CFL condition 4.30 are used: $\Delta t = 0.1$, $\Delta q = 2$ and $\Delta r = 2$. We also set $u_{\text{max}} = 5$, $\alpha = 0.7$, $p = 1.5$. The initial state is $X_0 = (q_l(0), r(0)) = (0.8, 140)$.

5.1 Linear cost: We first consider a linear cost $C(u) = \alpha u$ by setting $\beta = 0$. Fig. 1 and 2 give respectively the evolution of the available water level $r(t)$ and the evolution of the root biomass $q_r(t)$ when we use the optimal control found by our resolution method.

We note that there is no irrigation until the water level drops below the threshold. After a short latency period of 4 days when $r(t)$ reaches its minimum, the irrigation begins (around the 33$^{rd}$ day) and maintains $r(t)$ slightly above $r_0$ then stops when the senescence process begins (around the 87$^{th}$ day).

The optimal irrigation control is then compared with two other strategies: a naive control which consists in keeping the water level $r(t)$ at the threshold level $r_0$ (and actually corresponds to classical decision-aid tools for irrigation) and no irrigation at all.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Total water use (in mm)</th>
<th>Final root biomass (in g)</th>
<th>Final utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>no irrigation</td>
<td>0.0</td>
<td>49.1</td>
<td>73.7</td>
</tr>
<tr>
<td>optimal</td>
<td>203.5</td>
<td>181.7</td>
<td>130.1</td>
</tr>
<tr>
<td>naive</td>
<td>261.9</td>
<td>198.3</td>
<td>114.1</td>
</tr>
</tbody>
</table>

Table 1: Comparison of different irrigation strategies.

We observe that the final utility of the optimal strategy is clearly better than the naive one, as we have consumed far less irrigation water for almost the same final root biomass.
5.2 Quadratic cost: In this section, we conduct two tests based on a quadratic cost $C(u)$ with $\alpha = 0.7$: the first one with $\beta = 0.1$ and the other one with $\beta = 0.3$ in order to understand how the model reacts to the penalized cost. We compare the results with the linear case $\beta = 0$ and, as above, we show the evolution of $r(t)$ and $q_r(t)$ in Fig. 3 and 4. The other parameters remain the same i.e. $u_{max} = 5$, $p = 1.5$ and $X_0 = (q(0), r(0)) = (0.8, 140)$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Total water use (in mm)</th>
<th>Final root biomass (in g)</th>
<th>Final utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>203.5</td>
<td>181.7</td>
<td>130.1</td>
</tr>
<tr>
<td>0.1</td>
<td>184.4</td>
<td>170.4</td>
<td>106.4</td>
</tr>
<tr>
<td>0.3</td>
<td>65.3</td>
<td>91.3</td>
<td>82.1</td>
</tr>
</tbody>
</table>

Table 2: Comparison of the results for different quadratic costs.

Figure 3: Evolution of $r(t)$ for different values of $\beta$.

Figure 4: Evolution of $q_r(t)$ for different values of $\beta$.

6 Discussion

This study shows that the optimal problem of crop irrigation based on plant-soil dynamic modeling can be solved with the HJB equation. It is a notable and encouraging progress from a mathematical and numerical point of view compared to the previous approaches developed for this purpose ([13], [14], [15]). However, several extensions of this work could be investigated to broaden the scope of the potential applications of this method.

First, we only considered in this paper the deterministic case, which implies the assumption that the climatic scenario is known in advance. It may be a strong limitation, even though the method can be applied with an average of the historical climatic data and updated on a day-to-day basis during the crop growth cycle according to the observed climate variables. Therefore, it would be interesting to extend the method to a stochastic case, for which the HJB equation is also well-adapted and proper numerical schemes can be derived with known convergence properties ([10], [1]).

Moreover, another important case for applications would be the introduction of constraints. They are rather important in practical irrigation conditions since farmers have to face regulatory rules (for example water quotas) or logistical constraints (resulting from the irrigation material or manpower). The problem could potentially be solved by introducing some auxiliary variables, which would increase the dimension of the state vector.

Finally, the proposed model remains quite simple, and could clearly be improved for practical applications. There exist other processes that could be impacted by the water stress (like senescence or allocation [7]), which by consequence may induce stronger interactions in the model. Likewise, the soil model could benefit from the differentiation of several soil layers as in the Pilote model [12]. All these model refinements imply an increase of the number of state variables. Therefore, the impact on the performance of the method should be evaluated.

References


