A new design of $\mathcal{H}_\infty$ filtering for continuous-time Markovian jump systems with time-varying delay and partially accessible mode information

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In this paper, the delay-dependent $\mathcal{H}_\infty$ filtering problem for a class of continuous-time Markovian jump linear systems with time-varying delay and partially accessible mode information is investigated by an indirect approach. The generality lies in that the systems under consideration are subject to a Markov stochastic process with exactly known and partially unknown transition rates. By utilizing the model transformation idea, an input–output approach is employed to transform the time-delayed filtering error system into a feedback interconnection formulation. Invoking the results from the scaled small gain theorem, an improved version of bounded real lemma is obtained based on a Markovian Lyapunov–Krasovskii functional. The underlying full-order and reduced-order $\mathcal{H}_\infty$ filtering synthesis problems are formulated by a linearization technique. Via solving a set of linear matrix inequalities, the desired filters can therefore be constructed. The results developed in this paper are less conservative than existing ones in the literature, which are illustrated by two simulation examples.

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1. Introduction

State estimation has long been a significant and active research issue due to its theoretical and practical significance in engineering applications and signal processing. Many results on estimation and filtering design for different kinds of dynamic systems have been obtained [1–10]. Specifically, $\mathcal{H}_\infty$ filtering, whose main advantages are insensitive to the exact knowledge of the external noise signals and more robust to parameter uncertainties in systems [1,11], has attracted considerable attention. Therefore, recently, a number of studies on $\mathcal{H}_\infty$ filtering problem for discrete-time or continuous-time systems with delays and uncertainties have been reported in the open literature, e.g., [1,10–15].

On another research front line, as an important class of stochastic hybrid systems, Markovian jump linear systems (MJLSs) have been extensively studied due to their powerful modeling ability of Markov process in many fields, such as robot manipulator systems, aircraft control systems, medical and physical systems and so on [13,15,16]. Differing from the nondeterministic switching in switched systems, the modes evolution in MJLSs is determined by a Markov chain [17,18].
In traditional analysis and design, it is usually assumed that the mode information in the Markov chain is definitely known. In fact, it is questionable and generally expensive to obtain all the transition mode information even for a simple system [19–21]. Recently, some attention has been drawn to the more general cases with partially accessible mode information. It is noted that for the discrete-time case, the transition probabilities are all non-negative, and the sum of transition probabilities is 1 for each row in a transition probability matrix (TPM) [22], which facilitates the mathematical derivations in the discrete-time context. On the other hand, for the continuous-time case with partially accessible mode information, the difficulty is that the unknown elements may locate in diagonal of the transition rate matrix (TRM), which are non-positive. Therefore, it is more difficult and challenging to investigate the corresponding filtering analysis and design problems for continuous-time MJLSs with partially accessible mode information.

On the other hand, time-delays are frequently encountered in MJLSs. Recently, there have appeared some results on robust control and filtering design for MJLSs with time-delays and it has been shown that the delay-dependent conditions reflect the reality better in general [12,23–26]. In addition, it has also been shown that model transformation, which may incur additional dynamics to degrade the performance of original system, is a main source of design conservatism [27–30]. Recently, a new model transformation based on the input–output approach was proposed for time-delay systems [31–34]. Unlike the previous transformations, a new input–output model is introduced for the original system by employing a two-term approximation method. Then, the stability analysis and synthesis of systems with time-varying delay are transformed into a scaled small gain (SSG) problem [31,35–39]. However, to the authors’ best knowledge, there exist few related studies on delay-dependent H∞ filtering for continuous-time MJLSs with time-varying delay and partially accessible mode information based on the SSG techniques. Especially, it is expected that the conservatism of filtering design conditions shall be further reduced by utilizing the input–output approach, which inspires us for this study.

According to the issues mentioned above, we will propose a new delay-dependent H∞ filtering design method for continuous-time MJLSs in the presence of time-varying delay and partially accessible mode information based on SSG theorem. By introducing the lower bounds of the unknown diagonal elements, together with the convexification of unknown transition rates, the difficulty that the unknown elements appear in diagonal of TRM is overcome without introducing additional conservatism. A two-term approximation approach is used to transform the filtering error system into a feedback interconnection form, which contains a feedback subsystem with two constant delays and a feedback one with norm-bounded uncertainties. Then, based on a Markovian Lyapunov–Krasovskii functional combined with SSG theorem, new delay-dependent H∞ performance analysis conditions are derived. By applying a linearization procedure, both the full-order and reduced-order filtering design results can be constructed in terms of linear matrix inequalities. Two examples will be given to illustrate the advantages of the proposed results over the existing ones.

Notations: The notations used throughout the paper are standard. T1 ⊗ T2 represents the series connection of mapping T1 and T2. Rn and R+ × Rn denote, respectively, the n-dimensional Euclidean space, and the set of all n × m real matrices; P > 0 means that P is real symmetric and positive definite; Sym[A] is the shorthand notation for A + A′; 1 and 0 represent the identity matrix and a zero matrix, respectively; (Ω, F, P) denotes a complete probability space, in which Ω is the sample space, F is the σ-algebra of subsets of the sample space, and P is the probability measure on F; F[·] stands for the mathematical expectation; ||·|| denotes the Euclidean norm of a vector or its induced norm of a matrix; signals that are square integrable over [0,∞) is denoted by L2[0,∞) with the norm ∥·∥2. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem formulation and preliminaries

A continuous-time Markovian jump linear system (MJLS) with an interval time-varying delay in the state, defined in a complete probability space (Ω, F, P), is governed by state-space equation of the form:

\[
\begin{align*}
\dot{x}(t) &= A_r(t)x(t) + A_d(t)x(t-d(t)) + B_r(t)w(t) \\
y(t) &= C_r(t)x(t) + C_d(t)x(t-d(t)) + D_r(t)w(t) \\
z(t) &= L_r(t)x(t) + L_d(t)x(t-d(t)) + D_d(t)w(t) \\
x(0) &= \phi_1,
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector; \(w(t) \in \mathbb{R}^m\) denotes the disturbance input vector, which belongs to \(L_2[0,\infty)\); \(y(t) \in \mathbb{R}^q\) is the measured output; \(z(t) \in \mathbb{R}^r\) is the signal to be estimated, and \(d(t)\) is a time-varying delay satisfying \(0 \leq d_1 \leq d(t) \leq d_2 < \infty\) and \(d(t) \leq \mu < \infty\), where \(d_1\) and \(d_2\) are known constant scalars representing the lower and upper delay bounds, respectively. In (1), \(\phi_1\) is a vector-valued initial continuous function defined on the interval \([-d_2,0]\); the process \(r(t), t \geq 0\) is a continuous-time homogeneous Markov chain with right continuous trajectories and taking values in a finite set \(\mathcal{T} = \{1, \ldots, N\}\) with transition rate matrix (TRM) \(\Lambda = [\lambda_{ij}]_{N \times N}\) given by

\[
Pr[r(t+h) = j|r(t) = i] = \begin{cases} 
\lambda_{ij}h + o(h), & \text{if } i \neq j \\
1 + \lambda_{ii}h + o(h), & \text{if } i = j.
\end{cases}
\]

where \(h > 0\), \(\lim_{h \to 0} o(h)/h = 0\), and \(\lambda_{ij} \geq 0\), for \(i \neq j\), is the transition rate (TR) from mode \(i\) at time \(t\) to mode \(j\) at time \(t+h\), and \(\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}\). In the sequel, for each possible \(r(t) = i, i \in \mathcal{I}\), the system matrices of the \(i\)th mode are denoted by \((A_i, A_{di}, B_i, C_i, C_{di}, D_{ri}, D_{di}, L_i, L_{di}, D_{di})\), which are real and known matrices with appropriate dimensions.
The TRs of the stochastic process in this paper are considered to be partially available. For instance, in the case of a system (Σ) with four operation modes, the TRM may be as
\[
\begin{bmatrix}
\dot{\lambda}_{11} & \dot{\lambda}_{12} & \dot{\lambda}_{13} & \dot{\lambda}_{14} \\
\dot{\lambda}_{21} & \dot{\lambda}_{22} & \dot{\lambda}_{23} & \dot{\lambda}_{24} \\
\dot{\lambda}_{31} & \dot{\lambda}_{32} & \dot{\lambda}_{33} & \dot{\lambda}_{34} \\
\dot{\lambda}_{41} & \dot{\lambda}_{42} & \dot{\lambda}_{43} & \dot{\lambda}_{44}
\end{bmatrix}
\]
where the elements labeled with the hat ‘’ represent the unknown TRs. For notational clarity, \( \forall i \in \mathcal{I} \), we describe \( \mathcal{I} = \mathcal{I}_b \cup \mathcal{I}_{uk} \) as follows: \( \mathcal{I}_b := \{ j : \lambda_{ij} \text{ is known} \} \), and \( \mathcal{I}_{uk} := \{ j : \dot{\lambda}_{ij} \text{ is unknown} \} \), where \( j \) is the set of indices. Therefore, the filtering design problem is reformulated into the scaled small gain (SSG) problem of the new input–output system (Σ) and \( \mathcal{I}_{uk} \).

The objective of this paper is to design a filter of the following general structure to estimate \( z(t) \),
\[
(\Sigma): \dot{\hat{x}}(t) = A_{\hat{D}} \hat{x}(t) + B_{\hat{D}} y(t)
\]
\[
\hat{z}(t) = C_{\hat{D}} \hat{x}(t) + D_{\hat{D}} y(t),
\]
where \( \hat{x}(t) \in \mathbb{R}^n (n \leq n) \) is the filter state; \( \hat{z}(t) \in \mathbb{R}^q \) is the estimation of \( z(t) \); \( A_{\hat{D}} \in \mathbb{R}^{\hat{n} \times \hat{n}}, B_{\hat{D}} \in \mathbb{R}^{\hat{n} \times p} \), \( C_{\hat{D}} \in \mathbb{R}^{q \times \hat{n}} \) and \( D_{\hat{D}} \in \mathbb{R}^{q \times p} \), \( \forall \hat{i} \in \mathcal{I} \), are filter gains to be determined. It is noted that \( \hat{n} = n \) for the full-order filtering and \( \hat{n} < n \) for the reduced-order filtering.

Augmenting the model in (1) to include the states of the filter in (2), we obtain the following filtering error system:
\[
(\Sigma) : \dot{\tilde{x}}(t) = \tilde{A}_{\Sigma}(t) \tilde{x}(t) + \tilde{B}_{\Sigma} \tilde{w}(t)
\]
\[
\tilde{z}(t) = \Sigma_{\tilde{w}}(t) \tilde{x}(t) + \Sigma_{\tilde{w} \tilde{w}}(t) d(t) + \Sigma_{\tilde{w}}(t) \tilde{w}(t)
\]
where \( \tilde{x}(t) := [x^T(t) \, \dot{x}^T(t)]^T, \tilde{z}(t) := z(t) - \hat{z}(t) \) and
\[
\begin{align*}
\tilde{A}_{\Sigma} &= \begin{bmatrix} A_{\Sigma} & 0 \\ B_{\Sigma} C_{\Sigma} & A_{\Sigma} \end{bmatrix}, & \tilde{B}_{\Sigma} &= \begin{bmatrix} B_{\Sigma} \\ B_{\Sigma} D_{\Sigma} \end{bmatrix}, \\
\Sigma_{\tilde{w}} &= I_{\hat{n}} - D_{\Sigma} C_{\Sigma} - C_{\Sigma}, & \Sigma_{\tilde{w} \tilde{w}} &= I_{\hat{n}} - D_{\Sigma} C_{\Sigma} D_{\Sigma}, \\
\Sigma_{\tilde{w}}(t) &= \begin{bmatrix} I_{\hat{n}} - D_{\Sigma}(t) C_{\Sigma} & -C_{\Sigma} \end{bmatrix}, & \Sigma_{\tilde{w} \tilde{w}}(t) &= \begin{bmatrix} I_{\hat{n}} & 0_{\hat{n} \times \hat{n}} \end{bmatrix}.
\end{align*}
\]

More precisely, we introduce the following definitions for the filtering error system (Σ) in (3), which are essential for the later development.

**Definition 1** (Zhang et al. [25]). The filtering error system (Σ) in (3) is said to be stochastically stable if under \( w(t) = 0 \) and any initial condition \( \tilde{x}(0) \in \mathbb{R}^{(n + \hat{n})}, r(0) \in \mathcal{I} \),
\[
\lim_{t \to \infty} \mathcal{E} \left\{ \int_0^t \| \tilde{x}(\xi, \tilde{x}(0), r(0)) \|^2 d\xi \right\} < \infty.
\]

**Definition 2** (Zhang et al. [25]). Given a scalar \( \gamma > 0 \), the filtering error system (Σ) in (3) is said to be stochastically stable with an \( \mathcal{H}_\infty \) disturbance attenuation performance index \( \gamma \) if it is stochastically stable with \( w(t) = 0 \), and under zero initial condition,
\[
\mathcal{E} \left\{ \int_0^\infty \| \tilde{z}(\xi)(t) \| \| \tilde{z}(\xi)(t) \| d\xi \right\} < \gamma^2 \int_0^\infty \| w(t) \| w(t) dt, \quad \forall 0 \neq w(t) \in L_2[0, \infty).\]

For the filtering design based on (2), it is required that the original system (1) should be stochastically stable. The purpose of this paper is to design a delay-dependent \( \mathcal{H}_\infty \) filter such that the filtering error system (Σ) in (3) is stochastically stable with a prescribed \( \mathcal{H}_\infty \) performance index \( \gamma \) by an indirect approach. The key procedure with this approach is to introduce a new model transformation by employing a two-term approximation for MJLSs with time-varying delay, which converts system (Σ) into an interconnection structure. A particular direction of this transformation is that the \( \mathcal{H}_\infty \) filtering design problem is reformulated into the scaled small gain (SSG) problem of the new input–output model. More details on this approach can be found in [28,31,36]. Here, we just retrospect some preparatory notions on the SSG theorem.

Consider an interconnected system consisting of two subsystems,
\[
L_1 : \xi(t) = G \eta(t), \quad L_2 : \eta(t) = \Delta \xi(t),
\]
which is shown in Fig. 1, where the forward subsystem \( L_1 \) is known and time-invariant with operator \( G \) mapping \( \eta \) to \( \xi \), and the feedback one \( L_2 \) is unknown and time-varying with operator \( \Delta \in \mathcal{D} := \{ \Delta : \| \Delta \| \leq 1 \} \). As a direct ramification of
the SSG theorem [31,33,35,36], a sufficient condition for the robust stability of the interconnected system formed by \( L_1 \) and \( L_2 \) in (5) is provided as follows.

**Lemma 1.** Consider (5) and assume that \( L_1 \) is internally stable. The closed-loop system formed by \( L_1 \) and \( L_2 \) is robustly stable for all \( \Delta \in \mathcal{D} \) if \( \| T_{\eta} \circ G \circ T_{\eta}^{-1} \|_{\infty} < 1 \) holds for some matrices \( \{T_{\xi},T_{\eta}\} \in \mathcal{T} \) with \( \mathcal{T} := \{ (T_{\eta},T_{\xi}) \in \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times \bar{d}} : T_{\eta},T_{\xi} \) nonsingular; \( \| T_{\eta} \circ A \circ T_{\xi}^{-1} \|_{\infty} \leq 1 \} \).

### 3. Main results

In this section, we concentrate on transforming system (\( \Sigma \)) in (3) into an interconnection of two subsystems as in (5) and analyzing the scaled small gain (SSG) of the forward subsystem. A new delay-dependent \( \gamma \)-gain performance criterion will firstly be proposed for system (\( \Sigma \)) with complete known mode information, and then a bounded real lemma (BRL) and corresponding filtering synthesis for the underlying systems with partially accessible mode information will be developed.

#### 3.1. Model transformation

Considering the filtering error system (\( \Sigma \)) in (3), the term \( x(t-d(t)) \) is approximated by \( \frac{1}{2} [x(t-d_1) + x(t-d_2)] \), which brings to the approximation error,

\[
\eta_d(t) = x(t-d(t)) - \frac{1}{2} [x(t-d_1) + x(t-d_2)]
= \frac{1}{2} \int_{-d(t)}^{0} x(t+z) \, dz - \frac{1}{2} \int_{-d_1}^{0} x(t+z) \, dz
= \frac{1}{2} \int_{-d_1}^{0} \rho(z) \xi_d(t+z) \, dz,
\]

where \( \xi_d(t) := \dot{x}(t) \), and

\[
\rho(z) = \begin{cases} 
1 & \text{if } z \leq -d(t), \\
-1 & \text{if } z > -d(t).
\end{cases}
\]

Normalizing \( \eta_d(t) \) in (6) by multiplying 2/d with \( d = d_2 - d_1 \), (6) can be rewritten as,

\[
\eta_d(t) := A_d(t) \xi_d(t) = \frac{1}{d} \int_{-d_1}^{0} \rho(z) \xi_d(t+z) \, dz,
\]

where \( \eta_d(t) := (2/d) \eta_d(t) \), and operator \( A_d : \xi_d \rightarrow \eta_d \), which includes the uncertainties pulled out from \( x(t-d(t)) \) in the original time-delay system (\( \Sigma \)).

Substituting (7) into system (\( \Sigma \)) in (3), the filtering error system (\( \Sigma \)) is converted into a feedback system with the forward sub-system (\( \mathcal{L}_{d_1} \)) and the feedback one (\( \mathcal{L}_{d_2} \)), described by

\[
(\mathcal{L}_{d_1}) : \begin{cases} 
\dot{x}(t) = A_d x(t) + \frac{1}{2} A_dE[x(t-d_1) + x(t-d_2)] + \frac{d}{2} A_d \eta_d(t) + B_1 w(t) \\
\end{cases}
(\mathcal{L}_{d_2}) : \begin{cases} 
\dot{\xi}_d(t) = A_d x(t) + \frac{1}{2} A_dE[x(t-d_1) + x(t-d_2)] + \frac{d}{2} A_d \eta_d(t) + B_1 w(t) \\
\eta_d(t) = A_d(t) \xi_d(t), \quad x(t) = [d_1]^{-1} \xi(t), \quad t \in [-d_2,0].
\end{cases}
\]

In light of the transformation (6), the original system (\( \Sigma \)) is transformed into a feedback interconnection form, which contains a forward sub-system (\( \mathcal{L}_{d_1} \)) with two known constant delays and a feedback one (\( \mathcal{L}_{d_2} \)) with delay uncertainties. The stability problem of system (\( \Sigma \)) can therefore be interpreted as a robust performance analysis problem for the nominal system (\( \mathcal{L}_{d_1} \)) in the face of norm-bounded uncertainties \( A_d(t) \). Of course, the stability of the transformed system in (8)
implies the stability of the original time-delay system in (3). Specifically, the following result can be concluded which, in
the meantime, provides a possible choice of the scaling matrices \( \{T_n,T_{n2}\} \in T \).

**Lemma 2.** For any given invertible matrix \( X \), the operator \( \Lambda_d \) in (8) enjoys the property \( \|X\circ\Lambda_d\circ X^{-1}\|_\infty \leq 1 \).

**Proof.** By virtue of Jensen inequality [28], considering zero initial condition and exchanging the order of integration, it follows that

\[
E\left\{ \int^t_0 \eta_d^T(\tau)X^T\eta_d(\tau) \, d\tau \right\} = \frac{1}{d^2}
\]

\[
E\left\{ \int^t_0 \left[ \int^{\tau-d_1}_{-d_2} \rho(\beta)\xi_d(\tau+\beta) \, d\beta \right]^T \right\}
\]

\[
S\left[ \int^{\tau-d_1}_{-d_2} \rho(\beta)\xi_d(\tau+\beta) \, d\beta \right] \, d\tau \}
\]

\[
\leq \frac{1}{d^2} E\left\{ \int^t_0 d \int^{\tau-d_1}_{-d_2} \xi_d^T(\tau+\beta)SS_d^T(\tau+\beta) \, d\beta \, d\tau \right\}
\]

\[
= \frac{1}{d^2} E\left\{ \int^{\tau-d_1}_{-d_2} \int^t_0 \xi_d^T(\tau+\beta)SS_d(\tau+\beta) \, d\beta \, d\tau \right\}
\]

\[
= \frac{1}{d^2} E\left\{ \int^{\tau-d_1}_{-d_2} \int^t_0 \xi_d^T(\tau)SS_d(\tau+\beta) \, d\beta \, d\tau \right\}
\]

\[
\leq \frac{1}{d^2} E\left\{ \int^{\tau-d_1}_{-d_2} \int^t_0 \xi_d^T(\tau)SS_d(\tau) \, d\beta \, d\tau \right\}
\]

where \( S := X^T X \), which implies that \( \|X\circ\Lambda_d\circ X^{-1}\|_\infty \leq 1 \). This completes the proof. \( \square \)

**Remark 1.** In view of Lemma 2, it follows that the feedback subsystem \( (L_{d2}) \) satisfies \( \|X\circ\Lambda_d\circ X^{-1}\|_\infty \leq 1 \). Then, according to Lemma 1, if there exists the scaling matrix \( (X,X) \in T \) to make the SSG condition \( \|X\circ G_d\circ X^{-1}\|_\infty < 1 \) hold, the input–output stability of system (3) is obtained subsequently. This leads to a new BRL for the transformed system in (8), which will be presented in the next subsection.

### 3.2. New delay-dependent BRL

In the following, based on an input–output approach, we first analyze the SSG of \( (L_{d1}) \), and present a delay-dependent \( \mathcal{H}_\infty \) performance criterion for the interconnected system in (8) with completely known transition rates (TRs). Then, by introducing the lower bounds for the unknown diagonal elements in the transition rate matrix (TRM), a new BRL for the underlying system with partially accessible mode information is further proposed.

**Proposition 1.** The filtering error system in (8) with completely known mode information is stochastically stable with a guaranteed \( \mathcal{H}_\infty \) performance \( \gamma \), if there exist positive-definite symmetric matrices \( P_i \in \mathbb{R}^{m_i \times m_i} \), and \( (S, Q_{1i}, Q_{2i}, Q_{3i}, R_1, R_2, Z_1, Z_2) \in \mathbb{R}^{n \times m} \), for each mode \( i \in \tilde{I} \), such that the following matrix inequalities hold:

\[
\Phi := \begin{bmatrix}
\gamma^2 & \Phi_{12} & \Phi_{13} \\
* & -\gamma^2 & \Phi_{23} \\
* & * & \Phi_{33}
\end{bmatrix} < 0,
\]

(9)

\[
Q_1 := \sum_{j=1}^n \lambda_{ij}(Q_{1j}+Q_{3j})-R_2 \leq 0,
\]

(10)

\[
Q_2 := \sum_{j=1}^n \lambda_{ij}Q_{2j}-R_1 \leq 0,
\]

(11)

\[
Q_3 := \sum_{j=1}^n \lambda_{ij}Q_{3j}-R_2 \leq 0.
\]

(12)
where

\[ Y = \begin{bmatrix}
  y_1 + \sum_{j=1}^{N} \lambda_j p_j & \frac{1}{2} P A_{d1} + E^T Z_1 & \frac{1}{2} P A_{d2} & \frac{1}{2} P A_{d1}
  \\
  * & y_2 & -\frac{1}{2} (1-\mu) Q_{11} + \frac{1}{2} Z_2 & -\frac{1}{2} (1-\mu) Q_{11}
  \\
  * & * & -\frac{1}{2} (1-\mu) Q_{11} - \frac{1}{2} Z_2 & -\frac{1}{2} (1-\mu) Q_{11}
  \\
  * & * & * & -\frac{1}{2} (1-\mu) Q_{11} - \frac{1}{2} Z_2
\end{bmatrix}, \]

\[ y_1 = P A + A^T P + E^T (Q_{11} + Q_{21} + Q_{22} + d_1 R_1 + d_2 R_2 - Z_1) E, \]

\[ y_2 = -\frac{1}{2} (1-\mu) Q_{11} - Z_2 - \frac{1}{2} Z_2, \]

\[ \phi_{12} = [E^T P_1, 0_{m \times 3n}]^T, \]

\[ \phi_{13} = [d_1 A_1 E^T Z_1, d_1 A_1^T E^T S Z_1^T], \]

\[ \phi_{14} = [T_1, E, E^T S Z_1^T], \]

\[ \phi_{22} = [E^T Z_1, 0], \]

\[ \phi_{23} = \text{diag}(-Z_1, -Z_2, -S, -I_4), \]

\[ \mathbf{S} = W^T S W, \quad \mathbf{W} = [0_{n \times 4n} I_4]. \]

**Proof.** Consider the following Markovian Lyapunov–Krasovskii functional (LKF) for the filtering error system in (8),

\[ V(\mathbf{X}(t), r(t), t) = \sum_{s=1}^{4} V_s(\mathbf{X}(t), r(t), t), \]

with

\[ V_1(\mathbf{X}(t), r(t), t) := \mathbf{X}^T(t) P(t) \mathbf{X}(t), \]

\[ V_2(\mathbf{X}(t), r(t), t) := \int_{t-d(t)}^{t} \mathbf{X}^T(x) E^T Q_1(\mathcal{R}(t)) \mathbf{X}(x) \, dx + \int_{t-d_1}^{t} \mathbf{X}^T(x) E^T Q_2(\mathcal{R}(t)) \mathbf{X}(x) \, dx \]

\[ + \int_{t-d_2}^{t} \mathbf{X}^T(x) E^T Q_3(\mathcal{R}(t)) \mathbf{X}(x) \, dx, \]

\[ V_3(\mathbf{X}(t), r(t), t) := \int_{-d_1}^{0} \int_{t+\beta}^{t} \mathbf{X}^T(x) E^T R_1 \mathbf{X}(x) \, dx \, d\beta \]

\[ + \int_{-d_2}^{0} \int_{t+\beta}^{t} \mathbf{X}^T(x) E^T R_2 \mathbf{X}(x) \, dx \, d\beta, \]

\[ V_4(\mathbf{X}(t), r(t), t) := d_1 \int_{-d_1}^{0} \int_{t+\beta}^{t} \mathbf{X}^T(x) E^T Z_1 \mathbf{X}(x) \, dx \, d\beta \]

\[ + d_2 \int_{-d_2}^{0} \int_{t+\beta}^{t} \mathbf{X}^T(x) E^T Z_2 \mathbf{X}(x) \, dx \, d\beta. \]

Let \( \mathcal{D} \) be the weak infinitesimal generator of random process \( \{\mathbf{X}(t), r(t)\} \), for each \( r(t) = i \in \mathcal{I} \), and define

\[ \mathcal{D}[V(\mathbf{X}(t), r(t), t)] := \lim_{\delta \to 0} \frac{1}{\delta} [E(V(\mathbf{X}(t+\delta), r(t+\delta), t+\delta)|\mathbf{X}(t), r(t) = i) - V(\mathbf{X}(t), r(t), t)], \]

\[ V_s := \mathcal{D}[V_s(\mathbf{X}(t), r(t), t)], \quad s = 1, 2, 3, 4. \]

Then, we have

\[ V_1 = 2 \mathbf{X}^T(t) P \mathbf{X}(t) + \mathbf{X}^T(t) \left( \sum_{j=1}^{N} \lambda_j p_j \right) \mathbf{X}(t), \]

\[ V_2 \leq \mathbf{X}^T(t) E^T Q_1 \mathbf{X}(t) - (1-\mu) \mathbf{X}^T(t-d(t)) E^T Q_1 \mathbf{X}(t-d(t)) \]

\[ + \int_{t-d(t)}^{t} \mathbf{X}^T(x) E^T \left( \sum_{j=1}^{N} \lambda_j q_{1j} \right) \mathbf{X}(x) \, dx \]

\[ + \mathbf{X}^T(t) E^T Q_2 \mathbf{X}(t) - \mathbf{X}^T(t-d_1) E^T Q_2 \mathbf{X}(t-d_1) \]

\[ + \int_{t-d_1}^{t} \mathbf{X}^T(x) E^T \left( \sum_{j=1}^{N} \lambda_j q_{2j} \right) \mathbf{X}(x) \, dx \]
\[ + \mathbf{x}^T(t)E^I Q_3 \mathbf{x}(t) + \int_{t-d_2}^{t-d_1} \mathbf{x}^T(\tau)E^I \left( \sum_{j=1}^{N} \lambda_j Q_{3j} \right) \mathbf{x}(\tau) d\tau \]

\[- \mathbf{x}^T(t-d_2)E^I Q_3 \mathbf{x}(t-d_2) + \int_{t-d_1}^{t} \mathbf{x}^T(\tau)E^I \left( \sum_{j=1}^{N} \lambda_j Q_{3j} \right) \mathbf{x}(\tau) d\tau, \tag{17} \]

\[ \mathcal{V}_3 = d_1 \mathbf{x}^T(t)E^I R_1 \mathbf{x}(t) - \int_{t-d_1}^{t} \mathbf{x}^T(\tau)E^I R_1 \mathbf{x}(\tau) d\tau \]

\[ + d_2 \mathbf{x}^T(t)E^I R_2 \mathbf{x}(t) - \int_{t-d_1}^{t} \mathbf{x}^T(\tau)E^I R_2 \mathbf{x}(\tau) d\tau \]

\[- \int_{t-d_2}^{t-d_1} \mathbf{x}^T(\tau)E^I R_2 \mathbf{x}(\tau) d\tau, \tag{18} \]

\[ \mathcal{V}_4 = d_1^2 \mathbf{x}^T(t)E^I Z_1 \mathbf{x}(t) - d_1 \int_{t-d_1}^{t} \mathbf{x}^T(\tau)E^I Z_1 \mathbf{x}(\tau) d\tau \]

\[ + d_2^2 \mathbf{x}^T(t)E^I Z_2 \mathbf{x}(t) - d \int_{t-d_1}^{t} \mathbf{x}^T(\tau)E^I Z_2 \mathbf{x}(\tau) d\tau \]

\[- d \int_{t-d_2}^{t-d_1} \mathbf{x}^T(\tau)E^I Z_2 \mathbf{x}(\tau) d\tau. \tag{19} \]

Using Jensen inequality in [28], we have that

\[ \mathcal{V}_4 \leq d_1^2 \mathbf{x}^T(t)E^I Z_1 \mathbf{x}(t) + d_2^2 \mathbf{x}^T(t)E^I Z_2 \mathbf{x}(t) \]

\[- (\mathbf{x}(t)-\mathbf{x}(t-d_1))^T E^I Z_1 E(\mathbf{x}(t)-\mathbf{x}(t-d_1)) \]

\[- (\mathbf{x}(t-d_2)-\mathbf{x}(t-d_1))^T E^I Z_2 E(\mathbf{x}(t-d_2)-\mathbf{x}(t-d_1)) \]

\[- (\mathbf{x}(t-d_2)-\mathbf{x}(t-d_1))^T E^I Z_2 E(\mathbf{x}(t-d_2)-\mathbf{x}(t-d_2)). \tag{20} \]

Then, it follows from (14) under \( w(t) = 0 \) that

\[ \mathcal{D}[\mathcal{V}(\mathbf{x}(t), \mathbf{r}(t), t)] = \sum_{i=1}^{4} \mathcal{V}_i \leq \zeta^T(t) [Y + A_1^T E^I (d_1^2 Z_1 + d_2^2 Z_2) E A_1] \zeta(t) + Q(t), \tag{21} \]

where

\[ \zeta(t) := [\mathbf{x}^T(t) \mathbf{x}^T(t-d_1) \mathbf{x}^T(t-d_2)]^T, \]

\[ Q(t) := \int_{t-d_1}^{t} \mathbf{x}^T(\tau)E^I Q_1 \mathbf{x}(\tau) d\tau + \int_{t-d_2}^{t} \mathbf{x}^T(\tau)E^I Q_2 \mathbf{x}(\tau) d\tau \]

\[ + \int_{t-d_2}^{t-d_1} \mathbf{x}^T(\tau)E^I Q_3 \mathbf{x}(\tau) d\tau, \]

and \( Y, A_1, Q_1, Q_2, \) and \( Q_3 \) are defined in (13).

To proceed by the SSS theorem, by introducing the following index and applying the Schur complement to the condition in (9), together with (10)--(12), we obtain,

\[ J := \mathcal{E} \left\{ \int_{0}^{\infty} \left[ \zeta^T(\alpha) S \zeta(\alpha) - \eta_3(\alpha) \mathcal{S} \eta_3(\alpha) \right] d\alpha \right\} \]

\[ \leq \mathcal{E} \left\{ \int_{0}^{\infty} [\mathcal{D}[\mathcal{V}(\mathbf{x}(t), \mathbf{r}(t), t)] + \zeta^T(\alpha) S \zeta(\alpha) - \eta_3(\alpha) \mathcal{S} \eta_3(\alpha)] d\alpha \right\} \]

\[ \leq \mathcal{E} \left\{ \int_{0}^{\infty} \zeta^T(\alpha) [Y + A_1^T E^I (d_1^2 Z_1 + d_2^2 Z_2 + S) E A_1] \zeta(\alpha) d\alpha \right. \]

\[ \left. + \int_{0}^{\infty} Q(t) d\alpha \right\} < 0. \tag{22} \]

where \( \mathcal{S} \) is defined in (13). Thus, it is easy to see that (22) implies

\[ \mathcal{E} \left\{ \int_{0}^{\infty} \zeta^T(\alpha) S \zeta(\alpha) d\alpha \right\} < \mathcal{E} \left\{ \int_{0}^{\infty} \eta_3(\alpha) \mathcal{S} \eta_3(\alpha) d\alpha \right\}. \tag{23} \]

Defining \( S = X^T X \), it follows from (23) that \( \|X \circ G_d \circ X^{-1}\|_\infty < 1 \).

According to Lemma 2, the feedback subsystem satisfies \( \|X \circ A_d \circ X^{-1}\|_\infty \leq 1 \), such that based on Lemma 1, the filtering error system with \( w(t) = 0 \) is stochastically stable.

Now we shall establish the \( \mathcal{H}_\infty \) performance criterion of the filtering error system under zero initial condition.
Choose a Markovian LKF for the filtering error system in (8) as

$$
\nabla(\mathbf{x}(t), \mathbf{r}(t), t) := V(\mathbf{x}(t), \mathbf{r}(t), t) + \frac{1}{\alpha} \int_{-d_1}^{d_1} \int_{t}^{t+\beta} \mathbf{z}_d^T(\xi) \mathbf{z}_d(\xi) \, d\xi \, d\beta.
$$

(24)

Furthermore,

$$
\begin{align*}
\mathcal{D}[\nabla(\mathbf{x}(t), \mathbf{r}(t), t)] &= \mathcal{D}[V(\mathbf{x}(t), \mathbf{r}(t), t)] \\
&= \mathcal{D}[V(\mathbf{x}(t), \mathbf{r}(t), t)] + \mathcal{E}[\mathbf{z}_d^T(t) \mathbf{z}_d(t)] \\
&= \mathcal{D}[\nabla(\mathbf{x}(t), \mathbf{r}(t), t)] + \mathcal{E}[\mathbf{z}_d^T(t) \mathbf{z}_d(t)] \\
&= \mathcal{D}[\nabla(\mathbf{x}(t), \mathbf{r}(t), t)] + \mathcal{E}[\mathbf{z}_d^T(t) \mathbf{z}_d(t) - \eta_d^T(t) \eta_d(t)].
\end{align*}
$$

(25)

By applying Jensen inequality to the last term in (25), we have

$$
\begin{align*}
\mathcal{D}[\nabla(\mathbf{x}(t), \mathbf{r}(t), t)] &\leq \mathcal{D}[V(\mathbf{x}(t), \mathbf{r}(t), t)] + \mathcal{E}[\mathbf{z}_d^T(t) \mathbf{z}_d(t)] \\
&= \mathcal{D}[\nabla(\mathbf{x}(t), \mathbf{r}(t), t)] + \mathcal{E}[\mathbf{z}_d^T(t) \mathbf{z}_d(t) - \eta_d^T(t) \eta_d(t)].
\end{align*}
$$

(26)

Consider the following index:

$$
J := \mathcal{E} \left\{ \int_{0}^{\infty} \mathbf{z}^T(\mathbf{z}(t) - \gamma^2 \mathbf{w}(t) \mathbf{w}(t)) \, dt \right\}. 
$$

(27)

Under zero initial condition, we have

$$
\begin{align*}
J &\leq \mathcal{E} \left\{ \int_{0}^{\infty} [\mathcal{D}[\nabla(\mathbf{x}(t), \mathbf{r}(t), t)] + \mathbf{z}_d^T(t) \mathbf{z}_d(t) - \gamma^2 \mathbf{w}(t) \mathbf{w}(t)] \, dt \right\} \\
&\leq \mathcal{E} \left\{ \int_{0}^{\infty} \mathbf{z}_d^T(t) \mathbf{z}_d(t) \mathbf{w}(t) \mathbf{w}(t) \right\} \, dt + \int_{0}^{\infty} \mathcal{Q}(t) \, dt,
\end{align*}
$$

(28)

where

$$
\Phi = \begin{bmatrix}
\mathbf{T} - \mathbf{S} & \mathbf{\Phi}_{12} & \mathbf{\Phi}_{13} \\
\mathbf{\Phi}_{12}^T & \mathbf{\Phi}_{22} & \mathbf{\Phi}_{23} \\
\mathbf{\Phi}_{13}^T & \mathbf{\Phi}_{23} & \mathbf{\Phi}_{33}
\end{bmatrix},
$$

with \( \mathbf{\Phi}_{12}, \mathbf{\Phi}_{13}, \mathbf{\Phi}_{23}, \) and \( \mathbf{\Phi}_{33} \) defined in (13).

Based on the Schur complement, the conditions in (9)-(12) imply that \( \Phi < 0 \) and \( \mathcal{Q}(t) < 0 \). Therefore,

$$
\mathcal{E} \left\{ \int_{0}^{\infty} \mathbf{z}_d^T(\mathbf{z}(t) \mathbf{w}(t)) \, dt \right\} < \gamma^2 \int_{0}^{\infty} \mathbf{w}(t) \mathbf{w}(t) \, dt.
$$

(29)

This completes the proof. \( \square \)

**Remark 2.** Based on an input–output approach together with the Markovian LKF given in (14), new delay-dependent conditions on \( \mathcal{H}_\infty \) performance analysis are obtained for the filtering error system in (8) with completely known TRs in Proposition 1. The input–output approach based on SSG theorem provides a unified framework to deal with time-varying delays, which aims to reduce the conservatism of filtering design. In addition, considering the transformed model in (8) with two constant time-delays, it is noted that the delay partitioning technique in [13,24,26] can be employed to analyze the SSG of (L4) to further reduce the conservatism. To emphasize the superiority of the input–output method and make the presentation more lucid, we only propose the results in Proposition 1 but without using delay-partitioning method.

In Proposition 1, the stability analysis of the filtering error system is recast as an SSG problem of the transformed system in the face of delay uncertainties. The following theorem presents a sufficient condition for the \( \mathcal{H}_\infty \) performance analysis of the filtering error system in (8) with partially accessible mode information.

**Theorem 1.** The filtering error system in (8) with partially accessible mode information is stochastically stable with a guaranteed \( \mathcal{H}_\infty \) performance \( \gamma \), if there exist positive-definite symmetric matrices \( P_i \in \mathbb{R}^{(n + \ell_1) \times (n + \ell_1)} \) and \( S, Q_{d1}, Q_{d2}, Q_{s1}, R_{d1}, R_{d2}, Z_{d1}, Z_{d2} \in \mathbb{R}^{n \times n} \), for each mode \( i \in I \), such that the following matrix inequalities hold:

$$
\mathcal{F} \geq \begin{bmatrix}
\mathbf{T} - \mathbf{S} & \mathbf{\Phi}_{12} & \mathbf{\Phi}_{13} \\
\mathbf{\Phi}_{12}^T & \mathbf{\Phi}_{22} & \mathbf{\Phi}_{23} \\
\mathbf{\Phi}_{13}^T & \mathbf{\Phi}_{23} & \mathbf{\Phi}_{33}
\end{bmatrix} < 0, \quad j \in I_{\ell_{1k}},
$$

(30)
\[
\begin{align*}
Q_{k1}^{(i)} + Q_{k3}^{(i)} - R_2 - \lambda_k^{(i)} (Q_{1j} + Q_{3j}) & \leq 0, \quad j \in T_{uk}^{(i)} \\
Q_{k2}^{(i)} - R_1 - \lambda_k^{(i)} Q_{2j} & \leq 0, \quad j \in T_{uk}^{(i)} \\
Q_{k3}^{(i)} - R_2 - \lambda_k^{(i)} Q_{3j} & \leq 0, \quad j \in T_{uk}^{(i)} 
\end{align*}
\]
if \( i \in T_k^{(i)} \), \hspace{1cm} (31)

\[
\begin{align*}
Q_{k1}^{(i)} + Q_{k3}^{(i)} - R_2 + \lambda_d^{(i)} (Q_{1j} + Q_{3j}) - (\lambda_d^{(i)} + \lambda_k^{(i)}) (Q_{1j} + Q_{3j}) & \leq 0, \quad j \in T_{uk}^{(i)} \\
Q_{k2}^{(i)} - R_1 + \lambda_d^{(i)} Q_{2j} - (\lambda_d^{(i)} + \lambda_k^{(i)}) Q_{2j} & \leq 0, \quad j \in T_{uk}^{(i)} \\
Q_{k3}^{(i)} - R_2 + \lambda_d^{(i)} Q_{3j} - (\lambda_d^{(i)} + \lambda_k^{(i)}) Q_{3j} & \leq 0, \quad j \in T_{uk}^{(i)} 
\end{align*}
\]

if \( i \in T_{uk}^{(i)} \), \hspace{1cm} (32)

where

\[
\begin{align*}
\overline{T}_i := \begin{bmatrix}
\bar{T}_1 & \frac{1}{2} \bar{P}_d \bar{A}_{di} + E^T Z_1 & \frac{1}{2} \bar{P}_d \bar{A}_{di} & \frac{1}{2} \bar{P}_d \bar{A}_{di} \\
0 & \frac{1}{2} (1-\mu) Q_{11} + \frac{1}{2} Z_2 & -\frac{1}{2} (1-\mu) Q_{11} \\
0 & 0 & \frac{1}{2} (1-\mu) Q_{11} - \frac{1}{2} Z_2 & -\frac{1}{2} (1-\mu) Q_{11} \\
0 & 0 & 0 & -\frac{1}{2} (1-\mu) Q_{11} - \frac{1}{2} Z_2
\end{bmatrix}
\end{align*}
\]

\[
Q_{k1}^{(i)} := \sum_{j \in T_{uk}^{(i)}} \hat{\lambda}_i Q_{1j}, \quad Q_{k2}^{(i)} := \sum_{j \in T_{uk}^{(i)}} \hat{\lambda}_i Q_{2j}, \quad Q_{k3}^{(i)} := \sum_{j \in T_{uk}^{(i)}} \hat{\lambda}_i Q_{3j},
\]

(33)

with

\[
\begin{align*}
T_1 & := \begin{cases}
Y_1 + P_k^{(i)} - \lambda_k^{(i)} P_j, & j \in T_{uk}^{(i)}, \quad \text{if } i \in T_k^{(i)}, \\
Y_1 + P_k^{(i)} + \lambda_d^{(i)} P_j - (\lambda_d^{(i)} + \lambda_k^{(i)}) P_j, & j \in T_{uk}^{(i)}, \quad \text{if } i \in T_k^{(i)},
\end{cases} \\
P_k^{(i)} & := \sum_{j \in T_{uk}^{(i)}} \hat{\lambda}_i P_j,
\end{align*}
\]

and \( \overline{S}, \Phi_{12}, \Phi_{13}, \Phi_{23}, \Phi_{33}, Y_1, \) and \( Y_2 \) are defined in Proposition 1; \( \lambda_d^{(i)} \) is a given lower bound for the unknown diagonal element \( \hat{\lambda}_d \).

**Proof.** Based on Proposition 1, it is shown that the filtering error system in (8) subject to completely known TRs is stochastically stable with a prescribed performance \( \gamma \) if (9)–(12) hold. Since the diagonal elements in the TRM may contain unknown ones, we shall separate the proof of Theorem 1 into two cases, \( i \in T_k^{(i)} \) and \( i \in T_{uk}^{(i)} \).

(i) \( i \in T_k^{(i)} \)

In this case, \( i \in T_k^{(i)} \) implies that \( \hat{\lambda}_i \) is known, then it is straightforward that \( \lambda_k^{(i)} \leq 0 \). Here, we only need to consider \( \lambda_k^{(i)} < 0 \), since if \( \lambda_k^{(i)} = 0 \), condition (9) can be readily viewed as a direct result.

Notice that the term \( \sum_{j=1}^{N} \hat{\lambda}_i P_j \) in (9) can be treated as

\[
\sum_{j=1}^{N} \hat{\lambda}_i P_j = P_k^{(i)} + \sum_{j \in T_{uk}^{(i)}} \hat{\lambda}_j P_j = P_k^{(i)} - \lambda_k^{(i)} \sum_{j \in T_{uk}^{(i)}} \frac{\hat{\lambda}_j}{-\lambda_k} P_j,
\]

(34)

where \( P_k^{(i)} := \sum_{j \in T_{uk}^{(i)}} \hat{\lambda}_j P_j \), and the elements \( \hat{\lambda}_j, j \in T_{uk}^{(i)} \) are unknown. Since \( 0 \leq \hat{\lambda}_j / -\lambda_k \leq 1 \) and \( \sum_{j \in T_{uk}^{(i)}} \frac{\hat{\lambda}_j}{-\lambda_k} = 1 \), (34) becomes

\[
\sum_{j=1}^{N} \hat{\lambda}_j P_j = \sum_{j \in T_{uk}^{(i)}} \frac{\hat{\lambda}_j}{-\lambda_k} [P_k^{(i)} - \lambda_k^{(i)} P_j].
\]

(35)

Thus, for \( 0 \leq \hat{\lambda}_j \leq -\lambda_k^{(i)} \), the left-hand side of inequality (9) can be rewritten as

\[
\Phi = \sum_{j \in T_{uk}^{(i)}} \frac{\hat{\lambda}_j}{-\lambda_k} [\overline{T} - \overline{S} \Phi_{12} \Phi_{13} \Phi_{23} \Phi_{33}] = \sum_{j \in T_{uk}^{(i)}} \frac{\hat{\lambda}_j}{-\lambda_k} \overline{T},
\]

(36)

where \( \overline{T} \) is defined in (33) with \( \overline{T}_1 = Y_1 + P_k^{(i)} - \lambda_k^{(i)} P_j \), and \( \overline{S}, \overline{T}, \Phi_{12}, \Phi_{13}, \Phi_{23}, \) and \( \Phi_{33} \) are defined in (13). Then, (9) holds if and only if \( \overline{\Phi} < 0 \) in (36), which implies that, in the presence of unknown elements \( \hat{\lambda}_j, i \in T_k^{(i)}, j \in T_{uk}^{(i)} \), inequality (30) is equivalent to (9).

(ii) \( i \in T_{uk}^{(i)} \)

Following a similar argument as in (i), we only consider \( \hat{\lambda}_i < -\lambda_k^{(i)} \) here, since if \( \hat{\lambda}_i = -\lambda_k^{(i)} \), then the ith row of the TRM is completely known.
Equivalently, for this case, the term $\sum_{j=1}^{N} \hat{\lambda}_{i} p_{j}$ in (9) can be expressed as

$$
\sum_{j=1}^{N} \hat{\lambda}_{i} p_{j} = T_{k}^{(i)} + \sum_{j=1, j \neq i}^{N} \hat{\lambda}_{i} p_{j} = T_{k}^{(i)} + \hat{\lambda}_{i} p_{i} + \left(-\hat{\lambda}_{i} - \lambda_{k}^{(i)}\right) \sum_{j=1, j \neq i}^{N} \frac{\hat{\lambda}_{ij}}{\lambda_{k}^{(i)}} p_{j},
$$

(37)

where $T_{k}^{(i)} = \sum_{j=1}^{N} \hat{\lambda}_{i} p_{j}$.

Likewise, since $0 \leq \frac{\lambda_{ij}}{\lambda_{k}^{(i)}} \leq 1$ and $\sum_{j=1, j \neq i}^{N} \frac{\lambda_{ij}}{\lambda_{k}^{(i)}} = 1$, we have

$$
\sum_{j=1}^{N} \hat{\lambda}_{i} p_{j} = \sum_{j=1, j \neq i}^{N} \frac{\lambda_{ij}}{\lambda_{k}^{(i)}}\{T_{k}^{(i)} + \hat{\lambda}_{i} p_{i} + \left(-\hat{\lambda}_{i} - \lambda_{k}^{(i)}\right) p_{j}\}.
$$

(38)

Correspondingly, we can rewrite the left-hand side of the inequality in (9) as

$$
\Phi = \sum_{j=1, j \neq i}^{N} \frac{\lambda_{ij}}{\lambda_{k}^{(i)}}\left\{ T_{k}^{(i)} \begin{bmatrix} \Phi_{12} & \Phi_{13} \\ \ast & -\gamma^{2} I \end{bmatrix} \begin{bmatrix} \Phi_{23} \\ \ast \ast \Phi_{33} \end{bmatrix} \right\} < 0, \quad j \in T_{nk}^{(i)}, \ j \neq i.
$$

(39)

where $T$ is defined in (33) with $T_{k} = Y_{i} + T_{k}^{(i)} + \hat{\lambda}_{i} p_{i} - (\hat{\lambda}_{i} + \lambda_{k}^{(i)}) p_{i}$, and $T_{k}, T_{k}^{*}, \Phi_{12}, \Phi_{13}, \Phi_{23}$, and $\Phi_{33}$ are defined in (13). Deduced from (39), $\Phi < 0$ is equivalent to

$$
\Phi = \left[ \begin{array}{ccc} T_{k}^{(i)} & \Phi_{12} & \Phi_{13} \\ \ast & -\gamma^{2} I & \Phi_{23} \\ \ast & \ast & \Phi_{33} \end{array} \right] < 0, \quad j \in T_{nk}^{(i)}, \ j \neq i.
$$

(40)

For tractability, by introducing a lower bound $\lambda_{d}^{(i)}$ for the unknown element $\lambda_{d}$, we have

$$
\lambda_{d}^{(i)} \leq \hat{\lambda}_{i} < -\lambda_{k}^{(i)},
$$

(41)

which implies that $\lambda_{d}$ may take any value in $[\lambda_{d}^{(i)}, -\lambda_{k}^{(i)} + \epsilon]$ for some sufficiently small $\epsilon \leq 0$. Then $\hat{\lambda}_{i}$ can be directly written as a convex combination

$$
\hat{\lambda}_{i} = -\kappa \lambda_{k}^{(i)} + \kappa \lambda_{d}^{(i)} + (1 - \kappa) \lambda_{d}^{(i)},
$$

(42)

where $0 \leq \kappa \leq 1$. Since $\hat{\lambda}_{i}$ in (42) depends on $\kappa$ linearly, and (40) therefore needs only to be satisfied for $\kappa = 0$ and $\kappa = 1$, that is, (40) holds if and only if the following inequalities in (43)–(44) simultaneously hold:

$$
\left[ \begin{array}{ccc} T_{k}^{(i)} & \Phi_{12} & \Phi_{13} \\ \ast & -\gamma^{2} I & \Phi_{23} \\ \ast & \ast & \Phi_{33} \end{array} \right] < 0, \quad j \in T_{nk}^{(i)}, \ j \neq i,
$$

(43)

where $T$ is defined in (33) with $T_{k} = Y_{i} + T_{k}^{(i)} - \lambda_{k}^{(i)} p_{i}(P_{i} - P_{j})$, and

$$
\left[ \begin{array}{ccc} T_{k}^{(i)} & \Phi_{12} & \Phi_{13} \\ \ast & -\gamma^{2} I & \Phi_{23} \\ \ast & \ast & \Phi_{33} \end{array} \right] < 0, \quad j \in T_{nk}^{(i)}, \ j \neq i,
$$

(44)

where $T$ is defined in (33) with $T_{k} = Y_{i} + T_{k}^{(i)} + \lambda_{d}^{(i)} (P_{i} - P_{j}) - \lambda_{k}^{(i)} p_{i}$. Since $\epsilon$ is small enough, (43) holds if and only if

$$
\left[ \begin{array}{ccc} T_{k}^{(i)} & \Phi_{12} & \Phi_{13} \\ \ast & -\gamma^{2} I & \Phi_{23} \\ \ast & \ast & \Phi_{33} \end{array} \right] < 0, \quad j \in T_{nk}^{(i)}, \ j \neq i,
$$

(45)

where $T$ is defined in (33) with $T_{k} = Y_{i} + T_{k}^{(i)} - \lambda_{k}^{(i)} p_{i}$, which is implied by (44) when $j = i, j \in T_{nk}^{(i)}$. Hence, the inequality (9) can be replaced by (30) in the context $\forall j \in T_{nk}^{(i)}$.

In addition, following a similar procedure presented above, the conditions in (31)–(32) can also be obtained based on (10)–(12), respectively.

In summary, with the presence of unknown elements in the TRM, one can readily conclude that the filtering error system in (8) is stochastically stable with a prescribed $\mathcal{H}_{\infty}$ performance index $\gamma$ if (30)–(32) hold.

**Remark 3.** In order to render the unknown diagonal elements numerically tractable, in Theorem 1, the lower bounds for the unknown diagonal elements are introduced. Together with the property that the sum of each row is zero in a TRM,
a convex combination in (42) can thus be obtained. Thanks to the convex combination of \( \gamma_{d}^{(0)} \) and \( \gamma_{k}^{(0)} \), sufficient conditions for the \( \mathcal{H}_\infty \) performance analysis have been derived in Theorem 1. Yet, the obtained conditions are no loss of generality, since the lower bound \( \gamma_{d}^{(0)} \) of \( \gamma_{d} \) is allowed to be arbitrarily small.

In the following, we will give the filtering design result in the presence of partially accessible mode information, which relies heavily on the delay-dependent BRL presented in Theorem 1.

### 3.3. Delay-dependent \( \mathcal{H}_\infty \) filtering design

In this subsection, we consider both the full-order and reduced-order \( \mathcal{H}_\infty \) filtering designs. It is shown that the parametrized representation of the filter gains can be obtained in terms of the feasible solutions to a set of linear matrix inequalities (LMIs).

**Theorem 2.** The filtering error system in (8) with partially accessible mode information is stochastically stable with a guaranteed \( \mathcal{H}_\infty \) performance \( \gamma \), if there exist positive-definite symmetric matrices \( P_i = [P_{ij}^{(1)}, P_{ij}^{(2)}] \in \mathbb{R}^{(n + \hat{n}) \times (n + \hat{n})} \), \( \{S, Q_{11}, Q_{21}, Q_{31}, R_1, R_2, Z_1, Z_2 \} \in \mathbb{R}^{n \times n} \), and matrices \( \mathbf{A}_i \in \mathbb{R}^{n \times n} \), \( \mathbf{B}_i \in \mathbb{R}^{p \times n} \), \( \mathbf{C}_i \in \mathbb{R}^{n \times n} \), and \( \mathbf{D}_i \in \mathbb{R}^{q \times p} \), for each mode \( i \in \mathcal{I} \), such that the conditions (31)–(32) and the following LMIs hold:

\[
\hat{\phi}_{11} = \begin{bmatrix}
\Psi_1 & \Psi_2 & \Psi_3 + Z_1 & \Psi_4 & \Psi_5 & \Psi_6 & d\Psi_3 \\
* & -\gamma^2 I & \Psi_7 & \Psi_8 & \Psi_9 & \Psi_{10} & d\Psi_5 \\
* & * & -\frac{1}{2}(1-\mu)Q_{11} + \frac{1}{2}Z_2 & \Psi_{11} & \Psi_{12} & \Psi_{13} & d\Psi_3 \\
* & * & * & -\frac{1}{2}(1-\mu)Q_{11} - \frac{1}{2}Z_2 & \Psi_{14} & \Psi_{15} & d\Psi_5 \\
* & * & * & * & -d\gamma^2(1-\mu)Q_{11} - d\gamma^2Z_2 - S
\end{bmatrix} < 0, \quad j \in T_{uk},
\]

where

\[
\hat{\phi}_{12} = [B_i^T P_{ij}^{(1)} + D_i^T \mathbf{B}_i^T \mathbf{H}_i^{(2)} B_i^T H P_{ij}^{(2)} + D_i^T \mathbf{B}_i^T \mathbf{H}_i^{(2)} \mathbf{O}_{m \times 3n}]^T.
\]

\[
H = \{I_0 \ 0_{n \times (n-\hat{n})}\}^T,
\]

\[
\hat{\phi}_{13} := [d_1 \mathbf{A}_i^T E^T Z_1 \ d_2 \mathbf{A}_i^T E^T Z_2 \ \mathbf{A}_i^T E^T S \ \mathbf{Z}_i^T],
\]

\[
\hat{\phi}_{23} := [d_1 B_i^T Z_1 \ d_2 B_i^T Z_2 \ B_i^T S \ D_i - D_i D_i],
\]

\[
\hat{\phi}_{33} := \text{diag}(-Z_1, -Z_2, -S, -I_0),
\]

\[
\mathbf{A}_i := [A_i \ 0_{n \times n} \ \frac{1}{2} A_{di} \ \frac{1}{2} A_{di} \ \frac{1}{2} A_{di}],
\]

\[
\mathbf{Z}_i := [L_i - D_i C_i - C_i \ \frac{1}{2} L_{di} - \frac{1}{2} D_i C_{di} \ \frac{1}{2} L_{di} - \frac{1}{2} D_i C_{di} \ \frac{1}{2} L_{di} - \frac{1}{2} D_i C_{di}],
\]

\[
\Psi_1 = \Gamma_i + \frac{1}{2} P_{ij}^{(1)} - \frac{1}{2} P_{ij}^{(1)}, \quad j \in T_{uk}^{(i)}
\]

\[
\Psi_2 = \frac{1}{2} H P_{ij}^{(2)} + \frac{1}{2} P_{ij}^{(2)} - \frac{1}{2} H P_{ij}^{(2)}, \quad j \in T_{uk}^{(i)}
\]

\[
\Psi_3 = \frac{1}{2} P_{ij}^{(1)} A_{di} + \frac{1}{2} H \mathbf{B}_i^T C_{di},
\]

\[
\Psi_5 = \frac{1}{2} P_{ij}^{(2)} H^T A_{di} + \frac{1}{2} \mathbf{B}_i^T C_{di},
\]

\[
\Psi_6 = -\frac{1}{2} (1-\mu)Q_{11} - Q_{21} - Z_1 - \frac{1}{2} Z_2,
\]

\[
\Gamma_1 := \text{Sym}(P_{ij} A_i + H \mathbf{B}_i C_i) + Q_{11} + Q_{21} + Q_{31} + d_1 R_1 + d_2 R_2 - Z_1,
\]

\[
\Gamma_2 := H \mathbf{A}_i + A_i^T H P_{ij}^{(2)} + C_i^T \mathbf{B}_i^T,
\]

\[
\rho_{k}^{(1)} = \sum_{j \in T_{uk}^{(i)}} \lambda_{ij} P_{ij}^{(1)}, \quad \rho_{k}^{(2)} = \sum_{j \in T_{uk}^{(i)}} \lambda_{ij} P_{ij}^{(2)}.
\]
Moreover, if the above conditions have a set of feasible solutions \( \{P_i\}_{i=1}^\infty \), then an admissible \( n \)-order filter in the form of (2) can be obtained as

\[
A_i = P_i^{-1} \overline{A}_i, \quad B_i = P_i^{-1} \overline{B}_i, \quad C_i = C, \quad D_i = D_i.
\]

(48)

**Proof.** It follows from Theorem 1 that if we can show (30)-(32), then the claimed result follows. For simplicity in filter synthesis procedure, we first partition the Lyapunov matrices \( P_i \) in Theorem 1 as

\[
P_i = \begin{bmatrix} P_{i1} & HP_{i2} \\ * & P_{i3} \end{bmatrix},
\]

where \( H := [I, \, 0]_i^{(n\times n)} \), \( P_{i1} \in \mathbb{R}^{n \times n} \), \( P_{i2} \in \mathbb{R}^{d \times n} \), and \( P_{i3} \in \mathbb{R}^{d \times d} \). Then, similar to [14], performing a congruent transformation to \( P_i \) by \( \text{diag}(I_n, P_{i2}P_{i3}^{-1}) \) yields,

\[
\begin{bmatrix} P_{i1} & HP_{i2} \\ * & P_{i3}P_{i2}^{-1}P_{i3}^{-1} \end{bmatrix} = \begin{bmatrix} P_{i1} & HP_{i2} \\ * & P_{i3} \end{bmatrix}.
\]

(50)

Thus, without loss of generality, we can directly specify the Lyapunov matrices as

\[
P_i = \begin{bmatrix} P_{i1} & HP_{i2} \\ * & P_{i3} \end{bmatrix}.
\]

(51)

It is noted that in this way the matrix variables \( P_{i2} \) are set as Markovian and can be absorbed directly by the filter gain variables \( A_i \) and \( B_i \) by introducing

\[
\overline{A}_i = P_i^{-1} \overline{A}_i, \quad \overline{B}_i = P_i^{-1} \overline{B}_i, \quad i \in I.
\]

(52)

On the other hand, \( P_{i1} > 0 \) implies that \( P_{i1} \) is nonsingular. Then, the filter gains can be constructed by (48). This completes the proof. \( \square \)

**Remark 4.** Theorem 2 provides a new delay-dependent condition on \( \mathcal{H}_\infty \) filtering synthesis problem for continuous-time MJLSs in (1) with time-varying delay and partially accessible mode information. Following the similar arguments as in [14], the condition in Theorem 2 can be readily extended to the cases in which the systems contain parametric uncertainties. It is also worth mentioning that the condition given in Theorem 2 depends on the derivative of time-varying delay with \( d(t) \leq \mu < \infty \). However, it can be readily generalized for the case of delay-derivative-independent by setting \( Q_{i1} = 0, \, i \in I \) in Theorem 2 for the underlying systems.

4. Simulation studies

In this section, two simulation examples are provided to demonstrate the effectiveness and less conservatism of the proposed approach.

**Example 1.** Consider a continuous-time Markovian jump linear time-delay system in the form of (1), borrowed from [25] with some modifications. The system is with two modes and the following parameters:

\[
\begin{bmatrix} A_1 & A_{d1} & B_1 \\ C_1 & C_{d1} & D_{11} \\ L_1 & L_{d1} & D_{21} \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & -0.2 & 0.1 & 0.6 & 1 \\ 0.3 & -2.5 & 2 & 0.5 & -1 & -0.8 & 0 \\ -0.1 & 0.3 & -3.8 & 0 & 1 & -2.5 & 1 \\ 0.8 & 0.3 & 0 & 0.2 & -0.3 & -0.6 & 0.2 \\ 0.5 & -0.1 & 1 & 0 & 0.1 & 0.2 & 0.1 \end{bmatrix},
\]

\[
\begin{bmatrix} A_2 & A_{d2} & B_2 \\ C_2 & C_{d2} & D_{12} \\ L_2 & L_{d2} & D_{22} \end{bmatrix} = \begin{bmatrix} -2.5 & 0.5 & -0.1 & 0 & -0.3 & 0.6 & -0.6 \\ 0.1 & -3.5 & 0.3 & 0.1 & 0.5 & 0 & 0.5 \\ -0.1 & 1 & -2 & -0.6 & 1 & -0.8 & 0 \\ -0.5 & 0.2 & 0.3 & 0 & -0.6 & 0.2 & 0.5 \\ 0 & 1 & 1.6 & 0.2 & 0.1 & 0 & -0.1 \end{bmatrix}.
\]

The objective is to design a filter of the form (2) such that the resulting filtering error system (3) is stochastically stable with a guaranteed \( \mathcal{H}_\infty \) performance. To this end, suppose transition rates (TRs) \( \lambda_{11} = -1.2, \lambda_{22} = -0.3 \), and choose the lower and upper delay bounds, respectively, as \( d_1 = 0 \) and \( d_2 = 0.7 \) with \( \mu = 0.2 \). It is noted that the result given in [12] is not applicable to the filtering design for the above system with time-varying delay and it has also been found there is no feasible solution based on the method proposed in [25]. Nevertheless, by applying Theorem 2 proposed in this paper, we indeed obtain the feasible solutions of \( \gamma_{\text{min}} = 0.2709 \), for the full-order filter, \( \gamma_{\text{min}} = 0.3252 \) for the 2-order filter, and
$\gamma_{\min} = 0.3946$ for the 1-order filter, respectively. A more detailed comparison of the obtained performances based on Theorem 2 for different cases is summarized in Table 1, where $\hat{\mu}$ denotes that $\mu$ is unknown, which indicates the delay-derivative-independent filtering design results, as mentioned in Remark 4.

The above example has clearly demonstrated that the results proposed in this paper are more general and less conservative than the existing ones proposed in [12,25]. To illustrate the effectiveness of the results proposed in this paper more comprehensively, in the following we present another example.

**Example 2.** Consider a continuous-time Markovian jump linear time-delay system in (1) with four modes, and the system parameters are given as follows:

\[
\begin{bmatrix}
A_1 & A_{d1} & B_1 \\
C_1 & C_{d1} & D_{11} \\
L_1 & L_{d1} & D_{21}
\end{bmatrix} = \begin{bmatrix}
-3.5 & 0.8 & -0.9 & -1.3 & 1 \\
-0.6 & -3.3 & -0.7 & -2.1 & 0 \\
0.8 & 0.3 & 0.2 & -0.3 & 0.2 \\
0.5 & -0.1 & 0 & 0.1 & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
A_2 & A_{d2} & B_2 \\
C_2 & C_{d2} & D_{12} \\
L_2 & L_{d2} & D_{22}
\end{bmatrix} = \begin{bmatrix}
-2 & -1 & 0 & 1 & 0.7 \\
0 & -2 & 1 & 0 & -0.1 \\
0.9 & -2.1 & 1.5 & 0 & 0.6 \\
0 & 1 & -0.3 & 0.5 & -0.1
\end{bmatrix},
\]

\[
\begin{bmatrix}
A_3 & A_{d3} & B_3 \\
C_3 & C_{d3} & D_{13} \\
L_3 & L_{d3} & D_{23}
\end{bmatrix} = \begin{bmatrix}
-3 & 1 & -0.9 & 0 & 0.9 \\
-0.6 & -0.7 & 1.2 & -0.7 & -0.3 \\
-0.3 & 0.1 & 0.3 & -0.1 & 0.3
\end{bmatrix},
\]

\[
\begin{bmatrix}
A_4 & A_{d4} & B_4 \\
C_4 & C_{d4} & D_{14} \\
L_4 & L_{d4} & D_{24}
\end{bmatrix} = \begin{bmatrix}
-2 & 0 & -1 & 0 & 0.8 \\
0 & -0.9 & -1 & -1 & -0.2 \\
0.8 & -2.2 & 1.2 & -0.1 & 0.5 \\
1 & 0 & 0.2 & 0.1 & -0.2
\end{bmatrix}.
\]

Three different cases for the transition rate matrix (TRM) are given in Table 2, where the elements labeled with the hat “$\hat{\cdot}$” represent the unknown TRs. In Case 2, we restrict the unknown diagonal element $\lambda_{22}$ with a lower bound $\lambda_{d}^{(2)} = -1.5$, and also assign $\lambda_{d}^{(1)} = -1.3$, $\lambda_{d}^{(2)} = -1.5$, $\lambda_{d}^{(3)} = -2.5$, and $\lambda_{d}^{(4)} = -1.2$ a priori for Case 3, respectively.

By applying Theorem 2 with the lower and upper delay bounds $d_1 = 0.1$, and $d_2 = 0.65$, respectively, a detailed comparison of the obtained minimum $\mathcal{H}_{\infty}$ performance indices $\gamma_{\min}$ for both full-order and reduced-order filters with different delay-derivatives and three TRM cases is shown in Table 3, where $\hat{\mu}$ denotes that $\mu$ is unknown, which indicates the delay-derivative-independent filtering design results, as mentioned in Remark 4. The results given in Table 3 clearly show that the filtering design method proposed in this paper is effective.

Specifically, in the following, we consider the time-varying delay $0.1 \leq d(t) \leq 0.5$, with $d(t) \leq 0.9$. By applying Theorem 2, the feasible solutions of $\gamma_{\min} = 0.5870$ for the full-order filter and $\gamma_{\min} = 0.8509$ for the reduced-order filter are obtained.

<table>
<thead>
<tr>
<th>Filter-order</th>
<th>$\mu = 0.2$</th>
<th>$\mu = 0.5$</th>
<th>$\mu = 0.9$</th>
<th>$\hat{\mu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1 = 0.7$</td>
<td>0.2709</td>
<td>0.3390</td>
<td>0.4355</td>
<td>0.4373</td>
</tr>
<tr>
<td>2-order</td>
<td>0.3252</td>
<td>0.3927</td>
<td>0.5155</td>
<td>0.5193</td>
</tr>
<tr>
<td>1-order</td>
<td>0.3946</td>
<td>0.4766</td>
<td>0.6119</td>
<td>0.6356</td>
</tr>
<tr>
<td>$d_2 = 0.9$</td>
<td>0.3229</td>
<td>0.4379</td>
<td>0.8915</td>
<td>1.3834</td>
</tr>
<tr>
<td>2-order</td>
<td>0.3684</td>
<td>0.4979</td>
<td>1.2484</td>
<td>2.9093</td>
</tr>
<tr>
<td>1-order</td>
<td>0.4590</td>
<td>0.5894</td>
<td>1.7210</td>
<td>5.0887</td>
</tr>
</tbody>
</table>

Table 2

Three different TRMs.

<table>
<thead>
<tr>
<th>Case 1: completely known</th>
<th>Case 2: partially known</th>
<th>Case 3: completely unknown</th>
</tr>
</thead>
</table>
| $\begin{bmatrix}
-1.3 & 0.2 & 0.8 & 0.3 \\
0.3 & -1.3 & 0.5 & 0.5 \\
0.1 & 0.9 & -2.5 & 1.5 \\
0.4 & 0.2 & 0.6 & -1.2
\end{bmatrix}$ | $\begin{bmatrix}
-1.3 & 0.2 & \lambda_{11} & \lambda_{14} \\
\lambda_{21} & \lambda_{22} & 0.5 & 0.5 \\
0.1 & \lambda_{32} & -2.5 & \lambda_{34} \\
0.4 & 0.2 & 0.6 & -1.2
\end{bmatrix}$ | $\begin{bmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\
\lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} \\
\lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44}
\end{bmatrix}$ |
respectively, under Case 2 shown in Table 2. The filter gains are given as follows:

\[
\begin{align*}
A_f^1 & = \begin{bmatrix} -8.3334 & -7.5832 & -13.9418 \\ 4.9212 & -35.0598 & -34.5833 \\ -0.5429 & -0.0237 & -0.0340 \end{bmatrix}, \\
C_f^1 & = \begin{bmatrix} -18.0870 & 19.5453 & -9.2342 \\ -0.3476 & -0.6985 & -0.4796 \\ 0.1918 & -1.1420 & 0.0311 \end{bmatrix}, \\
A_f^2 & = \begin{bmatrix} -3.5271 & -3.4687 & 3.9715 \\ -0.2084 & -15.4139 & 10.7591 \\ 0.1722 & 0.8969 & -0.8439 \end{bmatrix}, \\
C_f^2 & = \begin{bmatrix} 33.4322 & -78.8817 & 27.5667 \\ 178.9554 & -394.3563 & 138.1362 \\ -1.0083 & -0.1646 & 0.0383 \end{bmatrix}, \\
A_f^3 & = \begin{bmatrix} 8.3477 & -10.2293 & -8.0686 & -0.2902 \\ -2.7626 & -0.7768 & -0.8577 & -0.3148 \\ -7.0410 & 2.6561 & 0.9081 & -0.4145 \\ -4.4234 & 0.2000 & -0.9782 & 0.0022 \end{bmatrix}, \\
C_f^3 & = \begin{bmatrix} 0.5 & -0.3 & 0 & 0 \\ 0.5 & -0.3 & 0 & 0 \\ 0.5 & -0.3 & 0 & 0 \\ 0.5 & -0.3 & 0 & 0 \end{bmatrix}, \\
A_f^4 & = \begin{bmatrix} 8.3477 & -10.2293 & -8.0686 & -0.2902 \\ -2.7626 & -0.7768 & -0.8577 & -0.3148 \\ -7.0410 & 2.6561 & 0.9081 & -0.4145 \\ -4.4234 & 0.2000 & -0.9782 & 0.0022 \end{bmatrix}, \\
C_f^4 & = \begin{bmatrix} 0.5 & -0.3 & 0 & 0 \\ 0.5 & -0.3 & 0 & 0 \\ 0.5 & -0.3 & 0 & 0 \\ 0.5 & -0.3 & 0 & 0 \end{bmatrix},
\end{align*}
\]

for the full-order case, and

\[
\begin{align*}
A_f^1 & = \begin{bmatrix} -14.3477 & -10.2293 \\ -8.0686 & -0.2902 \end{bmatrix}, \\
A_f^2 & = \begin{bmatrix} -2.7626 & -0.7768 \\ -0.8577 & -0.3148 \end{bmatrix}, \\
A_f^3 & = \begin{bmatrix} -7.0410 & 2.6561 \\ 0.9081 & -0.4145 \end{bmatrix}, \\
A_f^4 & = \begin{bmatrix} -4.4234 & 0.2000 \\ -0.9782 & 0.0022 \end{bmatrix},
\end{align*}
\]

for the reduced-order case. The feasible solutions for the other two TRM cases shown in Table 2 are omitted for brevity.

With the above obtained filters and for three different TRM cases shown in Table 2, the time responses of the full-order and reduced-order estimation errors under one of possible mode evolutions are shown in Figs. 2 and 3, respectively, where the initial condition is selected as \( x(0) = [0.5 \ -0.3 \ 0 \ 0]^T \), time-varying delay \( d(t) = 0.3 + 0.2 \sin(4.5t) \), and the disturbance

<table>
<thead>
<tr>
<th>TRMs</th>
<th>Filter-order</th>
<th>( \mu = 0.2 )</th>
<th>( \mu = 0.5 )</th>
<th>( \mu = 0.9 )</th>
<th>( \dot{\mu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>Full-order</td>
<td>0.5458</td>
<td>0.6186</td>
<td>0.6606</td>
<td>0.6611</td>
</tr>
<tr>
<td></td>
<td>Reduced-order</td>
<td>0.7354</td>
<td>0.8557</td>
<td>0.9362</td>
<td>0.9368</td>
</tr>
<tr>
<td>Case 2</td>
<td>Full-order</td>
<td>0.7121</td>
<td>0.7724</td>
<td>0.8282</td>
<td>0.8292</td>
</tr>
<tr>
<td></td>
<td>Reduced-order</td>
<td>0.9106</td>
<td>1.0939</td>
<td>1.1861</td>
<td>1.1865</td>
</tr>
<tr>
<td>Case 3</td>
<td>Full-order</td>
<td>0.7507</td>
<td>0.8556</td>
<td>0.9136</td>
<td>0.9161</td>
</tr>
<tr>
<td></td>
<td>Reduced-order</td>
<td>0.9258</td>
<td>1.1395</td>
<td>1.2231</td>
<td>1.2236</td>
</tr>
</tbody>
</table>

Fig. 2. Responses of \( \pi(t) \) for three TRM cases (full-order).
input $w(t) = 10e^{-0.15t} \sin(0.015t)$. The simulation results further indicate that, whether the TRs are known or not, the designed filters are feasible and effective.

5. Conclusions

The delay-dependent $H_\infty$ filtering design problem for a class of continuous-time Markovian jump linear systems (MJLSs) with time-varying delay and partially accessible mode information has been investigated in this paper. The partially known transition rates have been dealt with by a convex combination method. A new input–output model has been presented by employing a novel approximation for delayed state. On the basis of scaled small gain theorem and constructing a Markovian Lyapunov–Krasovskii functional, an improved bounded real lemma has been derived. Meanwhile, the filtering design results are established by a linearization technique for both full-order and reduced-order cases. Finally, two illustrative examples are presented to demonstrate the effectiveness and less conservatism of the proposed approach over the existing results.

It is noted that the extensions of the proposed method to the controller design for continuous-time MJLSs with multiple time-varying delays and defective mode information, which simultaneously involves the exactly known, partially unknown and uncertain transition rates, deserve further investigation. Applications of the proposed theoretical results to some real-world complex systems such as the networked control systems (NCSs) [40], vertical take-off landing (VTOL) helicopter systems [41], short-term interest rate (STIR) in financial economics systems [42] etc., are also part of our future works.

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