Some typical classes of t-norms and the 1-Lipschitz condition

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Introduction

- Triangular norms (t-norms) model conjunction in a fuzzy semantics
- There are several classes of t-norms.
- The Lipschitz condition is a relevant property in mathematics, particularly in theory.
- Lipschitz is stronger than continuity but weaker than derivability and has as main vantage guarantee that iterative processes, or equivalently, that differential equation, have a unique solution.
- So, this condition is reasonable to dynamic process and therefore can be used in several field of computer sciences. For example, in the convergence of multilayer feedforward neural networks when is used the backpropagation training algorithm, the process to find the weight matrices with statical parameters is dynamic and interactive.
- Thus, the Lipschitz condition seem be a reasonable requirement for t-norms and their dual t-conorms in fuzzy neural networks.
In this paper we will try to determine which t-norms of the class of continuous Archimedean t-norms (and their nilpotent and strict subclasses) satisfies the 1-Lipschitz condition.

We will prove that the unique nilpotent t-norm satisfying this condition is the Lukasiewicz.

We will prove that the product t-norm (the fundamental strict t-norm) satisfy the 1-Lipschitz condition and argument why we belief that it is the unique t-norm in this class satisfying the condition.

Thus, since each continuous Archimedean t-norm or is strict or is nilpotent and 1-Lipschitz condition implies in continuity, we can conjecture with strong evidences that the unique two Archimedean t-norms satisfying the 1-Lipschitz condition are the Lukasiewicz and the product.

We will shows that the family of Dubois-Prade t-norms, a subclass of continuous but not Archimedean class of t-norms, also satisfy this condition.
A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a **triangular norm**, t-norm in short, if it satisfy the follow properties:

1. **Symmetry**: for each $x, y \in [0, 1]$, $T(x, y) = T(y, x)$,
2. **Associativity**: for each $x, y, z \in [0, 1]$,
   \[ T(x, T(y, z)) = T(T(x, y), z), \]
3. **Monotonicity**: for each $x_1, y_1, x_2, y_2 \in [0, 1]$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ then $T(x_1, y_1) \leq T(x_2, y_2)$ and
4. **One identity**: for each $x \in [0, 1]$, $T(x, 1) = x$.

These properties of t-norms are sufficient to guarantee that each t-norm generalize the classical conjunction when the values are the boolean ones, i.e. $T(0, 0) = 0$, $T(0, 1) = 0$, $T(1, 0) = 0$ and $T(1, 1) = 1$ for each t-norm.
Example of t-norms

- Some basic t-norms are:
  - Gödel or minimum: \( T_G(x, y) = \min\{x, y\} \)
  - Lukasiewicz: \( T_L(x, y) = \max\{x + y - 1, 0\} \)
  - Product: \( T_P(x, y) = xy \)
  - Weak:
    \[
    T_{\text{Weak}}(x, y) = \begin{cases} 
    \min\{x, y\}, & \text{if } \max\{x, y\} = 1 \\
    0, & \text{otherwise}
    \end{cases}
    \]
  - Dubois and Prade: For each \( \alpha \in [0, 1] \) define
    \[
    T_\alpha(x, y) = \frac{xy}{\max\{x, y, \alpha\}}
    \]
  - Notice that if \( \alpha = 0 \) then \( T_\alpha = T_G \) and if \( \alpha = 1 \) then \( T_\alpha = T_P \).
Classes of t-norms

- \( x \in (0, 1) \) zero divisor if \( T(x, y) = 0 \) for some \( y \in (0, 1) \).
- A t-norm is **continuous** if it is continuous in the usual topology of \([0, 1]\) (and \([0, 1] \times [0, 1]\)).
- A t-norm \( T \) is **Archimedean** if for each \( x, y \in (0, 1) \) there exists a positive integer \( n \) such that \( T^n(x) < y \), where

\[
T^1(x) = T(x, x) \text{ and } T^{k+1}(x) = T(x, T^k(x)).
\]
- A continuous t-norm is Archimedean iff for each \( x \in (0, 1) \), \( T(x, x) < x \). Continuous Archimedean t-norms with zero divisors are called **nilpotents** and are called **strict** otherwise.
- Thus, a t-norm \( T \) is strict iff for each \( x, y, z \in [0, 1] \) such that \( x < y \) and \( 0 < z \), \( T(x, z) < T(y, z) \).
- A t-norm is **idempotent** iff \( T(x, x) = x \) for each \( x \in [0, 1] \).
- Thus, we can classify the t-norms as continuous and not-continuous, the continuous t-norms as Archimedean and as not Archimedean, and the continuous Archimedean t-norms as strict or as nilpotent.
t-norms satisfying the Lipschitz conditions

- A t-norm \( T \) satisfy the k-Lipschitz condition for a constant \( k > 0 \) if for each \( w, x, y, z \in [0, 1] \),

\[
| T(w, x) - T(y, z) | \leq k(| w - y | + | x - z |). \tag{1}
\]

- Because one identity property, there not exists t-norms satisfying (1) for \( k < 1 \).

- Let \( m \geq 1 \). If \( T \) satisfy (1) for \( k = m \) then also satisfy (1) for any \( k > m \).

- Therefore, 1 with \( k = 1 \) is the more strong Lipschitz condition possible for t-norms.

- However, even is few the knowledge on the class of t-norm satisfying the 1-Lipschitz condition. Still less is know the relationship between the classes previously seen in this subsection with this condition.
Automorphisms

- Bijective and monotonic functions $\rho : [0, 1] \rightarrow [0, 1]$ are called **automorphisms**.
- Examples of automorphisms are the exponents:

$$e^r(x) = x^r$$

for any $r \in \mathbb{R}^+$. Notice, that $(e^r)^{-1} = e^{\frac{1}{r}}$, and therefore also is an automorphism.

- Proposition: Let $\rho : [0, 1] \rightarrow [0, 1]$ be a monotonic function. Then $\rho$ is bijective iff $\rho(1) = 1$, $\rho(0) = 0$ and $\rho$ is continuous and strictly increasing.

- Corollary: Let $\rho$ be an automorphism. If $x, y \in (0, 1)$ and $x < y$ then $\rho(x) < \rho(y)$.

- $(Aut([0, 1], \circ))$ is a group.
Automorphism preserving t-norms

Let $T$ be a t-norm and $\rho$ be an automorphism. We said that $\rho$ preserve $T$, if for each $x, y \in [0, 1]$, $T(\rho(x), \rho(y)) = \rho(T(x, y))$.

Let $\rho : [0, 1] \rightarrow [0, 1]$ be an automorphism and $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a t-norm. Define $T^\rho : [0, 1] \times [0, 1]$ by

$$T^\rho(x, y) = \rho^{-1}(T(\rho(x), \rho(y)))$$

$T^\rho$ is a t-norm

We said that a class $\mathcal{T}$ of t-norms is closed under a class of automorphisms $\mathcal{A}$ if for each $T \in \mathcal{T}$ and each $\rho \in \mathcal{A}$, $T^\rho \in \mathcal{T}$.

The more usual classes of t-norms (continuous, Archimedean, whit zero divisors, nilpotent, etc.) are closed under the class of all automorphisms. However it is not holds for the case of t-norms satisfying the 1-Lipschitz condition.
Proposition: A t-norm $T$ is nilpotent iff there exists an automorphism $\rho$ such that $T = T_L^\rho$.

Proposition: $T_L$ satisfy the 1-Lipschitz condition.

Theorem: The unique nilpotent t-norm which satisfy the 1-Lipschitz condition is $T_L$.

Corollary: Let $\mathcal{L}$ be the class of t-norms satisfying the 1-Lipschitz condition. The unique automorphism which is closed under the class $\mathcal{L}$ is the identity.

But it not implies that some large proper subclasses of $\mathcal{L}$ will be closed for a non-trivial class of automorphisms.
Relating with the class of strict t-norms

- Proposition: A t-norm \( T \) is strict iff there exists an automorphism \( \rho \) such that \( T = T_\rho \).
- Proposition: \( T_P \) satisfy the 1-Lipschitz condition.
- Notice that if \( \rho \) preserve \( T \) trivially \( T^\rho = T \). So,
- Proposition: Let \( T \) be a strict t-norm which satisfy the 1-Lipschitz condition and \( \rho \) an automorphism which preserve \( T \). Then the strict t-norm \( T^\rho \) also satisfy the 1-Lipschitz condition.
- Since, for each strict t-norm \( T \) there exists an automorphism \( \rho \) such that \( T = T_\rho \), seem that for \( T \) satisfy the 1-Lipschitz condition, \( \rho \) must preserve product and therefore \( T \) would be equal to \( T_P \).
- So, we conjecture that the unique strict t-norm which satisfy the 1-Lipschitz condition is the product. Therefore, if this conjecture is correct as we belief, the unique archimedean t-norms satisfying the Lipschitz condition are \( T_L \) and \( T_P \).
Continuous non-Archimedean t-norms

Proposition: Let $\alpha \in [0, 1)$. Then $T_\alpha$ is a continuous non-Archimedean t-norm.

Therefore, the Dubois-Prade t-norms (without $T_P$) is a family of continuous non-Archimedean t-norm which satisfy the Lipschitz condition.

But, this family is not the unique t-norms in the class of non-Archimedean t-norms which satisfy the 1-Lipschitz condition.

Proposition: Let $\rho$ be an automorphism which preserve product (i.e. $T_P$) and $\alpha \in [0, 1)$. Then $T_{\alpha\rho}$ also is a non Archimedean t-norm which preserve the 1-Lipschitz condition.
Final Remarks

► We shown that the unique automorphism which preserve the 1-Lipschitz condition of any t-norms is the identity.
► However, for the more general case there are several automorphisms (for example concave automorphisms) which preserve the Lipschitz condition, still the constant could be changed.
► The objective of this paper was to relate the 1-Lipschitz condition with usual classes of t-norms. In this sense, the result obtained are only conclusive with the class of nilpotent t-norms. For Strict and non-Archimedean t-norms the result are partial.
► For the class of strict t-norms were given evidences to belief that there exists an unique t-norm in the class satisfying the 1-Lipschitz condition.
► So, up to least of $T_L$ and $T_P$, seem a necessary condition for a t-norm satisfy the 1-Lipschitz condition is that it be continuous and non-Archimedean.