NEW ALGORITHMS FOR NUMERICAL SOLUTION OF NON-LINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF THIRD ORDER USING HAAR WAVELETS

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ABSTRACT: This paper deals with the extension of earlier work [3] (designed for Fredholm and Volterra integral equations), [16] (designed for Fredholm and Volterra integro-differential equations of first-order) and [4] (designed for Fredholm and Volterra integro-differential equations of second-order) to third-order nonlinear Fredholm as well as nonlinear Volterra integro-differential equations. The approach used in this paper make use of hidden valuable dimensions of Haar wavelets. The proposed method provides strong generic ground, thus yielding solution of both Fredholm and Volterra integro-differential equations of third-order and second kind. Four numerical examples are used to illustrate the accuracy of the proposed method.

Keywords: Haar wavelets, Collocation method, Third order Fredholm integro-differential equations of second kind, Third order Volterra integro-differential equations of second kind.

INTRODUCTION

Integro-differential equations (IDEs) have many applications in natural sciences and engineering. Numerous applications of IDEs can be found in [16,17,10,13,7,11] and the references therein. In many cases analytical solution of IDEs is herculean task, therefore numerical analysts has focused on exploring accurate and efficient numerical methods. Recent work in the context of solution of these type of problems include differential transform method [6, 8], Newton Tau method [9], decomposition method [1], Haar wavelets method [12], hybrid Legendre polynomials and block-pulse functions approach [14], triangular functions [6], single-term Walsh series method [15], block-pulse functions [5], wavelet-Galerkin method [2] and compact finite difference method [19] etc.

The motivation of the present paper is to extend the scope of earlier work [3,16,4] in order to propose a new method with reasonable accuracy and efficiency. The added advantage of the new method in comparison to Haar wavelet method [12] is that it does not involve numerical integration. The integrand is approximated with the Haar wavelet basis and exact integration is performed. In the present work, we will consider two types of IDEs. The first type is third-order Fredholm integro-differential equation of second kind and which is given as follows:

\[
\begin{align*}
\int_0^x K(x,t,u(t),u'(t),u''(t))dt \\
\end{align*}
\]

and the second type is third-order Volterra integro-differential equation of second kind given as follows:

\[
\begin{align*}
\int_0^x K(x,t,u(t),u'(t),u''(t))dt \\
\end{align*}
\]

subject to the initial conditions \(u(0) = u_0, u'(0) = u_1, u''(0) = u_2\). The kernel function \(K(x,t,u(t),u'(t),u''(t))\) given in the above equations is a nonlinear function defined on \([0,1] \times [0,1]\). \(f(x)\) and \(g(x)\) are known functions defined on \([0,1]\) and \(u(x)\) is the unknown function representing solution of the IDEs.

The organization of the rest of the paper is as follows. In Section 2, Haar wavelets and their integrals are described. In Section 3 formulation of the method based on Haar wavelets is defined for Fredholm and Volterra IDEs. Numerical results are reported in Section 4 and conclusions are drawn in Section 5.

2 Haar Waves

The scaling function for the family of Haar wavelets is defined on the interval \([0,1]\) and is given as follows:

\[
h_1(x) = \begin{cases} 
1 & \text{if } x \in [0,1) \\
0 & \text{elsewhere}
\end{cases}
\]

The mother wavelet for Haar wavelets family is also defined on the interval \([0,1]\) and is given as:

\[
h_2(x) = \begin{cases} 
1 & \text{if } x \in \left[0, \frac{1}{2}\right) \\
-1 & \text{if } x \in \left[\frac{1}{2}, 1\right) \\
0 & \text{elsewhere}
\end{cases}
\]

All the other functions in Haar wavelet family are defined as:

where

\(\alpha = k/m, \beta = (k+0.5)/m, \gamma = (k+1)/m, i = 3,4,\ldots,2M\).

The integer \(m = 2^j\) where \(j = 0,1,\ldots,J, M = 2^j\) and integer \(k = 0,1,\ldots,m-1\). The integer \(j\) indicates the level of the wavelet and \(k\) is the translation parameter. The maximal level of
The Haar wavelet functions are given by \( i = m+k+1 \). The Haar wavelet functions are orthogonal to each other because of the following relation:

\[
b \int h(x) \, h_k(x) \, dx = 0, \quad \text{whenever } j \neq k.
\]

Any function \( f(x) \) which is square integrable in the interval \((0,1)\) can be expressed as an infinite sum of Haar wavelets as follows:

\[
f(x) = \sum_{i=1}^{\infty} a_i \, h_i(x).
\]

The above series terminates at finite terms if \( f(x) \) is piecewise constant or can be approximated as piecewise constant during each subinterval. The following notations are introduced:

\[
p_{i,1}(x) = \int_{0}^{x} h_i(z) \, dz, \quad p_{i,2}(x) = \int_{x}^{\infty} h_i(z) \, dz.
\]

These integrals can be evaluated using Eq. (5) and are given as follows:

\[
p_{i,1}(x) = \begin{cases} x-\alpha & \text{for } x \in [\alpha, \beta) \\ \gamma-x & \text{for } x \in [\beta, \gamma) \\ 0 & \text{elsewhere} \end{cases}
\]

\[
p_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2 & \text{for } x \in [\alpha, \beta) \\ \frac{1}{4n^2} - \frac{1}{2}(\gamma-x)^2 & \text{for } x \in [\beta, \gamma) \\ 0 & \text{elsewhere} \end{cases} \quad \text{for } x \in [\gamma, 1)
\]

3 Numerical Method

The collocation points for Haar wavelets approximations is defined as follows:

\[
x_p = (p-0.5)/2M, \quad p = 1,2,\ldots,2M.
\]

\[
t_q = (q-0.5)/N, \quad q = 1,2,\ldots,2N.
\]

Any square integrable function \( f(x) \) can be approximated using Haar wavelets as follows:

\[
f(x) = \sum_{i=1}^{2M} a_i \, \text{haa} \_r_i(x).
\]

Substituting the collocation points given in Eq. (12), we obtain the following linear system of equations:

\[
F(x_p) = \sum_{i=1}^{2M} a_i \, \text{haa} \_r_i(x_p), \quad p = 1,2,\ldots,2M.
\]

This is a \( 2M \times 2M \) linear system of equations whose solution for the unknown coefficients \( a_i \) can be calculated using the following theorem.

**Theorem 1.** The solution of the system (15) is given as follows:

\[
a_i = \frac{1}{2M} \sum_{j=1}^{2M} (x_j),
\]

\[
a_i = \frac{1}{\rho} \left( \sum_{p=\beta}^{\gamma} f(x_p) - \sum_{p=\rho+1}^{\gamma} f(x_p), \quad i = 2,3,\ldots,2M. \right.
\]

Where \( \alpha = (\sigma-1)+1, \beta = \rho(\sigma-1)+\rho/2, \gamma = \rho \sigma, \rho = 2M/\tau, \sigma = i-\tau, \tau = 2\log_2(\rho^{-1}) \)

**Proof.** See [3].

A real-valued function \( F(x) \) of two real variables \( x \) and \( t \) can be approximated using two-dimensional Haar wavelets basis as:

\[
F(x,t) \approx \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{ij}(x) \, \text{haa} \_r(t), \quad p = 1,2,\ldots,2M, \quad q = 1,2,\ldots,2N
\]

In order to calculate the unknown coefficients \( b_{ij} \), the collocation points defined in Eqs. (12) and (13) are substituted in Eq. (18). Hence, we obtain the following \( 4M \times 4M \) linear system with unknowns \( b_{ij} \):

\[
F_{pq} = \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{ij} \, \text{haa} \_r(x_p) \, \text{haa} \_r(t_q), \quad p = 1,2,\ldots,2M, \quad q = 1,2,\ldots,2N
\]

where for simplicity we have introduced the notation \( F_{pq} \) for the value of \( F(x_p, t_q) \). The solution of this system can be calculated using the following theorem.

**Theorem 2.** The solution of the system (19) is given below:

\[
b_{i,1} = \frac{1}{2M \times 2N} \sum_{p=\rho}^{\gamma} \sum_{q=1}^{2N} F_{pq}, \quad i=1,2,\ldots,2M
\]

\[
b_{i,2} = \frac{1}{\rho \times \rho} \left( \sum_{p=\rho}^{\gamma} \sum_{q=1}^{2N} F_{pq} - \sum_{p=\rho}^{\gamma} \sum_{q=1}^{2N} F_{pq}, \quad i=1,2,\ldots,2M, \quad j=1,2,\ldots,2N \right.
\]

\[
b_{j,1} = \frac{1}{\rho \times \rho} \left( \sum_{p=\rho}^{\gamma} \sum_{q=1}^{2N} F_{pq} - \sum_{p=\rho}^{\gamma} \sum_{q=1}^{2N} F_{pq}, \quad j=1,2,\ldots,2N, \quad i=1,2,\ldots,2M \right.
\]

\[
b_{j,2} = \frac{1}{\rho \times \rho} \left( \sum_{p=\rho}^{\gamma} \sum_{q=1}^{2N} F_{pq} - \sum_{p=\rho}^{\gamma} \sum_{q=1}^{2N} F_{pq}, \quad j=1,2,\ldots,2N, \quad i=1,2,\ldots,2M, \quad \gamma = 2\log_2(\rho^{-1}) \right.
\]

Where \( a_i = \rho_1(\sigma-1)+1, \beta_i = \rho_2(\sigma-1)+\rho_i/2, \gamma_i = \rho_i \sigma, \rho_i = 2M/\tau, \sigma_i = i-\tau, \tau = 2\log_2(\rho^{-1}) \)

and similarly,

\[
a_{ij} = \rho_1(\sigma-1)+1, \beta_{ij} = \rho_2(\sigma-1)+\rho_i/2, \gamma_{ij} = \rho_i \sigma, \rho_i = 2M/\tau, \sigma_i = i-\tau, \tau = 2\log_2(\rho^{-1}) \right.
\]
Proof. See [3].
In the following two subsections the proposed numerical methods are constructed for two different types of IDEs, namely third order Fredholm IDEs and Volterra IDEs of second type. In each case, the kernel function $K(x)$ is approximated with a two-dimensional Haar wavelet basis as given in Eq. (18).

3.1 Third order Fredholm integro-differential equation of second kind
Consider the third order Fredholm IDE of second type (1). The function $u'''(x)$ is developed into Haar series as:

$$u'''(x) = \sum_{i=1}^{2M} \alpha_i h_i(x).$$

(26)

By integrating Eq. (26), we obtain

$$u''(x) = u_0''' + \sum_{i=1}^{2M} a_i p_i(x).$$

(27)

Again by integrating Eq. (27), we obtain

$$u'(x) = u_0'' + xu_0''' + \sum_{i=1}^{2M} a_i p_{i,1}(x).$$

(28)

Again by integrating Eq. (28), we obtain

$$(x) = u_0 + xu_0'' + \frac{1}{2} x^2 u_0''' + \sum_{i=1}^{2M} a_i p_{i,2}(x).$$

(29)

The Kernel function $K(x,t,u(t),u'(x),u''(x),u'''(x))=F(x,t)$ is approximated using two-dimensional Haar wavelet approximation given in Eq. (18). Substituting this approximation of the Kernel function in the third order Fredholm IDE of second type (1), we obtain the following:

$$u'''(x) + g(x)u''(x) + w(x)u'(x) + s(x)u(x) = f(x) + \frac{1}{\rho} \sum_{i=1}^{2M} \sum_{j=1}^{2M} b_{ij} h_i(t) h_j(t) dt.$$  

(30)

Using the property of Haar wavelets, Eq. (31) reduces to the following:

$$u'''(x) + g(x)u''(x) + w(x)u'(x) + s(x)u(x) = f(x) + \frac{2M}{\rho} \sum_{i=1}^{2M} b_{i,1} h_i(x) dt.$$  

(32)

Substituting the collocation points, we obtain the following system of equations:

$$u'''(x_r) + g(x_r)u''(x_r) + w(x_r)u'(x_r) + s(x_r)u(x_r) = f(x_r) + \frac{2M}{\rho} \sum_{i=1}^{2M} b_{i,1} h_i(x_r).$$  

(33)

Now the coefficients $b_{i,1}$ can be replaced with their expressions given in Eqs. (20)-(21), and we have the following:

$$u''''(x) = f(x_r) - g(x_r)u''(x_r) - w(x_r)u'(x_r) - s(x_r)u(x_r) + \frac{1}{2M \times 2M} \sum_{p=1}^{2M} \sum_{q=1}^{2M} F(x_p, t_q) h_{1}(x_r) + \frac{2M}{\rho} \sum_{i=2}^{2M} \frac{1}{\rho^2} \sum_{p=\alpha q}^{\beta q} \sum_{p=\beta q}^{\alpha q} h_{1}(x_r),$$

$$r = 1, 2, \ldots, 2M.$$  

(34)

For the sake of simplicity, we introduce the following notations:

$$(x_r) = (x_r) = u_r, \quad r = 1, 2, \ldots, 2M.$$  

(35)

$$u'(x_r) = u'(x_r) = u'_r, \quad r = 1, 2, \ldots, 2M.$$  

(36)

$$u''(x_r) = u''(x_r) = u''_r, \quad r = 1, 2, \ldots, 2M.$$  

(37)

$$u'''(x_r) = u'''(x_r) = u'''_r, \quad r = 1, 2, \ldots, 2M.$$  

(38)

With these new notations, the above system (34) can be written as following:

$$u''''_r = f(x_r) - g(x_r)u''_r - w(x_r)u'_r - s(x_r)u_r + \frac{1}{2M \times 2M} \sum_{p=1}^{2M} \sum_{q=1}^{2M} F(x_p, t_q) h_{1}(x_r) +$$

$$\frac{2M}{\rho} \sum_{p=\alpha q}^{\beta q} \sum_{p=\beta q}^{\alpha q} h_{1}(x_r), \quad r = 1, 2, \ldots, 2M.$$  

(39)

Using theorem 1 we can express $u''', \quad r = 1, 2, \ldots, 2M, \quad u'_r, \quad r = 1, 2, \ldots, 2M$, and $u_r, \quad r = 1, 2, \ldots, 2M$ in terms of $u''''_r, \quad r = 1, 2, \ldots, 2M$ and given as follows:

$$u''_r = u_0'' + \frac{1}{2M} \sum_{j=1}^{2M} u_j'''_1 p_{1,1}(x_r) + \frac{2M}{\rho} \sum_{i=2}^{\beta} \frac{1}{\rho} \sum_{p=\alpha}^{\beta} u_{p''''} - \sum_{p=\beta + 1}^{\beta} u_{p'''} p_{i,1}, \quad r = 1, 2, \ldots, 2M.$$  

(40)

$$u'_r = u_0' + u_0'' x_r + \frac{1}{2M} \sum_{j=1}^{2M} u_j'''_1 p_{1,2}(x_r) + \frac{2M}{\rho} \sum_{i=2}^{\beta} \frac{1}{\rho} \sum_{p=\alpha}^{\beta} u_{p''''} - \sum_{p=\beta + 1}^{\beta} u_{p'''} p_{i,2}, \quad r = 1, 2, \ldots, 2M.$$  

(41)

$$u_r = u_0 + u_0'' x_r + \frac{1}{2M} \sum_{j=1}^{2M} u_j'''_1 x_r^2 + \frac{2M}{\rho} \sum_{i=2}^{\beta} \frac{1}{\rho} \sum_{p=\alpha}^{\beta} u_{p''''} - \sum_{p=\beta + 1}^{\beta} u_{p'''} p_{i,3}, \quad r = 1, 2, \ldots, 2M.$$  

(42)

Now Eq. (39) represents a $2M \times 2M$ nonlinear system with unknowns $u''''_r, \quad r = 1, 2, \ldots, 2M$. We use Broyden’s method to solve this system. The solution of this system gives us the values of $u''''_r$ at collocation points. We can easily obtain the solution of Eq. 1 at collocation points using the following equation.
\( u_r = u_0 + u_0' x_r + \frac{1}{2} u_0'' x_r^2 + \frac{1}{2 M} \sum_{j=1}^{2M} u_j''' p_{1,3}(x_r) + \frac{2M}{i=2} \frac{1}{\rho} \) \\
\left( \sum_{p=\alpha}^{\beta_1} \sum_{p=\beta_1+1}^{\beta_2} u_p'''(\beta) p_{i,3}, \ r=1,2,\ldots,2M. \right) \tag{43}

3.2 Volterra integro-differential equation of third order: 
Consider the third order Volterra IDE of second type (2). 
Haar functions approximation for the given function \( u'''(x) \) is developed as given in Eq. (26). The Kernel function \( K(x,t,u(t),u'(x),u''(x),u'''(x)) = F(x,t) \) is approximated using two-dimensional Haar wavelets approximation given in Eq. (18). Using these expressions in the Volterra IDE of second type (2), we obtain the following equation:

\[ u'''(x) + g(x) u''(x) + w(x) u'(x) + s(x) u'(x) = f(x) + \int_0^x \sum_{i=1}^{2M} \sum_{j=1}^{2M} b_i j h_i(x) h_j(t) \, dt. \]  
\[ \tag{44} \]

This equation reduces to

\[ u'''(x) + g(x) u''(x) + w(x) u'(x) + s(x) u'(x) = f(x) + \sum_{i=1}^{2M} \sum_{j=1}^{2M} b_i j h_i(x) p_j,1 \]  
\[ \tag{46} \]

Substituting the collocation points, we obtain

\[ u'''(x_r) + g(x_r) u''(x_r) + w(x_r) u'(x_r) + s(x_r) u'(x_r) = f(x_r) + \sum_{i=1}^{2M} \sum_{j=1}^{2M} b_i j h_i(x_r) p_j,1 \]  
\[ \tag{47} \]

\[ u'''(x_r) + g(x_r) u''(x_r) + w(x_r) u'(x_r) + s(x_r) u'(x_r) + g b_{1,1} h_{1,1}(x_r) p_{1,1}(x_r) + \sum_{i=2}^{2M} b_i j h_i(x_r) p_{i,1}(x_r) + \sum_{i=2}^{2M} b_i j h_i(x_r) p_{i,1}(x_r) \]  
\[ \tag{48} \]

Now the coefficients \( b_i \)'s can be replaced with their expressions given in Eqs. (20)-(23), we have

\[ u'''(x_r) = f(x_r) - g(x_r) u''_r - w(x_r) u' r - s(x_r) u_r + \frac{1}{2 M} \sum_{p=1}^{2M} F(x_p,t_q) h_1(x_r) p_{1,1}(x_r) + \frac{2M}{i=2} \frac{1}{\rho_1} \sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{\beta_2} F(x_p,t_q) - \frac{2M}{i=2} \frac{1}{\rho_1} \sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{\beta_2} F(x_p,t_q) - \frac{\beta_1}{p=\beta_1+1} \sum_{q=1}^{\beta_2} F(x_p,t_q) h(x_r) p_{1,1}(x_r) + \]  
\[ \tag{51} \]

\[ u'''(x) = 8 e^x - \frac{1}{16} e^x + \frac{1}{16} \int_0^1 e^{-2t} u^2(t) \, dt \]  
\[ \tag{52} \]
with the initial conditions \( u''(0) = 4, \ u'(0) = 2, \ u(0) = 1 \) and the exact solution \( u(x) = e^{2x} \).

In Table 2 we have shown the maximum absolute errors at the collocation points as well as the experimental rate of convergence. The table shows that the maximum absolute errors decrease with the increase in number of collocation points.

### Table 2: Maximum absolute and rate of convergence for Ex. 2

<table>
<thead>
<tr>
<th>J</th>
<th>2M</th>
<th>Maxi absolute errors</th>
<th>Rate of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4.7267e + 002</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>1.4091e + 002</td>
<td>1.7461</td>
</tr>
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<td>3</td>
<td>16</td>
<td>3.8351e + 003</td>
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<td>4</td>
<td>32</td>
<td>9.9967e + 004</td>
<td>1.9397</td>
</tr>
<tr>
<td>5</td>
<td>64</td>
<td>2.5515e + 004</td>
<td>1.9701</td>
</tr>
<tr>
<td>6</td>
<td>128</td>
<td>6.4447e + 005</td>
<td>1.9852</td>
</tr>
</tbody>
</table>

**Test Problem 3.** Consider the Nonlinear Volterra IDE [18]:

\[
u''(x) = \frac{3}{2} e^{x^2} - \frac{1}{2} e^{3x} + \int_0^x e^{-t} u^3(t) dt
\]  

(53)

with the initial conditions \( u''(0) = 1, \ u'(0) = 1, \ u(0) = 1 \) and the exact solution \( u(x) = e^{x} \).

In Table 3 we have shown the maximum absolute errors at the collocation points as well as the experimental rate of convergence. The table shows that the maximum absolute errors decrease with the increase in number of collocation points.

### Table 3: Maximum absolute and rate of convergence for Ex. 3

<table>
<thead>
<tr>
<th>J</th>
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<th>Rate of Convergence</th>
</tr>
</thead>
<tbody>
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<td>–</td>
</tr>
<tr>
<td>2</td>
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<td>1.7036</td>
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<td>4</td>
<td>32</td>
<td>1.3175e + 004</td>
<td>1.9240</td>
</tr>
</tbody>
</table>

**Test Problem 4.** Consider the Nonlinear Volterra IDE [18]:

\[
u''(x) = -\frac{1}{2} x - 8\cos(2x) - \frac{1}{8} \sin(4x) + \int_0^x (1-u^2(t)) dt
\]  

(54)

with the initial conditions \( u''(0) = 0, \ u'(0) = 2, \ u(0) = 0 \) and the exact solution \( u(x) = \sin(2x) \).

### Table 4: Maximum absolute and rate of convergence for Ex. 4

<table>
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<th>2M</th>
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### 5 CONCLUSION

A new method based on Haar wavelets is proposed for the numerical solution of nonlinear third order Fredholm IDEs and Volterra IDEs of second kind. Two dimensional Haar wavelets basis are used for approximating the kernel functions of IDEs. The algorithms are validated numerically. Better accuracy and rapid convergence of the new approach has been achieved.

### REFERENCES


