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The Probability That an Element of a Group Fixes a Set and Its Graph Related to Conjugacy Classes

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ABSTRACT

Let G be a metacyclic 2-group. The commutativity degree is the probability that two random elements commute in G, denoted as $P(G)$.This concept is used to determine the abelianness of a group and there are three ways to find this probability. The probability can be obtained by finding the multiplication table, conjugacy classes and recently using centralizers. In this paper, we use the conjugacy classes as a way to compute our results. The concept of commutativity degree has been generalized by many authors; one of these generalizations is the probability that a group element fixes a set. Thus, the main objective of this paper is to find the probability of an element of a group G fixes Ω where Ω is a set consisting of (a,b), where a and b are commuting elements in G of size two and the group G acts on the set of all subset of Ω by conjugation. The results that are obtained from the probability can be conducted with graph theory more precisely graph related to conjugacy classes. Hence, our second objective is to find the graph related to conjugacy class for the mentioned probability. The work in this article is done for all metacyclic 2-groups of nilpotency class two and class at least three of negative type.

KEYWORDS: Commutativity degree, metacyclic 2-group, graph, conjugacy classes, group action.

INTRODUCTION

Throughout this paper, G denotes a non-abelian metacyclic 2-group. The probability that two randomly selected elements commute in G is called the commutativity degree of G, denoted by $P(G)$. The probability is defined as follows:

$$P(G) = \frac{\left|\left\{(x, y) \in G \times G \,\middle|\, xy = yx\right\}\right|}{|G|^2}$$

The above probability is less than or equal to 5/8 for finite non-abelian groups [1,2]. Gustafson [1] and MacHale [2] showed that this probability can be computed using conjugacy classes. Numerous researches have been done on the commutativity degree and many results have been achieved.

As mentioned, this probability has been generalized by lots of researches and these generalizations have also been extended by many authors. A part from that, we use one of those generalizations which is the probability that a group element fixes a set that was firstly introduced by Omer *et al.* [3]. This probability can be obtained by calculated the conjugacy classes under some group action on a set.

In the following context, we state some basic concepts that are needed in this paper. These basic concepts can be found in one of the references [4, 5].

Definition1.1[4]A group G is called a metacyclic if it has a cyclic normal subgroup K such that the quotient group G/H is also cyclic.

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Definition1.2 [5]Let G be a finite group. Then, G acts on itself by if there is a function$G \times G \to G$, such that

$i.(gh)x = g(hx), \forall g,h,x \in G.$

$ii.1_G x = x, \forall x \in G.$

Next, we provide some concepts related to metacyclic *p*-groups. Throughout this article, *p* denotes a prime.

In 1973, King [6] gave the presentation of metacyclic *p*-groups, as in the following:

$$G \cong \langle a,b : a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^n, [a,b] = a^m \rangle, where\, \alpha,\beta > 0, m > 0, n \le p^{\alpha}, p^{\alpha} \,|\, n(m-1)$$

However, Beuerle [7] in 2005 separated the classification into two parts, namely for the non-abelian metacyclic *p*-groups of class two and classat least three. Based on [7], the metacyclic p -groups of nilpotency class two are then partitionedinto two families of non-isomorphic *p*-groups stated as follows:

$$G \cong \langle a,b : a^{p^{\alpha}} = 1, b^{p^{\beta}} = 1, [a,b] = a^{p^{\alpha-\gamma}} \rangle, where\, \alpha,\beta,\gamma \in \square, \alpha \ge 2\gamma and\, \beta \ge \gamma \ge 1.$$

$$G \cong Q_{8.}$$

Meanwhile, the metacyclic *p*-groups of nilpotency class of at least three (*p* is an odd prime) are partitioned into the following groups:

$$G \cong \langle a,b : a^{p^{\alpha}} = 1, b^{p^{\beta}} = 1, [b,a] = a^{p^{\alpha-\gamma}} \rangle, where\, \alpha,\beta,\gamma \in \square, \alpha - 1\gamma < 2 and\, \alpha \le \beta.$$

$$G \cong \langle a,b : a^{p^{\alpha}} = 1, b^{p^{\beta}} = 1, [b,a] = a^{p^{\alpha-\gamma}} \rangle, where\, \alpha,\beta,\gamma,\in \in \square, \alpha - 1\gamma < 2\gamma, \alpha \le \beta$$

and $\alpha \le \beta + \in$.

Moreover, metacyclic p -groups are also classified into two types, namely negative and positive [7]. The following notations are used in this paper are represented as follows:

$$G(\alpha,\beta,\in,\gamma,\pm) \cong \langle a,b : a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\in}}, [b,a] = a^t \rangle, \text{ where } \alpha,\beta,\gamma,\in \in \square,$$

$$t = p^{\alpha-\gamma} \pm 1.$$

. If $t = p^{\alpha-\gamma} + 1$, then the group is called a metacyclic of negative type and it is of positive type if $t = p^{\alpha-\gamma} - 1$. Thus, $G(\alpha,\beta,\in,\gamma,-)$ is denoted by the metacyclic group of negative type, while $G(\alpha,\beta,\in,\gamma,+)$ is denoted by the positive type. These two notations are shortened to $G(p,+)$ and $G(p,-)$ for metacyclic p-group of positive and negative type respectively [6, 7].

In addition, the metacyclic 2-groups of negative type of class at least three are partitioned into eight families [7].The followings are some of the negative types which are considered in the scope of this paper:

$$G \cong \langle a,b : a^{2^{\alpha}} = 1, b^2 = a^{2^{\alpha-1}}, [b,a] = a^{-2} \rangle, \text{ where } \alpha \ge 3.$$

$$G \cong \langle a,b : a^{2^{\alpha}} = 1, b^2 = 1, [b,a] = a^{-2} \rangle, \text{ where } \alpha \ge 3,$$

$$G \cong \langle a,b : a^{2^{\alpha}} = 1, b^2 = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle, \text{ where } \alpha \ge 3,$$

$$G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{-2} \rangle, \text{ where } \alpha \ge 3, \beta > 1,$$

$$G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle, \text{ where } \alpha \ge 3, \beta > 1,$$

$$G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = a^{2^{\alpha-1}}, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle, \text{ where } \alpha - \gamma > 1, \beta > \gamma > 1,$$

$$G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle, \text{where } \alpha - \gamma > 1, \beta \ge \gamma > 1.$$

In the following context, we state some basic concepts that arerelated to graph theory.

Definition 1.3 [8] A graph Γ consists of two sets vertices$V(\Gamma)$ and edges $E(\Gamma)$ together with relation of incidence. The directed graph is a graph whose edges are identified with ordered pair of vertices. Otherwise, Γ is called indirected. Two vertices are adjacent if they are joined by an edge.

Definition 1.4 [9] A complete graph K_n is a graph where each order pair of distinct vertices are adjacent.

Lemma 2.4[8] *If* Γ *is a simple graph with* $v \ge 3$ *and* $\deg(v) \ge \frac{|V(\Gamma)|}{2}$, *then* Γ *is Hamiltonian.*

This paper is structured as follows: In Section 2, we state some previous works that are related to the commutativity degree in particular related to the probability that a group element fixes a set. Whilst, our main results are introduced in Section 3, which include results on the probability mentioned in section 2 and results on graph related to conjugacy classes are also introduced in Section 3.

PRELIMINARIES

In this section, some previous works that are needed in this paper are included. This section is divided into two parts. The first part provides some works related to the probability that a group element fixes a set or a subgroup element. While, the second part states some previous researches on graph theory specifically graph related to conjugacy classes.

The Commutativity Degree

In this part, some works that are related to the probability that an element of a group fixes a set are stated as follows. We start with a new conceptintroduced by Sherman [9]in 1975, namely the probability of an automorphism of a finite group fixes an arbitrary element in the group.

Definition 2.1. [9] Let G be a group. Let *X* be a non-empty set of G (i.e., G is a group of permutations of X). Then the probability of an automorphism of a group fixes arandom element from X is defined as follows:

$$P_G(X) = \frac{\left|\{(g,x) | gx = x \, \forall g \in G, x \in X\}\right|}{|X||G|}.$$

In 2011, Moghaddam *et al.* [10]explored Sherman'sdefinition and introduced a new probability which is called the probability of an automorphismfixes a subgroup element of a finite group. This probability is stated as follows:

$$P_{A_G}(H,G) = \frac{\left|\{(\alpha,h) | h^{\alpha} : h \in H, \alpha \in A_G\}\right|}{|H||G|},$$

where A_G is the group of automorphisms of a group$G$. It is obvious that when $H = G$, then $P_{A_G}(G,G) = P_{A_G}(G)$.Omer *et al.* [3] found the probability that an element of a group fixes a set of size two of commuting element in G. Their results are listed in the following.

Definition 2.2[3] Let G be a group. Let *S* be a set of all subsets of commuting elements of size two in G, where G acts on *S* by conjugation. Then the probability of an element of agroup fixes a set is given as follows:

$$P_G(S) = \frac{\left|\{(g,s) | gS = S \, \forall g \in G, s \in S\}\right|}{|S||G|}$$

Theorem 2.3[3] *Let G be a finite group and let X be a set of elements of G of size* two*in the form of* (a,b) *where a and b commute. Let S be the set of all subsets of commuting elements of G of size* 2 *and G acts on S by conjugation. Then the probability that an element of a group fixesa set is given by:*

$$P_G(S) = \frac{K}{|S|},$$

where K is the number of conjugacy classes of S in G .

In addition, the probability that a group element fixes a set under some group action is also was found for some finite non-abelian 2-groups such as quasi-dihedral groups and semi-dihedral groups and others (for more details see [11]).

Graph Related to Conjugacy Classes

In this part, some earlier and recent publication that are related to graph related to conjugacy classes are stated as follows:

In 1990, Bertram *et al.* [12] introduced a graph which is called a graph related to conjugacy classes. The vertices of this graph are non-central conjugacy classes, thus $|V(\Gamma)| = K(G) - z(G)$. Where $K(G)$ is the number of conjugacy classes and $Z(G)$ presents the center of a group. In this graph, two vertices are adjacent if the cardinalities are not coprime.

Moreto *et al.* [13], classified the finite groups that their conjugacy classes lengths are set-wise relatively prime for any five distinct classes.

Recently, Bianchi *et al.* [14] studied the regularity of the graph related to conjugacy classes and provided some results. Later, Erfanian and Tolue [15] introduced a new graph which is called a conjugate graph. The vertices of this graph are non-central elements of a finite non-abelian group. Two vertices of this graph are adjacent if they are conjugate.

Furthermore, the idea of an orbit graph came from the group action on a set Ω. This graph was firstly introduced by Omer *et al.* [16]. The vertices of an orbit graph are $|V(\Gamma_G^{\Omega})| = \Omega - A$, where Ω can be disjoint union of distinct orbit under action of G on the set Ω, while $A = \{v \in \Omega \,|\, vg = gv, g \in G\}$. Two vertices of this graph are linked by an edge if and only if there exists $g \in G$ such that $g\omega_1 = \omega_2$, where $\omega_1, \omega_2 \in \Omega$.

Ilangovan and Sarmin [17], found some graph properties of graph related to conjugacy classes of two-generator two-groups of class two.

Most recent, Moradipour *et al.* [18] used the graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-groups.

RESULTS AND DISCUSSION

In this section, we provide our main results. The probability that a group element fixes a set can be computed by finding the conjugacy classes of Ω under the action. Thus, the probability is the ratio of conjugacy classes of Ω divides the order of Ω (the number of the elements in Ω). The main point of finding the conjugacy classes is that to associate our results with graph theory more specific with the graph related to conjugacy classes. Thus, the first part of this section concentrates on finding theprobability that an element of a group fixes a set, while the second part provides the graph for the first part.

The probability that group element fixes a set.

We start finding this probability for metacyclic 2-groups of negative type of nilpotency class of at least three.

Theorem 3.1 Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G \cong \langle a, b : a^{2^{\alpha}} = b^2 = a^{2^{\alpha-1}}, [b,a] = a^{-2} \rangle, \alpha \geq 3.$ *Let S be a set of elements of G of size two in the* *form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $P_G(\Omega)=\frac{3}{|\Omega|}$.*

Proof. If G acts on Ω by conjugation, then there exists $\psi: G\times\Omega\to\Omega$ such that $\psi_g(\omega)=g\omega g^{-1}, \omega\in\Omega, g\in G$. The elements of order two in G are $a^{\frac{2^\alpha}{2}}$, b, and $a^{\frac{2^\alpha}{8}i}b$, where i is odd, $0\le i\le 2^\alpha$. Thus, the elements of Ω are stated as follows: there is only one element in the form $\left(1,a^{\frac{2^\alpha}{2}}\right)$, four elements are in the form $\left(a^{\frac{2^\alpha}{2}},a^{2^{\alpha-3}i}b\right)$, where i is odd and $0\le i\le 2^\alpha$ and four elements are in the form $\left(1,a^{2^{\alpha-3}i}b\right)$ where i is odd and $0\le i\le 2^\alpha$. From which it follows that $|\Omega|=9$. If G acts on Ω by conjugation, then $cl(\omega)=g\omega g^{-1}, \omega\in\Omega, g\in G$. The conjugacy classes can be described as follows:

One conjugacy class of $\left\{\left(1,a^{2^{\alpha-3}i}b\right), 0\le i\le 2^\alpha\right\}, i$ is odd, one conjugacy class in the form $\left\{\left(a^{\frac{2^\alpha}{2}},a^{2^{\alpha-3}i}b\right), 0\le i\le 2^\alpha\right\}, i$ is odd and one conjugacy class in the form $\left(1,a^{\frac{2^\alpha}{2}}\right)$. Therefore, we have three conjugacy classes. Using Theorem 2.3 in [3], thus the probability that an element of a group fixes a set $P_G(\Omega)$ is $P_G(\Omega)=\frac{3}{|\Omega|}$ ■

Theorem 3.2 Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G\cong\langle a,b: a^{2^\alpha}=b^2=1, [b,a]=a^{-2}\rangle, \alpha\ge 3$. Let S be a set of elements of G of size two in the formof (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $P_G(\Omega)=\frac{5}{|\Omega|}$.

Proof. If G acts on Ω by conjugation, then there exists $\psi: G\times\Omega\to\Omega$ such that $\psi_g(\omega)=g\omega g^{-1}, \omega\in\Omega, g\in G$. The elements of order two in G are $a^{\frac{2^\alpha}{2}}$ and $a^{\frac{2^{\alpha-1}}{2}i}b$, where $0\le i\le 2^\alpha$. Hence, the elements of Ω are described as follows: there is one element in the form $\left(1,a^{\frac{2^\alpha}{2}}\right)$, four elements are in the form $\left(a^{\frac{2^\alpha}{2}},a^{\frac{2^\alpha}{4}i}b\right)$, $0\le i\le 2^\alpha$, and four elements are in the form $\left(1,a^{\frac{2^\alpha}{4}i}b\right)$, $0\le i\le 2^\alpha$. From which it follows that $|\Omega|=9$. If G acts on Ω by conjugation, then $\psi_g(\omega)=g\omega g^{-1}, \omega\in\Omega, g\in G$. The conjugacy classes are described as follows:

$\left\{\left(1,a^{\frac{2^\alpha}{4}i}b\right),0\le i\le 2^\alpha\right\},i$ is even, $\left\{\left(1,a^{\frac{2^\alpha}{4}i}b\right),0\le i\le 2^\alpha\right\},i$ is odd,

$\left\{\left(a^{\frac{2^\alpha}{2}},a^{\frac{2^\alpha}{4}i}b\right),0\le i\le 2^\alpha,\right\},i$ is even, $\left\{\left(a^{\frac{2^\alpha}{2}},a^{\frac{2^\alpha}{4}i}b\right),0\le i\le 2^\alpha\right\},i$ is odd, $\left(1,a^{\frac{2^\alpha}{2}}\right)$. Thus, there are five conjugacy classes when G acts on Ω. According to Theorem 3.1 in [3], then the probability that an element of a group fixes a set is $P_G(\Omega)=\frac{5}{|\Omega|}$. ∎

Theorem 3.3 *Let* G *be a metacyclic 2-group of negative type of nilpotency class of at least three,* $G\cong\langle a,b:a^{2^\alpha}=b^2=1,[b,a]=a^{2^{\alpha-1}-2}\rangle,\alpha\ge 3$. *Let* S *be a set of elements of* G *of size two in the form of* (a,b) *where* a *and* b *commute. Let* Ω *be the set of all subsets of commuting elements of* G *of size two and* G *acts on* Ω *by conjugation. Then* $P_G(\Omega)=\frac{5}{|\Omega|}$.

Proof: The proof is similar to the previous theorem. ∎

Theorem 3.4 *Let* G *be a metacyclic 2-group of negative type of nilpotency class of at least three,* $G\cong\langle a,b:a^{2^\alpha}=1,b^{2^\beta}=1,[b,a]=a^{-2}\rangle$, *where* $\alpha\ge 3,\beta>1$. *Let* S *be a set of elements of* G *of size two in the form of* (a,b) *where* a *and* b *commute. Let* Ω *be the set of all subsets of commuting elements of* G *of size two and* G *acts on* Ω *by conjugation. Then* $P_G(\Omega)=1$.

Proof. If G acts on Ω by conjugation, then there exists $\psi:G\times\Omega\to\Omega$ such that $\psi_g(\omega)=g\omega g^{-1},\omega\in\Omega,g\in G$. The elements of size two in Ω are in the form $\left(Z(G),a^{\frac{2^\alpha}{2}}b^{\frac{2^\beta}{2}}\right)$.

Thus, we have two elements in the form $\left(1,a^{\frac{2^\alpha}{2}}b^{\frac{2^\beta}{2}}\right)$, two elements are in the form $\left(a^{\frac{2^\alpha}{2}},a^{\frac{2^\alpha}{2}}b^{\frac{2^\beta}{2}}\right)$, one element is in the form $\left(b^{2^\beta},a^{\frac{2^\alpha}{2}}b^{\frac{2^\beta}{2}}\right)$ and one element is in the form $\left(1,a^{\frac{2^\alpha}{2}}\right)$. Thus, $|\Omega|=6$. If G acts on Ω by conjugation, then $\psi_g(\omega)=g\omega g^{-1},\omega\in\Omega,g\in G$. Since we have six different types of elements in Ω, therefore the number of conjugacy classes is the sameas $|\Omega|$. Using Theorem 3.1 in [3], thus $P_G(\Omega)=1$. ∎

Theorem 3.5 *Let* G *be a metacyclic 2-group of negative type of nilpotency class of at least three,* $G\cong\langle a,b:a^{2^\alpha}=1,b^{2^\beta}=1,[b,a]=a^{2^{\alpha-1}-2}\rangle$, *where* $\alpha\ge 3,\beta>1$. *Let* S *be a set of elements of* G *of size two in the form of* (a,b) *where* a *and* b *commute. Let* Ω *be the set of all subsets of commuting elements of* G *of size two and* G *acts on* Ω *by conjugation. Then* $P_G(\Omega)=1$.

Proof: The proof is similar to the previous theorem. ∎

Theorem 3.6 *Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = a^{2^{\alpha-1}}, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1, \beta > \gamma > 1$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $P_G(\Omega) = 1$.*

Proof: The proof is similar to that of Theorem 3.4 ■

Theorem 3.7 *Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1, \beta \geq \gamma > 1$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $P_G(\Omega) = 1$.*

Proof: The proof is similar to that of Theorem 3.4 ■

In the following, we find the probability that an element of G fixes a set of metacyclic 2-group of nilpotency class two.

Theorem 3.8 *Let G be a metacyclic 2-group of nilpotency class two,* $G \cong \langle a,b : a^{p^{\alpha}} = 1, b^{p^{\beta}} = 1, [a,b] = a^{p^{\alpha-\gamma}} \rangle$, where $\alpha, \beta, \gamma \in \mathrm{N}, \alpha \geq 2\gamma$ and $\beta \geq \gamma \geq 1$. *Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then*

$$P_G(\Omega) = \begin{cases} \dfrac{5}{|\Omega|}, if\ \beta = \gamma = 1, \alpha < 3, \\ 1, if\ \beta > 1, \gamma = 1, \alpha \geq 3. \end{cases}$$

Proof. If G acts on Ω by conjugation, then there exists $\psi : G \times \Omega \to \Omega$ such that $\psi_g(\omega) = g\omega g^{-1}, \omega \in \Omega, g \in G$. If $\beta = \gamma = 1$, then the elements of order two in G are all b's elements and $a^{\frac{2^{\alpha}}{2}}$. Therefore, the elements of Ω can be expressed as follows: there are four elements are in the form $(1, a^i b), 0 \leq i \leq 2^{\alpha}$, four elements are in the form $\left(a^{\frac{2^{\alpha}}{2}}, a^i b\right), 0 \leq i \leq 2^{\alpha}$ and one element is in the form $\left(1, a^{\frac{2^{\alpha}}{2}}\right)$. Form which it follows $|\Omega| = 9$. Thus when G acts on Ω by conjugation, then $cl(\omega) = \{g\omega g^{-1}, \omega \in \Omega, g \in G\}$. Hence,

$$\begin{cases} if\ \ g = a^i, then\ \omega^g = a^i \omega a^{-i} = (a^i)^{-1}\omega, \\ if\ \ g = a^i b^j, then\ \omega^{a^i b^i} = (a^i \omega a^{-i})^{b^j} = a^{(-1)^j i + (-1)^j - i}\omega. \end{cases}$$

Thereupon, the conjugacy classes in Ω are listed as follows:

$$\left\{\left(1,a^{i}b\right),\left(1,a^{-i}b\right)\right\},\quad \text{if } i \text{ is even},$$
$$\left\{\left(1,a^{i}b\right),\left(1,a^{-i}b\right)\right\},\quad \text{if } i \text{ is odd},$$
$$\left\{\left(a^{\frac{2^{\alpha}}{2}},a^{i}b\right),\left(a^{\frac{2^{\alpha}}{2}},a^{-i}b\right)\right\},\quad \text{if } i \text{ is even},$$
$$\left\{\left(a^{\frac{2^{\alpha}}{2}},a^{i}b\right),\left(a^{\frac{2^{\alpha}}{2}},a^{-i}b\right)\right\},\quad \text{if } i \text{ is odd},$$
$$\left\{\left(1,a^{\frac{2^{\alpha}}{2}}\right)\right\},\quad 0\le i\le 2^{\alpha}.$$

Thus, we have five conjugacy classes. According to [11], then the probability that a group element fixes a set is $P_G(\Omega)=\frac{5}{|\Omega|}$. In the case that $\beta=\gamma=1,\alpha\ge 3$, the proof then follows from that of Theorem 3.4.

■

Theorem 3.9 *Let G be a metacyclic 2-group of nilpotency class two, $G\cong\langle a,b:a^{4}=b^{2}=1,[b,a]=a^{-2}\rangle$, namely Q_8. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $P_G(\Omega)=1$.*

Proof: If G acts on Ω by conjugation, then there exists $\psi:G\times\Omega\to\Omega$ such that $\psi_g(\omega)=g\omega g^{-1},\omega\in\Omega,g\in G$ .The only element in Ω of size two is $(1,a^{2})$, thus when G acts on Ω by conjugation, we have only one conjugacy class. Using [3], the proof then follows. ■

Graph related to conjugacy classes.
In this part, we provide the graph related to conjugacy classes for the previous section, starting with graph related to conjugacy classes of metacyclic 2-groups of negative type of nilpotency class at least three.

Theorem 3.10 Let G be a metacyclic 2-group of negative type of nilpotency class at least three, $G\cong\langle a,b:a^{2^{\alpha}}=b^{2}=a^{2^{\alpha-1}},[b,a]=a^{-2}\rangle,\alpha\ge 3$. Let S S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $\Gamma_G^{\Omega}=K_2$.

Proof: According to [12], $|V(\Gamma_G^{\Omega})|=K(\Gamma_G^{\Omega})-A$, and based on Theorem 3.1, we have three conjugacy classes. Thus $|V(\Gamma_G^{\Omega})|=3-1$. According to [8], $deg(v)=|V(\Gamma_G^{\Omega})|-1$, thus $deg(v)=1$. According to [8], the degree of the graph Γ_G^{Ω}, $\deg(\Gamma_G^{\Omega})=\sum_{i=1}^{|V(\Gamma_G^{\Omega})|}\deg(v_i)=2|E(\Gamma_G^{\Omega})|$. Therefore,

$d\left(\Gamma_G^{\Omega}\right)=\sum_{i=1}^{2}1=2\,|E(\Gamma_G^{\Omega})|$. Thus, $|E(\Gamma_G^{\Omega})|=1$. Form which it follows that, $\Gamma_G^{\Omega}=K_2$.

Thus the graph has the following shape, namely K_2

K_2

As the above graph, we see that two vertices (i.e conjugacy classes) namely $\left(1,a^{2^{\alpha-3}i}b\right)$ and $\left(a^{\frac{2^{\alpha}}{2}},a^{2^{\alpha-3}i}b\right)$ are adjacent by an edge. The proof then follows.■

Corollary 3.11 *If G a metacyclic 2-group of negative type of nilpotency class of at least three, $G\cong\langle a,b:a^{2^{\alpha}}=b^2=a^{2^{\alpha-1}},[b,a]=a^{-2}\rangle,\alpha\geq 3$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation and $\Gamma_G^{\Omega}=K_2$, then the graph is connected and Hamiltonian.*

Proof: The graph is connected since we have only one complete component of K_2, and based on Lemma 1.3, the graph is Hamiltonian. ■

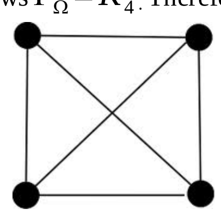
Theorem 3.12 *Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G\cong\langle a,b:a^{2^{\alpha}}=b^2=1,[b,a]=a^{-2}\rangle,\alpha\geq 3$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $\Gamma_{\Omega}=K_4$.*

Proof: According to [12], $|V\left(\Gamma_G^{\Omega}\right)|=K\left(\Gamma_G^{\Omega}\right)-A$, and based on Theorem 3.2, we have five conjugacy classes. Thus, $|V\left(\Gamma_G^{\Omega}\right)|=5-1$. According to [8], $deg(v)=|V(\Gamma_G^{\Omega})|-1$, thus $deg(v)=3$. According to [8], the degree of the graph Γ_G^{Ω} is

$$d\left(\Gamma_G^{\Omega}\right)=\sum_{i=1}^{\left|V\left(\Gamma_G^{\Omega}\right)\right|}\deg(v_i)=2\left|E\left(\Gamma_G^{\Omega}\right)\right|.$$ Therefore,

$$d\left(\Gamma_G^{\Omega}\right)=\sum_{i=1}^{4}3=2\,|E(\Gamma_G^{\Omega})|.$$

Thus, $|E(\Gamma_G^{\Omega})|=6$. Form which that follows $\Gamma_{\Omega}=K_4$. Therefore, the graph shape is drawn as follows:



K4

Thus, the four vertices (i.e non-central conjugacy classes) all are linked by edges. The proof then follows.■

Proposition 3.13*Let G be a metacyclic 2-group of nilpotency class of at least three, $G \cong \langle a,b : a^{2^{\alpha}} = b^2 = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle, \alpha \geq 3$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation and $\Gamma_{\Omega} = K_4$, then the graph connected and Hamiltonian.*

Proof: The graph is connected since we have only one complete graph of K_4. In accordance with Lemma 1.3, the graph is Hamiltonian. ■

Theorem 3.14 *Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G \cong \langle a,b : a^{2^{\alpha}} = b^2 = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle, \alpha \geq 3$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $\Gamma_{\Omega} = K_4$.*

Proof: The proof is similar to that of Theorem 3.12. ■

Theorem 3.15 *Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G \cong \langle a,b : a^{2^{\alpha}} = b^{2^{\beta}} = 1, [b,a] = a^{-2} \rangle, \alpha \geq 3, \beta > 1$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then Γ_{Ω} is an empty graph.*

Proof: Based on Theorem 3.4, there are six conjugacy classes and referring to [12], $|V(\Gamma_G^{\Omega})| = K(\Gamma_G^{\Omega}) - A$. Since all elements in G commute with all elements in Ω, thus the graph is an empty graph. This theorem works only when $\beta \leq 2$. ■

Theorem 3.16 *Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-1}-2} \rangle$, where $\alpha \geq 3, \beta > 1$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then Γ_{Ω} is an empty graph.*

Proof: The proof is similar to the previous theorem. ■

Theorem 3.17 *Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = a^{2^{\alpha-1}}, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1, \beta > \gamma > 1$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then Γ_{Ω} is an empty graph.*

Proof: The proof is similar to that of Theorem 3.16. ■

Theorem 3.18 *Let G be a metacyclic 2-group of negative type of nilpotency class of at least three, $G \cong \langle a,b : a^{2^{\alpha}} = 1, b^{2^{\beta}} = 1, [b,a] = a^{2^{\alpha-\gamma}-2} \rangle$, where $\alpha - \gamma > 1, \beta \geq \gamma > 1$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then Γ_{Ω} is an empty graph.*

Proof: The proof is similar to that of Theorem 3.14. ■

The graph related to conjugacy classes for metacyclic 2-groups of nilpotency class two when the group acts on a set is given as follows:

Theorem 3.19 *Let G be a metacyclic 2-group of nilpotency class two, $G \cong \langle a,b : a^{2^{\alpha}} = b^{2^{\beta}} = 1, [b,a] = a^{-2} \rangle, \alpha \geq 3$. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then $\Gamma_{\Omega} = \mathrm{K}_4$.*

Proof: The proof is similar to that of Theorem 3.10. ■

Theorem 3.20*Let G be a metacyclic 2-group of nilpotency class two, $G \cong \langle a,b : a^{2^{\alpha}} = b^{2^{\beta}} = 1, [b,a] = a^{-2} \rangle, \alpha \geq 3$, namely Q_8. Let S be a set of elements of G of size two in the form of (a,b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two and G acts on Ω by conjugation. Then Γ_{Ω} is an empty graph.*

Proof: According to Theorem 3.9, we only have one conjugacy class, thus the graph is an empty graph. ■

CONCLUSION

In this paper, the probability that a group elementfixes a set of size two for some finite non-abelian metacyclic 2-groups was found. As consequence of obtained results, we conclude that the probability that an element of metacyclic 2-group of nilpotency of class two and at least three is less than or equal to one. The results that were obtained in the first part of Section three were associated to the graph related to conjugacy classes, where we found that all graphs are Hamiltonian graphs. In addition, the graph related to conjugacy classes is a complete graph. This work can be extended by finding the probability that an element of a group fixes a set under some other group actions such as regular, transitive or faithful action. Thus the results obtained can also be associated to graph theory and many graph properties can be found.

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