Dynamic Systems

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Introduction

A critical factor facilitating the "cognitive revolution" of the early 1960's was the capability of formulating rigorous (computational or mathematical) models of how the inputs and outputs from mental processing systems change over time. The earliest approach was to view the mind as a dynamic cybernetic feedback system (see Miller, Gallanter, & Pribram, 1960). But this was later abandoned in favour of another approach, which was to view the mind as a rule-based, symbol processor (see Newell and Simon, 1972). Most recently, developments in neural and connectionist networks have revived interest in a dynamic systems approach. More broadly, dynamic systems theory has been adopted by a wide range of fields in cognitive science, including perceptual-motor behaviour, child development, speech and language, and artificial intelligence. Some excellent presentations of mathematical dynamic system theory include Beltrami (1987), Luenberger (1979), Padula and Arbib (1974), and Strogatz (1994).

Elements of Dynamic Systems

Generally speaking, *dynamic systems* are composed of three parts. The first part is the *state* of a system, which is a representation of all the information about the system at some particular moment in time. As an example, the state of a computer can be summarized by a long *n*-element list of the binary bits of information stored in the registers and memory banks at any moment. The state of a brain model can be summarized by a large *n*-dimensional vector of positive real numbers, representing all the neural activations at any moment. In general, the symbol $X(t) = [x_1(t), \dots, x_n(t)]$ will be used to denote the state of a system at time t.

The second part is the *state-space* of a system. This is a set that contains all of the possible states to which a system can be assigned. The state-space of a computer is the 2^n ensemble containing all of the possible configurations for the *n*-element binary valued lists. The state space of a brain model is the set of points contained in the positive region of the *n*-dimensional Cartesian vector space denoted R^n. The symbol Ω is used to denote the state space of a dynamic system, and $X(t) \in \Omega$.

The third part is the *state-transition function* that is used to update and change the state from one moment to another. For example, the state-transition function of a computer is defined by the production rules that change the bits of information from the state at one step $X(t)$ to the next step $X(t+1)$. The state-transition function for a brain model is a continuous function of time that maps the brain state $X(t)$ at time t to another state $X(t+h)$ at a later moment. The symbol $\boldsymbol{T}$ is used to denote the state-transition function that maps an initial state, $X(t)$ after some period of time h into a new state $X(t+h)$:

$$X(t+h) = \boldsymbol{T}\,(\,X(t)\,,\,t,\,t+h\,)$$

Whenever the state transition function is assumed to be a differentiable function of time, then we can define a *local generator* as the time derivative

$$d\boldsymbol{T}/dt = \lim_{h \to 0}\,(\,X(t+h)\text{-}X(t)\,)\,/\,h \;=\; \boldsymbol{f}\,(\,X(t)\,,\,t\,)\,.$$

In sum, given an initial starting state, $\boldsymbol{X}(0)$, the local generator is used to generate a *trajectory*, $\boldsymbol{X}(t)$ for all $t > 0$. Assuming that the local generator satisfies certain smoothness properties, then a local generator is guaranteed to produce a unique trajectory from any given starting position. The objective of dynamic systems analysis is to understand all the possible trajectories produced by a local generator.

Example Dynamic Systems

Before going into more details about the analysis of dynamic systems, it will be helpful to first describe some classic examples from the literature.

Logistic Growth Model

Consider the following simple dynamic model of growth. To be concrete, suppose we wish to model a student's probability of performing a task correctly as a function of training (e.g., successfully playing a piece of classic music on the piano). Define $p(t)$ as the probability of correctly performing the task, and assume that this preference changes from one time point, t, to a later time point, $t+h$, according to the following simple difference equation:

$$[p(t+h) - p(t)]/h = [\alpha\, p(t)] \cdot [1\text{-}p(t)]. \qquad (1a)$$

This model can be understood as a product of two parts. The second part, $[1\text{-}p(t)]$, represents the amount that remains to be learned. The first part, $[\alpha p(t)]$, can be interpreted as the learning rate, which is an increasing function of the amount already learned. Note that as the probability approaches zero or one, then the change approaches zero, so that the probability remains bounded between zero and one.

This difference equation can be reformulated as a differential equation by allowing the time interval, h, to approach zero in the limit:

$$d\,p(t)/dt = \lim_{h \to 0}\,[p(t+h)\text{-}p(t)]/h \;=\; \alpha\, p(t)[1\text{-}p(t)]. \qquad (1b)$$

By separating the variables and integrating, we can solve this differential equation to yield the solution (see Braun, 1975):

$$p(t) = [1 + c\,exp(-\alpha t)]^{-1} \qquad (1c)$$

where the constant *c* depends on the initial state, *p(0)*. Figure 1 is a time series plot that illustrates performance over time (with c = 100 so that *p(0)* starts very close to zero). As can be seen in this figure, this model produces a family of smooth S - shaped curves that gradually rises from the initial state and asymptote at the stable equilibrium point, p = 1. The growth rate, α, adjusts the steepness of the S - shaped curve.

Fig 1

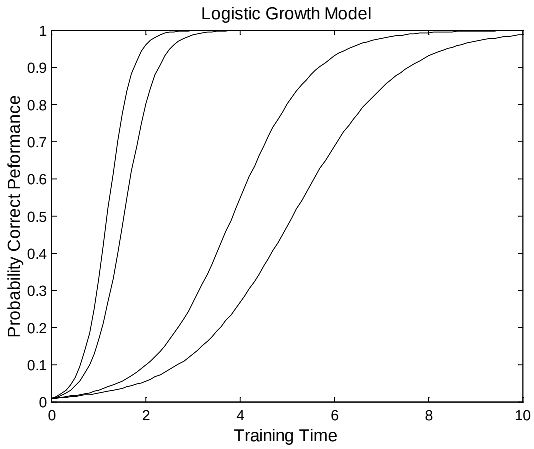

Figure 1: A time series plot showing the trajectories produced by the logistic growth model. The horizontal axis represents training time, and the vertical axis represents the probability of correct task performance.

Linear System Model

Next consider the problem of developing a simple dynamic model of motivation to perform at task (see Atkinson & Birch, 1970; Townsend and Busemeyer, 1989). Suppose a student is trying to decide how much effort to invest in a task over time, such as for example the amount of time to allocate toward achieving an athletic, academic, or social goal. Define *q(t)* as the amount of effort actually expended at time *t*, and define *p(t)* as the student's strength of preference for doing the task at time *t*.

First we assume that the rate of change in effort is directly influenced by the preference state during this time interval. But due to fatigue effects, the change in effort is decreased in proportion to the current amount of effort. These assumptions are incorporated into the following model:

$$dq(t)/dt = p(t) - \alpha\, q(t) \qquad (2a)$$

Second, we assume that the rate of change in preference is determined by two factors. One is the valence difference between the anticipated rewards for success and punishments for failure, denoted *v(t)*. Another is a satiation effect, which causes the preference to decrease in proportion to the current amount of effort consumed. The following model is used to represent these assumptions:

$$dp(t)/dt = v(t) - \beta\, q(t) \qquad (2b)$$

Given the initial states, $p(0)$ and $q(0)$, and given the valence differences, $v(t)$, $t \geq 0$, then Equations 1 and 2, define a simple dynamic system that can be used to model the behavior of a students effort on a task over time. The valence, $v(t)$, is called the input or forcing term of the system, and the coefficients, α and β, are called the *parameters* of the system.

Finding solutions to *coupled systems* of linear differential equations, such as Equations 1 and 2, is a topic covered in standard texts on ordinary differential equations (see Braun, 1975). In the special case when the initial state starts at $[p(0), q(0)] = [0,0]$, and the input valence is constant across time ($v(t) = v$), then the solution for effort is

$$q(t) = (v/\beta)\{1 - e^{(-\alpha/2)\,t}[\sin(\theta t) + (\alpha/2)\cos(\theta t)]\} \qquad (2c)$$

with $\theta^2 = \beta-(\alpha/2)^2$. (This can be checked by taking the time derivative of Equation 2c, and showing that it satisfies both Equations 1 and 2 and the initial conditions.)

Figure 2 is a time series plot, and it illustrates the behavior predicted by the model (with the parameters fixed at $\alpha = .25$, $\beta = 1$, and $v = .50$). The horizontal axis shows the time on the task, and the vertical axis shows the amount of effort spent at each time point. The curve initially oscillates up and down between zero and one, but it eventually settles down to an asymptotic level of effort at $q = .50$.

Fig 2

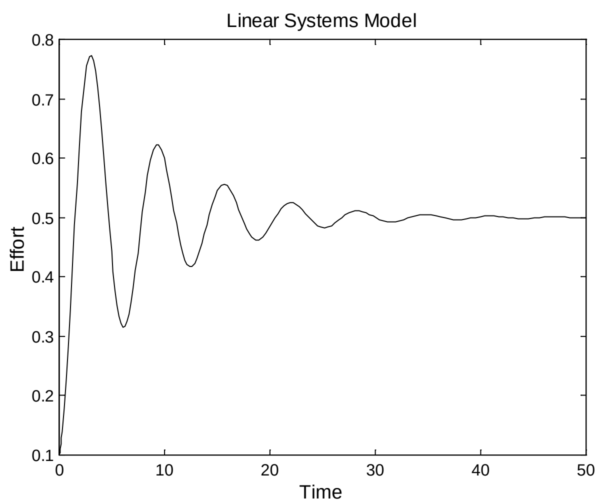

Figure 2: A time series plot showing the trajectory produced by the simple dynamic model of effort. The horizontal axis represents time on the task, and the vertical axis represents the amount of effort as a function of time.

Predator - Prey Model

Suppose a researcher is trying to decide how much effort to spend on generating and testing ideas. Define x_1 as the number of candidate ideas generated for testing, and define x_2 as the number of tested ideas. Note that an idea cannot be tested unless its generated, and once it has been tested, it is eliminated from the candidate pool. Consider the following simple nonlinear model of this process:

$$dx_1 / dt = \alpha x_1 - \beta x_1 x_2 - \gamma x_1^2 \quad (3a)$$

$$dx_2 / dt = \varphi x_1 x_2 - \lambda x_2 \quad (3b)$$

In Equation 3a, the coefficient, α, allows candidate ideas to grow over time, the coefficient β reflects depletion of candidate ideas that are generated and tested, and the last coefficient γ represents interference between ideas. In Equation 3b, the coefficient ϕ represents the increase in testing produced by a successful test, and the coefficient λ provides for a fatigue effect from testing ideas. All of the parameters are assumed to be positive valued.

Equations 3a and 3b form what the dynamic systems literature calls a predator-prey model. In this example, the idea testing process is playing the role of the predator, which is preying on the idea generating process. When dealing with nonlinear differential equations such as this, the most practical method for finding solutions is to use numerical integration routines commonly available in mathematical programming languages such as Matlab or Mathematica.

Figure 3 shows a time series plot for the ideas generated and tested (setting $\alpha = .2$, $\beta = .3$, $\delta = .1$, $\phi = .6$, $\lambda = .5$, and setting the initial state equal to $[x_1(0), x_2(0)] = [.1,.1]$). As seen in the figure, generated ideas must build up first, and testing ideas dominates later. Eventually, both processes approach a steady state.

Fig 3

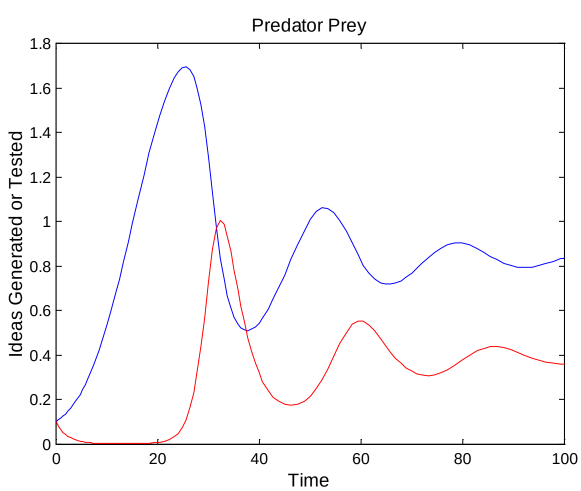

Figure 3: A time series plot showing the trajectories produced by the predatory - prey model of idea generating and testing. The horizontal axis represents time on the task, and the vertical axis represents the number of ideas generated and tested, where generation leads testing.

Properties of Dynamic Systems

Dynamic systems describe the general laws that a system obeys. Various special cases of the general law can be derived simply by changing either the initial state, *X(0)*, or the system parameters (e.g., α_i 's). For a given initial state and fixed set of parameter values, a dynamic system generates a unique trajectory or path through the state space as a function of time (like that shown in Figure 1). The primary goal of dynamic systems theory is to develop analytic methods for studying all the trajectories produced by a dynamic system, and to understand how these trajectories change as a function of the initial state and system parameters.

Phase Portraits and Attractors

Figure 4 illustrates these ideas using the simple dynamic model of effort as an example. The four panels shown in Figure 4 are called *phase portraits*. The two-dimensional plane defined within each portrait represents the state space of the system. The four different portraits were computed from Equations 2a and 2b by decreasing the parameter, α, in Equation 1 from $\alpha = 2$, to $\alpha = .5$, to $\alpha = 0$, and finally to $\alpha = -1$. The remaining parameters of the model were fixed equal to $\beta = 1$ and $v = 1$.

Fig 4

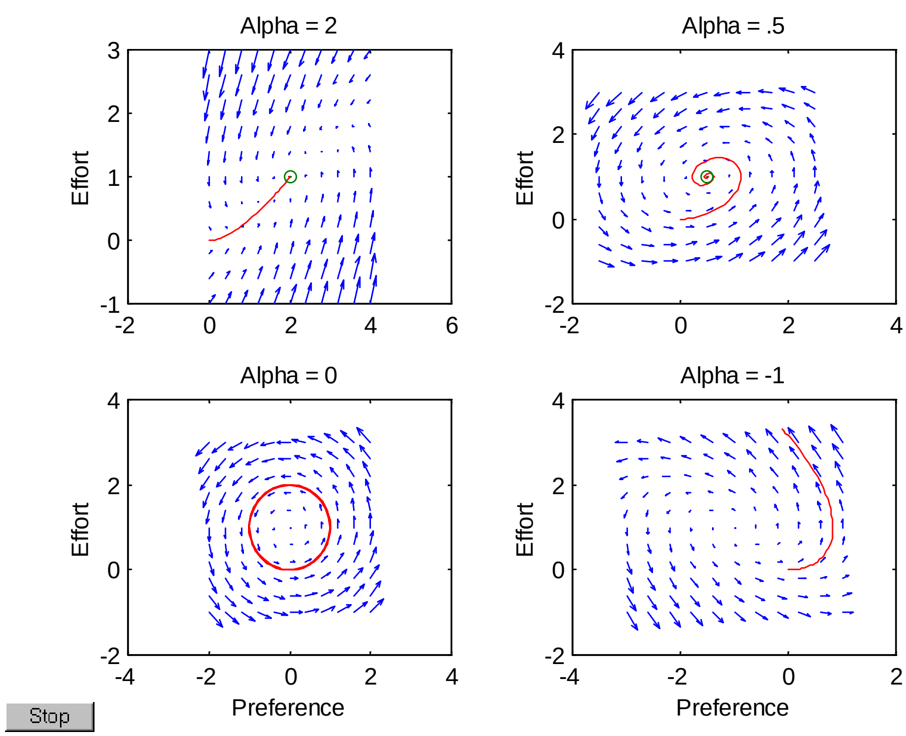

Figure 4: A display of four different phase portraits, one for each setting of the parameter α in Equations 2a and 2b. The horizontal axis within each portrait represents the preference state, and the vertical axis represents the level of effort. Each arrow is a vector indicating the direction and rate of change in the system at a particular point in the state space. The smooth curve within each portrait shows the trajectory produced by setting the initial state equal to zero for preference and effort.

Each arrow inside of each phase portrait is a velocity vector representing the directional rate of change in preference and effort that occurs at a particular state of the system. In other words, the head of an arrow indicates where the state will move in the next instant, given the current state indicated by tail of the arrow. Thus the arrows indicate the *flow* of the dynamic system. The smooth curve following the flow within each portrait indicates the trajectory produced by the system when the initial state of the system is started at [$p(0)$, $q(0)$] = [0 , 0]. Different trajectories would result from different choices of the initial starting state. In fact, each initial state defines a unique trajectory, and therefore the trajectories never cross (provided that the local generator satisfies certain smoothness properties).

The four panels illustrate how the dynamic properties of the model vary depending on the parameter α in the model. The top left panel shows all of the arrows flowing toward a stable equilibrium point located at state [p,q] = [2,1]. In this case, each trajectory moves steadily toward the equilibrium without any oscillation. The top right panel shows the arrows spiralling in toward a stable equilibrium point located at [p,q] = [.5,1]. In this case, each trajectory initially oscillates but eventually settles down on the equilibrium point (like Figure 2). The bottom left panel shows the arrows flowing in a circular manner around the center of the circle, with the center located at [p,q] = [0,1]. In this case, each trajectory oscillates up and down indefinitely, like a clock. Finally, the bottom right panel shows the arrows spiralling away from an unstable equilibrium point located at [p,q]=[-1,1]. In this case, the system shoots off, out of control, toward infinity.

All four of the phase portraits contain a special state called an *equilibrium point*, but the nature of the equilibrium point changes across the portraits. An equilibrium point, $\boldsymbol{X}^*$, has the special property that the local generator is zero at this point:

$$f(X^*) = 0.$$

Thus no change occurs when the system is in this state. The equilibrium points for the top left and top right portraits are called *stable* equilibrium points or *attractors* because the system eventually returns back toward these equilibrium points whenever the state of the system starts within a close proximity of these points (for more rigorous definitions, see the references mentioned above). The equilibrium point for the bottom right panel is an *unstable* equilibrium or *repellor* because if the system is placed an arbitrarily small distance away from that point, it eventually drifts further away and leaves the immediate neighbourhood surrounding the equilibrium.

Stability Analysis

The dynamic effort model, defined by Equations 2a and 2b, is an example of a linear system. A linear system enjoys the special property of allowing only a single equilibrium point (provided that the system is non-singular). The stability of equilibrium points for linear systems can easily determined by checking the eigenvalues of the linear equations (see Braun, 1975). Nonlinear systems, however, allow multiple equilibrium points, some of which may be stable, and others may be unstable, and a more general method for *stability analysis* is required to determine their properties.

There are several general mathematical techniques for studying the qualitative properties of equilibrium points. One of the most powerful methods is based on the construction of what is called a *Liapunov function* for the dynamic system. First of all, a Liapunov function maps each state of the system into a real number:

$$\boldsymbol{V} : \Omega \rightarrow \mathrm{R}.$$

Secondly, this function has the special property that its time derivative never increases:

$$d\,\boldsymbol{V}/dt \leq 0 \text{ for all } t.$$

In physics, the Liapunov function can be interpreted as the potential function of a conservative system, and in engineering it can be interpreted as the objective function that a control system is designed to minimize. If there is a Liapunov function for the system, and an equilibrium point, $\boldsymbol{X}^*$ is a local minimum of this function, then $\boldsymbol{X}^*$ is a stable attractor. The *basin of attraction* for $\boldsymbol{X}^*$ is largest possible region for which $\boldsymbol{X}^*$ serves as the attractor. In other words, once the system enters the basin of attraction for $\boldsymbol{X}^*$, then it never leaves, and it eventually converges in the limit toward the attractor. When a Liapunov function is defined over the entire state space, then the state space can be partitioned into a collection of attraction basins with a single stable attractor located within each basin (see Figure 5).

Fig 5

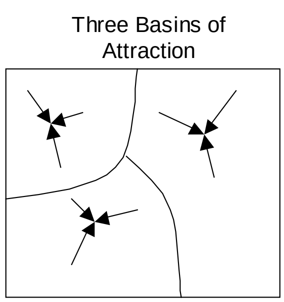

Figure 5: An illustration of a two dimensional state space that is divided into three basins of attraction. The arrows indicate the direction pointing downhill, where the Liapunov function is decreasing. The meeting point of the three arrows within each basin represents the stable attractor at the local minimum. The curves indicate the boundaries that separate each basin.

To illustrate the idea of a Liapunov function more concretely, reconsider the predator prey model described earlier. The top panel in Figure 6 shows the phase portrait for this model.

This nonlinear model has three equilibrium points: [0,0], $[(\alpha/\gamma),0]$, and finally, $[(\lambda/\phi), (\phi\alpha - \lambda\gamma)/\beta\phi]$, but only the last one is stable. For convenience, we will define the last equilibrium point as $[x_1^*, x_2^*] = [(\lambda/\phi), (\phi\alpha - \lambda\gamma)/\beta\phi]$. (In the case of Figure 4, this is located at the state $[x_1^*, x_2^*] = [.83,.39]$).

Using Liapunov's second method, we can show that the last equilibrium point is a stable attractor for all positive values of x_1 and x_2. The Liapunov function for this example is

$$\mathbf{V}(x_1,x_2) = \lambda (x_1/x_1^*) - \lambda \ln(x_1/x_1^*) + \alpha (x_2/x_2^*) - \alpha \ln(x_2/x_2^*). \qquad (4)$$

and the time derivative of this function is

$$d\mathbf{V}(x_1,x_2)/dt = -\lambda^2(\gamma / \varphi)[1-(x_1/x_1^*)]^2 \leq 0 ,$$

which is non-increasing for all positive values of the state variables. Also note that the partial derivative of $\boldsymbol{V}$ with respect to the state variable is zero at [x_1^*, x_2^*], which is required for this point to be a local minimum of $\boldsymbol{V}$.

Fig 6

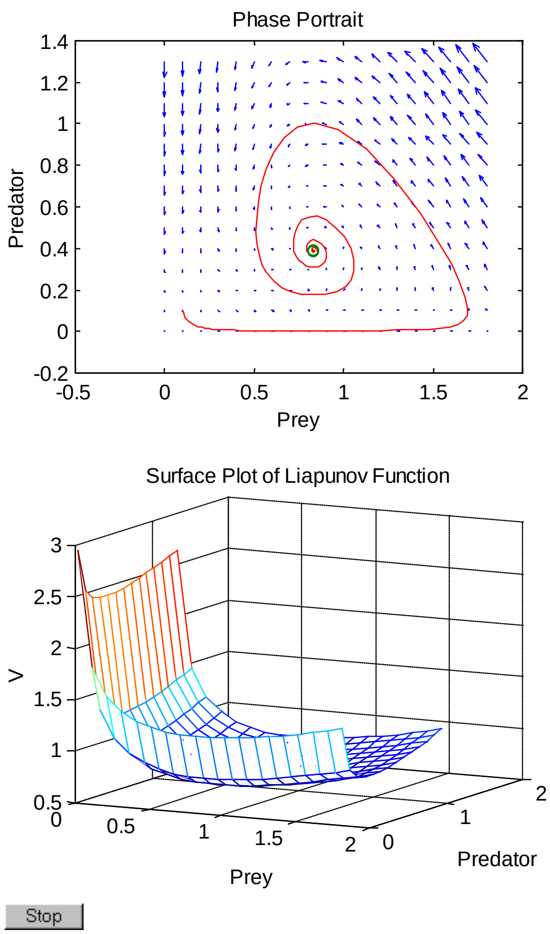

Figure 6: The top panel, illustrates the phase portrait produced by the predator - prey model of effort for generating and testing ideas. The bottom panel illustrates the Liapunov function corresponding to this model. In the bottom panel, the plane represents the state space of the model, and the surface on the vertical axis above the plane represents the value of the Liapunov function for each point in the state space.

The bottom panel in Figure 6 shows the surface of the Liapunov function over the state space (using the same parameters as in the top panel). Note that the surface has a minimum located at the equilibrium point [x_1*, x_2*] = [.83, .39]. Thus it satisfies all the requirements for being the stable attractor associated with the basin of attraction inside the positive region of the state space.

Bifurcation Analysis and Catastrophe

A *bifurcation* is said to occur if the equilibrium points undergo qualitative changes in their behavior as a result of small and continuous changes in model parameters. The parameter values at which these bifurcations occur are called bifurcation points. To illustrate the idea of bifurcation, consider the following example of a slightly more complex version of the predator prey model:

$$dx_1 / dt = (.4)\, x_1 - (.04)x_1^{\,2} - (.2)(x_1 x_2)/(1+x_1)$$

$$dx_2 / dt = \alpha\, x_2\, [1-(.5)(x_1/x_2)]$$

This example was written with only one free parameter, α, which is the focus of this bifurcation analysis. For this model, the location of the positive valued equilibrium point, [x_1*,x_2*] = [2.7, 5.4], is independent of the parameter α (see Beltrami, 1987, p. 156). Also, for large values of α, this equilibrium point is a stable attractor. The trajectory of the model looks very much like that shown in the top panel of Figure 6.

Fig 7

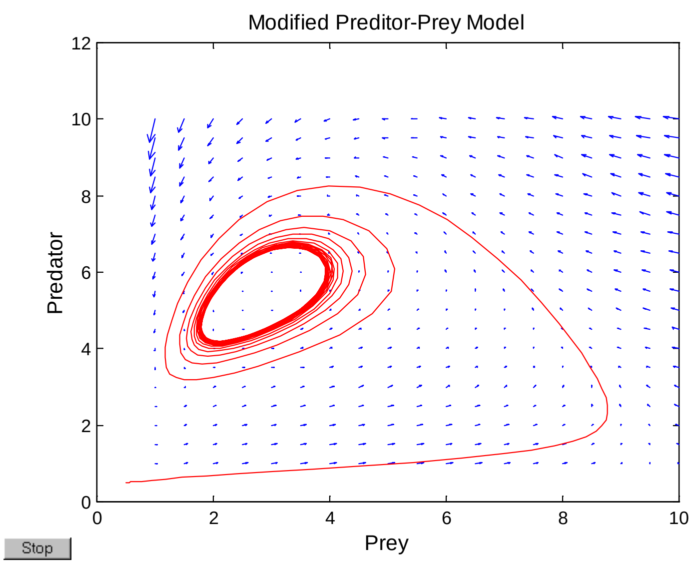

Figure 7: Phase portrait and trajectory produced by a modified predator – prey model that exhibits limit cycle behavior.

As the parameter α decreases, a qualitative change in the dynamics appears. The equilibrium point switches from an attractor to a repellor. Now a new behavior appears in which the asymptotic trajectory is attracted toward a *limit cycle* or asymptotic periodic orbit. In this case, the attractor is a set of points contained in the limit cycle. Figure 7 shows the phase portrait for this case, and the trajectory produced when the system is started at the initial state [$x_1(0),x_2(0)$] = [.5,.5] and $\alpha = .10$. This is an example of what is known as a *Hopf bifurcation*. This example forces us to broaden the concept of an attractor to include limit cycles.

Catastrophe theory (Zeeman, 1977) is concerned with bifucations that result in discontinuous jumps in stable equilibrium points. Figure 8 illustrates the idea of a catastrophe. In this figure, each of the five curves represent a Liapunov function defined over a uni-dimensional state space. The variations across curves are produced by making small changes in a single parameter of the dynamic system. The upward pointing arrow indicates the starting position of the system. The first curve on the far left demonstrates an attractor on the right to which the system converges in the limit. A birfucation occurs in the second curve, causing the second, third, and fourth curves to contain attractors on the left and right, and a repellor in the middle. However, given that the initial state lies inside the right basin, the system continues to converge to the attractor on the right. Finally, for the last curve on the right, the change in parameter has finally eliminated the attractor on the right, and now the attractor jumps suddenly into the left basin of attraction. Exactly the opposite jump occurs when the system starts on the left, producing a *hysteresis* effect.

Fig 8

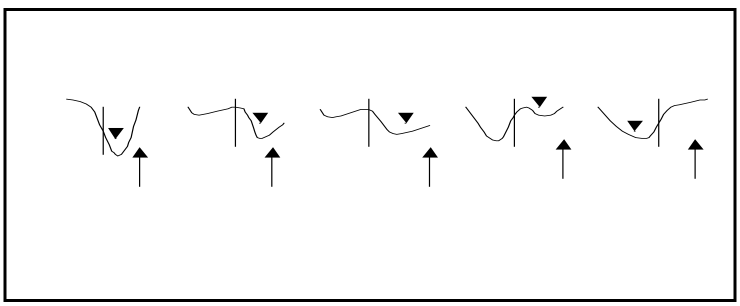

Figure 8: Five hypothetical Liapunov functions produced by small changes in a parameter of a dynamic system. The upward pointing arrow indicates the starting position. The Diamond indicates the final equilibrium state. Note that the equilibrium makes a sudden jump into a new basin of attraction in the last case

Chaotic Systems

Thus far we have seen three types of asymptotic behavior produced by dynamic systems: (a) the system is attracted toward a single attracting state, (b) the system is attracted toward a limit cycle and oscillates indefinitely along some periodic orbit, or (c) the system shoots outside of any bound toward infinity. However, it is possible for some dynamic systems to exhibit aperiodic behavior that does not fall into any of the above three categories. This new category of *strange attractors* is produced by chaotic dynamic systems.

Surprisingly, chaotic behavior can arise from what appears to be extremely simple dynamic models. Let us reconsider the discrete time version of the logistic model for growth, given by Equation 1a. (Recall that the continuous time version produced well behaved, and easily understood trajectories for all values of the parameter α).

Rearranging terms and letting $h = 1$ for the discrete time model yields:

$$p(t+1) - p(t) = \alpha\, p(t)[1-p(t)].$$

Simplifying further, we define $\beta = (1+\alpha)$, and $x(t) = [\alpha/(1+\alpha)]\ p(t)$ so that we can write the equation as

$$x(t+1) = \beta\, x(t)[1-x(t)] \qquad (5)$$

When $\beta < 1$ the system produced by Equation 5 decays to zero, when $1 < \beta < 3$, the system grows toward a stable equilibrium point like the continuous time model. However, as β increases above 3.0, the system becomes periodic, and for $\beta > 3.57$ the system breaks down and becomes aperiodic or chaotic. Figure 9 shows a time series plot of the behavior of the model with $x(0) = .1$, and β changing across panels from .9 to 1.2 to 3.0 and finally to 3.9. Although simple in form, Equation 5 is apparently capable of producing aperiodic or chaotic behavior.

Fig 9

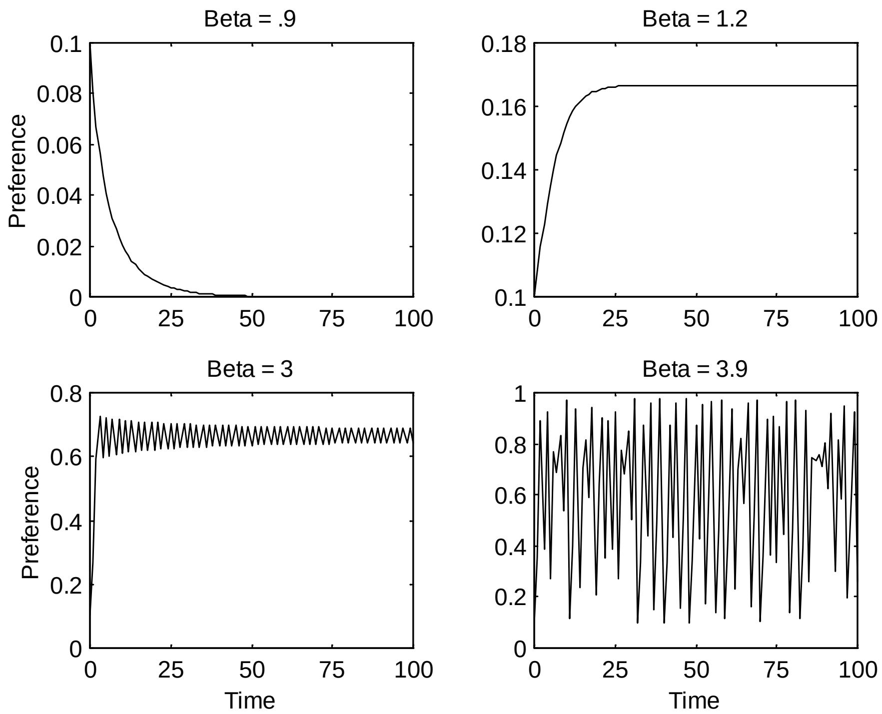

Figure 9: A display of four different time series plots produced by the discrete time logistic model by changing the parameter β in the model.

One of the defining features of a *chaotic dynamic system* is sensitive dependence on initial conditions. This means that an arbitrarily small change in the initial state eventually grows into a very large change in future states. This is sometimes referred to as the *butterfly effect* to convey the idea that the fluttering of a butterfly's wings in Brazil can eventually set off a tornado in Texas.

For example, if we set β = 4, then $x(0) = .1000$ yields $x(10000) = .2098$, but $x(0) = .1001$ yields $x(10000) = .9819$. Thus, although both of these trajectories were computed from the same equation and started from almost the same initial state, the trajectories they eventually produce are totally unrelated.

A rigorous method for identifying chaotic dynamic systems is based on an index, called the Liapunov index, λ:

$$\lambda = \lim_{n \to \infty} \sum_{i=0,n} \ln | d f(x_i) / dx | / n . \qquad (6)$$

The function, *f*, in Equation 6, refers to the local generator for a univariate discrete dynamic systems. For example, *f* is defined by the right hand side of Equation 5 for the logistic model, and $df(x)/dx = \beta(1-2x)$ in this case. This index was derived to provide a mathematical measurement of sensitivity to initial conditions. A dynamic system is chaotic when this index is positive ($\lambda > 0$).

Figure 10 shows the Liapunov index plotted as a function of β for the logistic model (using $x(0) = .10$ as the starting position, however, the pattern does not depend on this starting position). Note that the index is never positive until β > 3.57, at which point the model becomes chaotic. It is also interesting to not that the system occasionally returns to periodic behavior at a few values of β above 3.6!

Fig 10

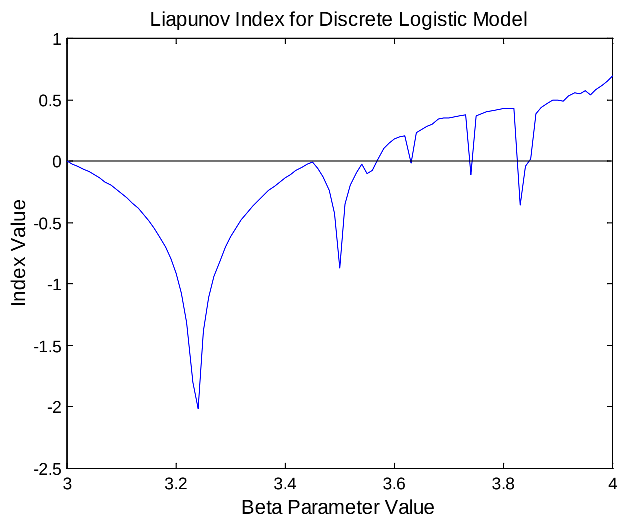

Figure 10: A plot of the Liapunov index as a function of the parameter β for the logistic model. Index values above zero imply a chaotic system.

The discrete time logistic model is the simplest example of a chaotic dynamic system. It is important to note, however, that chaotic systems are not limited to discrete time systems, and there are also many (and more complex) examples of continuous time chaotic systems that have been studied in the dynamic systems literature.

As a final comment, chaotic behavior may seem to be an undesirable property for a dynamic system. But upon further consideration, these models provide an alternative to stochastic dynamic models for describing the unpredictability and variability manifested by many complex biological systems.

Types of Dynamic Systems

Cognitive scientists have employed a wide range of different types of dynamic system, ranging from production rule models of computers (Newell and Simon, 1972) to artificial neural network models of the brain (Grossberg, 1982; Rumelhart and McClelland, 1986). These models differ according to some basic characteristics as described below.

1. Discrete versus continuous state space. The computer system has a *discrete* state space, which contains a countable number of possible states. The state space for the brain model is an example of a *continuous vector* space, which contains an uncountable number of states. The latter is also endowed with additional properties including multiplication of states ($aX \in \Omega$ for any real number a), addition of states ($X_1+X_2 \in \Omega$ for any $X_1 \in \Omega$ and $X_2 \in \Omega$), and distances between states ($\| X_1 - X_2 \|$). Dynamic systems theory usually assumes that state spaces are vector spaces.

2. Discrete versus continuous time indices. The computer model is an example of a discrete time system, in which time indices, t, are elements of a countable number of time steps indexed by the set of positive integers $\{0, 1, 2, 3, \ldots, t, t+1, \ldots\}$. The brain model is an example of a continuous time system, in which the time indices, t, are elements of an uncountable set indexed by the set of positive real numbers, $t \in [0,\infty)$. Classical dynamic systems were concerned with continuous time systems, but now both types are treated in a parallel manner.

3. Linear versus nonlinear systems. The computer model is an example of a discontinuous nonlinear system in which the production rules produce jumps from state to state. Some early neural models used continuous linear state transition functions, but more recent neural models use continuous nonlinear transition functions. In general, a dynamic system is linear if the local generator satisfies the following superposition property for two arbitrary states X_1 and X_2 :

$$\boldsymbol{f} (aX_1 + bX_2 , t) = a\boldsymbol{f} (X_1 ,t) + b\boldsymbol{f} (X_2 ,t)$$

In this case, the local generator can be written as a linear combination of the state variables:

$$\boldsymbol{f} (X(t) , t) = \sum_{i=1,n} \alpha_i(t)\, x_i(t) \, .$$

The coefficients (α_i , $i = 1, \ldots , n$) used to define the linear combination are called the system parameters. The dynamic model of effort (Equation 2a and 2b) is an example of a linear model. The logistic and predator prey models are both examples of nonlinear models.

4. Time invariant vs. time varying systems. Ideally, the computer is an example of a time invariant system in which the state transition function does not change over time. But a

brain model may require a time varying system to allow for growth, development, and aging. In general, the system is time invariant if the local generator can be written independent of the time index:

$$\mathbf{f}(X, t) = \mathbf{f}(X)$$

One way to redefine a time varying system as a time invariant system is to add an extra state variable, and set it equal to the time index. For example, the dynamic model of effort (Equation 2a and 2b) contains a possibly time varying input, $v(t)$. However, this two-dimensional time varying system [p,q] can be transformed into a three dimensional time invariant system [p, q, x_3] by defining a third state variable, $x_3 = t$. In this case, $v(x_3)$ is some known function of the third state variable x_3. The new three-dimensional system can then be described by three equations:

$$dq/dt = p - \alpha q$$

$$dp/dt = v(x_3) - \beta q$$

$$dx_3/dt = 1.$$

5. Deterministic versus stochastic systems. Ideally, the computer model is an example of a deterministic system: If we know the exact state of the system at time t, then we can perfectly predict the state of the system at the next time step $t+1$. Our simple model of task effort was also formulated as a deterministic system. However, models of the brain must account for the inherent unpredictability of human behavior. In the past, this was usually accomplished by allowing a subset of the variables in the state vector to be random rather than known variables, or by allowing the initial state vector to be a random rather than fixed vector. Recently, deterministic but chaotic dynamic systems have been explored as alternatives to stochastic dynamics systems. Battacharia and Waymire (1990) provide an introduction to stochastic dynamic systems theory.

Summary

Dynamic system theory is primarily concerned with continuous state, deterministic systems. It includes both discrete and continuous time systems as well as linear and nonlinear systems. Dynamic systems theory was originally developed for applications to problems arising in physics and engineering. Now cognitive scientists are making broad use of this approach, especially in applications of connectionist and artificial neural network models of cognition. The simple examples provided in this section were designed more for pedagogical purposes rather than scientific purposes. Nevertheless, the field of cognitive science has an abundance of highly successful and truly impressive applications of these ideas to various substantive areas including pattern recognition, motor behavior, cognitive development, learning, thinking and decision-making. There is little doubt that dynamic systems theory has much to contribute toward the scientific advancement of cognitive science. This section only touches on a few of the major concepts included in dynamic systems theory, but excellent textbooks and courses are commonly available to the interested reader wishing to acquire a deeper understanding.

In closing, serious students of cognitive science are encouraged to explore these powerful tools for explaining the dynamics of complex cognitive systems.

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Glossary

dynamic systems are used to describe how complex biological or artificial agents change and evolve over time.

nonlinear systems are dynamic systems whose state transition functions do not obey the superposition principle. In other words, the transition produced by the linear combination of states is not equal to the sum of the transitions produced by each state.

A *trajectory* is the path through the state space that a dynamic system generates over time.

phase portraits show the velocity vectors representing the directional rate of change produced by each state of a dynamic system.

An *attractor* is a set of points in the state space that the dynamic system converges toward as time approaches infinity.

Catastrophe theory is concerned with small changes in system parameters that result in discontinuous jumps in stable equilibrium points.

Chaotic systems are dynamic systems that produce aperiodic but bounded trajectories that are highly sensitive to initial conditions.