

ROOTS OF BERNSTEIN-SATO POLYNOMIALS FOR MONOMIAL IDEALS: A POSITIVE CHARACTERISTIC APPROACH

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ABSTRACT. We describe the roots of the Bernstein-Sato polynomial of a monomial ideal using reduction mod p and invariants of singularities in positive characteristic. We give in this setting a positive answer to a problem from [MTW] concerning the dependence on the characteristic for these invariants of singularities.

1. INTRODUCTION

The Bernstein-Sato polynomial (or b -function) of an arbitrary ideal in a polynomial ring was introduced in [BMS2] generalizing the case of principal ideals. For monomial ideals, it was shown there that this polynomial can be computed algorithmically in principle, using for example *Macaulay2*. In this paper we use a positive characteristic approach to give a description of the roots of the Bernstein-Sato polynomial in this case. This is a by-product of positive answers to questions in [MTW] on the dependence on the characteristic for some invariants of singularities.

In order to explain our approach, we recall the definition of the invariants from [MTW]. Let \mathfrak{a} be a nonzero ideal in a regular local ring R of characteristic $p > 0$. Suppose that J is a proper ideal in R whose radical contains \mathfrak{a} . For every positive integer e , let $J^{[p^e]}$ be the ideal generated by the p^e th powers of the elements of J , and

$$\nu_{\mathfrak{a}}^J(p^e) := \max\{\ell \geq 0 \mid \mathfrak{a}^\ell \not\subseteq J^{[p^e]}\}.$$

Suppose now that we start with ideals \mathfrak{a} and J in $\mathbb{Z}[X_1, \dots, X_n]$ such that \mathfrak{a} is contained in the radical of J , and let $b_{\mathfrak{a}}(s)$ be the Bernstein-Sato polynomial of \mathfrak{a} . We consider the invariants associated to the reductions mod p of \mathfrak{a} and J around the origin. It was shown in [MTW] that if $p \gg 0$, then $b_{\mathfrak{a}}(\nu_{\mathfrak{a}}^J(p^e)) \equiv 0 \pmod{p}$ for all e (see also Proposition 2.2 below).

One expects that in many cases there are polynomial formulas for the invariants mod p , formulas depending on a suitable congruence of p . More precisely, in good situations there should be a positive integer N , and for every i relatively prime to N there should be a polynomial $P_i \in \mathbb{Q}[t]$ such that $\nu_{\mathfrak{a}}^J(p^e) = P_i(p)$ whenever p is large enough and $p \equiv i \pmod{N}$. When this holds, Dirichlet's Theorem on the distribution of prime numbers implies that we get roots of the Bernstein-Sato polynomial over \mathbb{Q} : in fact, each $P_i(0)$ is

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such a root (see Remark 2.4). Our main results are that such formulas can be given for monomial ideals (see Theorem 4.1), and furthermore that all the roots can be obtained in this way (see Theorem 4.9).

The proofs of the above results give in particular a description of all roots of the Bernstein-Sato polynomial of a monomial ideal. As a consequence we deduce the following description mod \mathbb{Z} of the roots. Suppose that \mathfrak{a} is a proper nonzero ideal in $\mathbb{Z}[X_1, \dots, X_n]$ generated by monomials. The Newton polyhedron $P_{\mathfrak{a}}$ of \mathfrak{a} is the convex hull in \mathbb{R}_+^n of those u in \mathbb{N}^n such that the monomial X^u is in \mathfrak{a} . For every facet Q of $P_{\mathfrak{a}}$ that is not contained in a coordinate hyperplane, there is a unique linear function L_Q on \mathbb{R}^n having rational coefficients such that $Q = P_{\mathfrak{a}} \cap L_Q^{-1}(1)$. (Here a facet means a maximal-dimensional face.) We denote by m_Q the smallest positive integer such that $m_Q L_Q$ has integer coefficients.

Corollary 1.1. *The set consisting of the images in \mathbb{Q}/\mathbb{Z} of the roots of the Bernstein-Sato polynomial of \mathfrak{a} is equal to*

$$\left\{ \frac{m}{m_Q} + \mathbb{Z} \mid Q \text{ facet of } P_{\mathfrak{a}} \text{ and } 0 \leq m < m_Q \right\}.$$

One can deduce from our results a description of the roots of the Bernstein-Sato polynomial, and not just of their classes mod \mathbb{Z} (see Remark 4.6). Note that such a description has to be more involved, as the roots do not depend only on the integral closure of the ideal (or equivalently, on the Newton polyhedron). Another, more explicit description of the roots of the Bernstein-Sato polynomial of monomial ideals (with a direct, combinatorial proof) will appear in [BMS1].

A few words about the structure of the paper: in the next section we recall the definition of Bernstein-Sato polynomials and give some details in the general set-up about the connection between their roots and the invariants obtained by reduction mod p . In §3 we specialize to monomial ideals, and give a direct proof of the existence of the Bernstein-Sato polynomial in this case. This will be useful later, as it provides useful information about the roots (see Remark 3.5). The idea is based on the approach to computing Bernstein-Sato polynomials of monomial ideals from [BMS2]. In §4, we study the invariants of the reduction mod p for monomial ideals. We give a polynomial formula for these invariants (depending on a suitable congruence of p) and show that all the roots of the Bernstein-Sato polynomial can be obtained by our method. The fourth section is devoted to some examples. In the Appendix we show that if \mathfrak{a} is a monomial ideal, it is enough to consider only positive characteristic invariants that correspond to monomial ideals J .

2. BERNSTEIN-SATO POLYNOMIALS AND REDUCTION MOD p

We start by recalling some general facts about Bernstein-Sato polynomials. For proofs and details we refer to [BMS2].

Let $\mathfrak{a} \subseteq (X_1, \dots, X_n) \subseteq \mathbb{C}[X] = \mathbb{C}[X_1, \dots, X_n]$ be a nonzero ideal, and let f_1, \dots, f_r be nonzero generators of \mathfrak{a} . The Bernstein-Sato polynomial $b_{\mathfrak{a}}(s) \in \mathbb{C}[s]$ of \mathfrak{a} is the monic

generator of the ideal consisting of those $b(s)$ for which we have a relation

$$(1) \quad b(s_1 + \dots + s_r) \prod_j f_j^{s_j} = \sum_c P_c \cdot \prod_{j, c_j < 0} \binom{s_j}{-c_j} \prod_j f_j^{s_j + c_j},$$

where the above sum is over finitely many $c \in \mathbb{Z}^r$ such that $\sum_j c_j = 1$, and where $P_c \in \mathbb{C}[X, \partial_X, s]$ for all c . (Here $\prod_{j, c_j < 0}$ means that the product is over the j such that $c_j < 0$.) As usual, if $m > 0$, then the notation $\binom{s_j}{m}$ stands for $\frac{1}{m!} s_j (s_j - 1) \dots (s_j - m + 1)$.

In (1) one has to interpret the equality formally. Note that if $r = 1$, then this equation can be rewritten as

$$b(s) f^s = P(X, \partial_X, s) \cdot f^{s+1},$$

so we recover the usual definition of the Bernstein-Sato polynomial of a principal ideal (see [Bj], [Ka]).

The fact that there is a nonzero $b(s)$ as above is proved in [BMS2], where one also shows that it does not depend on the choice of generators and that all its roots are negative rational numbers. We refer to [BMS2] also for the motivation for the defining formula (1) in terms of V -filtrations.

In our case the ideal is defined over \mathbb{Z} , and the P_c can be defined over \mathbb{Q} by the following

Proposition 2.1. *If the ideal \mathfrak{a} is defined over a subfield K of \mathbb{C} , then the P_c in (1) can be defined also over K .*

Proof. There is a finitely generated K -subalgebra A of \mathbb{C} such that the P_c are defined over A . Choosing a maximal ideal of A and taking the image in the corresponding residue field K' of A , we get a relation in which the P_c are defined over a finite extension K' of K . We may assume that K'/K is Galois by enlarging K' if necessary. Averaging by the action of the Galois group (where the Galois group acts on the coefficients of P_c), we get the assertion. \square

From now on, we consider only ideals in $\mathbb{Z}[X]$, as this will be enough for our purpose. We will always work in a neighborhood of the origin. Note that there is a local notion of Bernstein-Sato polynomial, where we require (1) to hold only in some open subset containing the origin. In the rest of this section one could replace $b_{\mathfrak{a}}$ by this local version. However, in what follows we are interested in monomial ideals (which are homogeneous), so in this case there will be no distinction between local and global Bernstein-Sato polynomials.

If p is a prime number, the ideal \mathfrak{a} defines by reduction mod p (and localization) an ideal \mathfrak{a}_p in $R_p := \mathbb{F}_p[X]_{(X_1, \dots, X_n)}$, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ as usual. Recall that if I is an ideal in R_p and if $e \geq 1$, then $I^{[p^e]} = (g^{p^e} | g \in I)$. To simplify the notation, for $\mathfrak{a} \subseteq \text{Rad}(J) \subseteq (X_1, \dots, X_n)$, we will denote by $\nu_{\mathfrak{a}}^J(p^e)$ the largest ℓ such that $\mathfrak{a}_p^{\ell} \not\subseteq J_p^{[p^e]}$.

By Proposition 2.1, $b_{\mathfrak{a}}$ and all P_c in (1) have coefficients in \mathbb{Q} , and hence in $\mathbb{Z}[m^{-1}]$ for some integer m . Moreover, we may assume that for all c that appear in (1) and for

all j such that $c_j < 0$, $(-c_j)!$ divides m . If p does not divide m , then (1) will hold also after reduction mod p . We will apply this equality by letting s_1, \dots, s_r to be nonnegative integers. Note that in this case, if $c_j < 0$ and $s_j + c_j < 0$, then $\binom{s_j}{-c_j} = 0$, so the corresponding term in (1) vanishes.

The key to our approach is the following elementary observation from [MTW]. The proof below uses Proposition 2.1, and may be slightly easier than the original one, although both arguments are essentially the same.

Proposition 2.2. *Let \mathfrak{a} and J be nonzero ideals in $\mathbb{Z}[X]$ such that $\mathfrak{a} \subseteq \text{Rad}(J) \subseteq (X_1, \dots, X_n)$. Let m be as above. If p does not divide m and $e \geq 1$, then*

$$(2) \quad b_{\mathfrak{a}}(\nu_{\mathfrak{a}}^J(p^e)) = 0 \text{ in } \mathbb{F}_p.$$

Proof. We can find nonnegative integers a_1, \dots, a_r such that $\sum_j a_j = \nu_{\mathfrak{a}}^J(p^e)$ and $\prod_j f_j^{a_j} \notin J_p^{[p^e]}$. On the other hand, the hypothesis implies $\prod_j f_j^{b_j} \in J_p^{[p^e]}$ if $\sum_j b_j = \nu_{\mathfrak{a}}^J(p^e) + 1$ and $b_j \geq 0$ for all j .

Using (1) mod p for $s_j = a_j$ (together with the remark before this proposition) and the fact that $J_p^{[p^e]}$ is an $\mathbb{F}_p[X, \partial_X]$ -submodule of $\mathbb{F}_p[X]$, we deduce that $b_{\mathfrak{a}}(\nu_{\mathfrak{a}}^J(p^e)) = 0$ in \mathbb{F}_p . \square

This can be used to give roots of the Bernstein-Sato polynomial whenever one can solve the following problem from [MTW].

Problem 2.3. Under good conditions on \mathfrak{a} and J there should exist a positive integer N together with polynomials P_i of degree e in $\mathbb{Q}[t]$ for every $i \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that the following holds. If a prime p is sufficiently large and $p \equiv i \pmod{N}$, then $\nu_{\mathfrak{a}}^J(p^e) = P_i(p)$. Moreover, one should be able to choose N to depend only on \mathfrak{a} .

Remark 2.4. Note that if we have P_i as in the above problem, then $b_{\mathfrak{a}}(P_i(p)) = 0$ in \mathbb{F}_p for infinitely many primes p by Proposition 2.2 and Dirichlet's Theorem. Therefore $P_i(0)$ is a root of $b_{\mathfrak{a}}$. Here we may assume that P_i is defined over $\mathbb{Z}[m^{-1}]$, replacing m by a multiple if necessary. A basic question is which roots can be obtained in this way (Example 4.1 in [MTW] shows that there might be roots which are not detected by the above method).

We mention that the asymptotic behavior of $\nu_{\mathfrak{a}}^J(p^e)$ for $e \rightarrow \infty$ is measured by the F -threshold $c^J(\mathfrak{a}_p)$ introduced in [MTW]: this is defined by

$$(3) \quad c^J(\mathfrak{a}_p) := \lim_{e \rightarrow \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} = \sup_e \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}.$$

The set of such numbers for various J form the jumping coefficients for the test ideals $\{\tau(\mathfrak{a}_p^\alpha)\}_\alpha$ introduced by Hara and Yoshida in [HY]. On the other hand, in characteristic zero we have the multiplier ideals of \mathfrak{a} , and the corresponding jumping coefficients (see [La] for the theory of multiplier ideals). We refer to [MTW] for an overview of results and open questions relating the jumping coefficients of the multiplier ideals and the F -thresholds for the reduction mod p , when $p \gg 0$.

Remark 2.5. As follows from the above discussion, the “top degree” part in $\nu_{\mathfrak{a}}^J(p^e)$ is related to the jumping coefficients of the multiplier ideals of \mathfrak{a} , while the “free term” is related to the roots of the Bernstein-Sato polynomial. Recall that there is also a direct connection between these roots and the jumping coefficients. More precisely, the largest root of $b_{\mathfrak{a}}$ is $-\text{lc}(\mathfrak{a})$, where $\text{lc}(\mathfrak{a})$ is the log canonical threshold of \mathfrak{a} (this is the first nonzero jumping coefficient). Moreover, if $\lambda \in [\text{lc}(\mathfrak{a}), \text{lc}(\mathfrak{a}) + 1)$ is a jumping coefficient, then $-\lambda$ is a root of $b_{\mathfrak{a}}$. These results are proved in [Ko] and [ELSV] in the codimension one case, and in [BMS2] in general.

We turn now to the monomial case, in which we will give an explicit description of the whole picture. As we will see, understanding the F -thresholds is quite easy. For example, this follows from the result in [HY] saying that for monomial ideals the test ideals are the same as the multiplier ideals. On the other hand, we will see that the invariants $\nu_{\mathfrak{a}}^J(p^e)$ give much more information. In particular, by the method described above we will be able to recover all the roots of the Bernstein-Sato polynomial.

3. BERNSTEIN-SATO POLYNOMIALS OF MONOMIAL IDEALS

From now on we assume that \mathfrak{a} is an ideal generated by monomials, so we may take $f_j = X^{a_j} = \prod_i X_i^{a_{i,j}}$ for all j , where $a_j = (a_{1,j}, \dots, a_{n,j}) \in \mathbb{N}^n$. We consider the linear forms $\ell_i(s) = \sum_j a_{i,j} s_j$ on \mathbb{Z}^r .

In this section we give a direct proof of the existence of the Bernstein-Sato polynomial in this case. We first make use of the homogeneity of the ideal to reinterpret equation (1) as in [BMS2]. We consider, more generally, the ideal $I_{\mathfrak{a}} \subseteq \mathbb{C}[s_1, \dots, s_r]$ consisting of those polynomials F such that there is an equality of the form

$$(4) \quad F(s_1, \dots, s_r) f_1^{s_1} \dots f_r^{s_r} = \sum_c P_c \cdot \prod_{j, c_j < 0} \binom{s_j}{-c_j} f_1^{s_1+c_1} \dots f_r^{s_r+c_r},$$

with c and P_c as in (1).

On $\mathbb{C}[X, X^{-1}, s]$ we consider the \mathbb{Z}^n -grading given by $\deg(s_j) = 0$ and $\deg(X_i) = e_i$, the i th element of the standard basis of \mathbb{Z}^n . This induces a grading on $\mathbb{C}[X, X^{-1}, s] \prod_j f_j^{s_j}$ such that $\deg(h \prod_j f_j^{s_j}) = \deg(h)$. On $\mathbb{C}[X, \partial_X, s]$ we have the corresponding grading with $\deg(\partial_{X_i}) = -\deg(X_i)$ and the action on $\mathbb{C}[X, X^{-1}, s] \prod_j f_j^{s_j}$ is compatible with the gradings.

Note that $\prod_j f_j^{s_j} = \prod_i X_i^{\ell_i(s)}$ and $\prod_j f_j^{s_j+c_j} = \prod_i X_i^{\ell_i(s+c)}$. It follows that in (4) we may assume that $\deg(P_c) = -(\ell_1(c), \dots, \ell_n(c))$ for all c . If $P \in \mathbb{C}[X, \partial_X, s]$ has degree $m = (m_i)$, then

$$P \in \mathbb{C}[X_1 \partial_{X_1}, \dots, X_n \partial_{X_n}, s] \cdot \prod_i \xi_i^{|m_i|},$$

where $\xi_i = X_i$ if $m_i \geq 0$, and $\xi_i = \partial_{X_i}$ if $m_i \leq 0$.

For every $c \in \mathbb{Z}^r$, let

$$g_c := \prod_{j, c_j < 0} \binom{s_j}{-c_j} \cdot \prod_{i, \ell_i(c) > 0} \binom{\ell_i(s) + \ell_i(c)}{\ell_i(c)}.$$

Proposition 3.1. *With the above notation, we have*

$$I_{\mathfrak{a}} = \left(g_c \mid c \in \mathbb{Z}^r, \sum_j c_j = 1 \right) = \left(g_c \mid c \in \mathbb{Z}^r, \sum_j c_j \geq 1 \right).$$

Proof. We get the first equality by the above argument, considering the action of $\partial_{X_k}^{\ell_k(c)}$ on $\prod_i X_i^{\ell_i(s+c)}$. For the second equality we need to show that if $c \in \mathbb{Z}^r$ and $\sum_j c_j \geq 1$, then $g_c \in I_{\mathfrak{a}}$. We get this by induction on $\sum_j c_j$. Indeed, if $\sum_j c_j \geq 2$ and $c_{j_0} > 0$, let $c'_j = c_j$ for $j \neq j_0$ and $c'_{j_0} = c_{j_0} - 1$. Since $\ell_i(c) \geq \ell_i(c')$, it follows that $g_{c'}$ divides g_c . By the induction hypothesis, $g_{c'}$ lies in $I_{\mathfrak{a}}$, hence so does g_c . \square

It follows from the above description of the generators of $I_{\mathfrak{a}}$ that for any irreducible component Γ of $V(I_{\mathfrak{a}})_{\text{red}}$, there are subsets $A \subseteq \{1, \dots, r\}$ and $B \subseteq \{1, \dots, n\}$ together with α_j in $\mathbb{Z}_{\geq 0}$ for j in A and β_i in $\mathbb{Z}_{< 0}$ for i in B such that

$$(5) \quad \Gamma = \{u = (u_j) \in \mathbb{C}^r \mid u_j = \alpha_j \text{ for } j \text{ in } A, \ell_i(u) = \beta_i \text{ for } i \text{ in } B\}.$$

Moreover, after possibly enlarging A and B , we may assume that if j is not in A and u_j is constant on Γ , then u_j is not in $\mathbb{Z}_{\geq 0}$ for u in Γ , and if i is not in B and ℓ_i is constant on Γ , then $\ell_i(u)$ is not in $\mathbb{Z}_{< 0}$ for u in Γ .

Lemma 3.2. *With the above notation and assumption, if c in \mathbb{Z}^r is such that $c_j \geq -\alpha_j$ for all j in A and $\ell_i(c) \leq -\beta_i - 1$ for all i in B , then $\sum_j c_j \leq 0$.*

Proof. By Proposition 3.1, if $\sum_j c_j \geq 1$ then $\Gamma \subseteq g_c^{-1}(0)$. Therefore the assertion follows from the above maximality assumption. \square

Remark 3.3. We see that the converse of the above argument is also true: if A, B , the α_j and the β_i are such that the assertion in Lemma 3.2 holds and if Γ is given by (5), then $\Gamma \subseteq V(I_{\mathfrak{a}})_{\text{red}}$.

By definition, the Bernstein-Sato polynomial of \mathfrak{a} is the monic generator of $I_{\mathfrak{a}} \cap \mathbb{C}[s_1 + \dots + s_r]$. The existence of this polynomial follows from the next proposition.

Proposition 3.4. *With the above notation, the intersection $I_{\mathfrak{a}} \cap \mathbb{C}[s_1 + \dots + s_r]$ is nonzero.*

Proof. We need to show that for every irreducible component Γ of $V(I_{\mathfrak{a}})_{\text{red}}$, the map $u = (u_j) \in \Gamma \rightarrow \sum_j u_j$ is constant. This is equivalent to the assertion that if A and B are such that Γ is given by (5) and Lemma 3.2 holds, then for every u in \mathbb{C}^r such that $u_j = 0$ for all j in A and $\ell_i(u) = 0$ for all i in B , we have $\sum_j u_j = 0$. Since the vector space cut out by these equations is defined over \mathbb{Q} , we may assume that u is in \mathbb{Q}^r and moreover, that it is in \mathbb{Z}^r .

Suppose that $\sum_j u_j$ is nonzero. After possibly replacing u by $-u$, we may assume that $\sum_j u_j > 0$. In this case we get a contradiction by applying Lemma 3.2 for $c = u$. This completes the proof of the existence of the Bernstein-Sato polynomial $b_{\mathfrak{a}}$ in the monomial case. \square

Remark 3.5. It follows from the discussion before Proposition 3.1 that both $b_{\mathfrak{a}}$ and the P_c satisfying (4) have rational coefficients. The above proof shows that all roots of $b_{\mathfrak{a}}$ are rational, since Γ is an affine linear subspace defined by equations with rational coefficients. Moreover, we have obtained the following description of the roots: we need to consider all A, B, α_j and β_i such that the assertion in Lemma 3.2 holds. If Γ is defined by (5) and if it is nonempty, then $\sum_j u_j$ is constant on Γ , and its value gives a root of $b_{\mathfrak{a}}$. In addition, all the roots arise in this way.

4. ROOTS OF BERNSTEIN-SATO POLYNOMIALS FOR MONOMIAL IDEALS

We study now the invariants from §2 in the case when \mathfrak{a} is a monomial ideal. Proposition 6.1 in the Appendix shows that for every ideal $J \subseteq (X, \dots, X_n) \subseteq \mathbb{Z}[X_1, \dots, X_n]$ such that $\mathfrak{a} \subseteq \text{Rad}(J)$, there is a monomial ideal \tilde{J} such that if $p \gg 0$, then $\nu_{\mathfrak{a}}^J(p^e) = \nu_{\mathfrak{a}}^{\tilde{J}}(p^e)$ for every $e \geq 1$. Therefore in order to understand the functions $\nu_{\mathfrak{a}}^J$ for various J , it is enough to consider the case when J is a monomial ideal, too. Our goal is to give an affirmative answer to Problem 2.3 in this setting and to show that all roots of $b_{\mathfrak{a}}$ are given by our method. Moreover, along the way we will get a description of these roots.

We assume that \mathfrak{a} and J are proper nonzero monomial ideals in $\mathbb{Z}[X_1, \dots, X_n]$ such that \mathfrak{a} is contained in the radical of J . Since we deal with monomial ideals, we can define the function $\nu_{\mathfrak{a}}^J$ without taking the reduction mod p . If q is an arbitrary positive integer (not necessarily a prime power), we put

$$J^{[q]} := (X^{qw} \mid X^w \in J),$$

where $X^w = \prod_i X_i^{w_i}$ for $w = (w_1, \dots, w_n) \in \mathbb{N}^n$. We also define

$$\nu_{\mathfrak{a}}^J(q) := \max\{t \geq 0 \mid \mathfrak{a}^t \not\subseteq J^{[q]}\}.$$

Our first goal is to prove the following theorem. Note that it immediately gives a positive answer to Problem 2.3 for monomial ideals.

Theorem 4.1. *If \mathfrak{a} is a nonzero proper monomial ideal, then there is a positive integer N with the following property. If J is a monomial ideal whose radical contains \mathfrak{a} , then there are rational numbers $\alpha > 0$ and γ_j for $j = 0, \dots, N-1$, such that $\nu_{\mathfrak{a}}^J(q) = \alpha q + \gamma_j$ if $q \equiv j \pmod{N}$ and q is large enough.*

Remark 4.2. With the notation in the theorem, note that if j and N are relatively prime, then by Proposition 2.2 together with Dirichlet's Theorem, $b_{\mathfrak{a}}(\gamma_j) \equiv 0 \pmod{p}$ for infinitely many primes p . Therefore γ_j is a root of $b_{\mathfrak{a}}$ for every such j . Note also that if $p \gg 0$ is a prime, then α is equal to the F -pure threshold $c^J(\mathfrak{a}_p)$.

The proof of Theorem 4.1 will give, in fact, a description of α and of the γ_j . Moreover, Theorem 4.9 below will show that all roots of $b_{\mathfrak{a}}$ are given by some γ_j as above, for a suitable J and some j relatively prime to N .

We start with some preparations. Note first that we may write $J = \bigcap_{i=1}^d J_i$, where each J_i is generated by powers of variables $X_j^{b_j}$ for $j \in I_i \subseteq \{1, \dots, n\}$. This can be checked, for example, by induction on n . If Theorem 4.1 holds for every J_i , then it holds for J : it is clear that we have $\nu_{\mathfrak{a}}^J(q) = \max_i \nu_{\mathfrak{a}}^{J_i}(q)$ for all q . The assertion follows from the fact that if a finite set S is such that for every i in S the function $m \rightarrow h_i(m)$ is affine linear for m large enough and in a suitable congruence class, then so is the function $m \rightarrow \max_{i \in S} h_i(m)$. Therefore from now on we may assume that $J = (X_i^{b_i} | i \in I)$ for some $I \subseteq \{1, \dots, n\}$.

Recall that we denote the generators of \mathfrak{a} by X^{a_1}, \dots, X^{a_r} , where $a_j = (a_{1,j}, \dots, a_{n,j})$. We have $\mathfrak{a}^t \subseteq J^{[q]}$ if and only if for all $\beta = (\beta_j) \in \mathbb{N}^r$ with $\sum_j \beta_j = t$, there is i in I such that $qb_i \leq \sum_{j=1}^r a_{i,j} \beta_j$. Hence $\nu_{\mathfrak{a}}^J(q) = \tau(I; (qb_i - 1)_{i \in I})$, where for $w = (w_i) \in \mathbb{N}^{|I|}$, we put

$$\tau(I; w) := \max \left\{ \sum_{j=1}^r \beta_j \mid \beta \in \mathbb{N}^r, \ell_i(\beta) \leq w_i \text{ for } i \text{ in } I \right\}.$$

Recall that $\ell_i(\alpha) = \sum_{j=1}^r a_{i,j} \alpha_j$. We denote by ℓ_i also the extension of this linear function to \mathbb{Q}^r .

In order to simplify the notation we show that we may assume $I = \{1, \dots, n\}$, after replacing, if necessary, J by $J' := J + (X_i^M | i \notin I)$ for M sufficiently large. Indeed, note first that if b' is in $\mathbb{N}^{|I|}$, then $\tau(I; b')$ is finite, so the set

$$\{\beta \in \mathbb{Q}_+^r \mid \ell_i(\beta) \leq b'_i \text{ for } i \text{ in } I\}$$

is bounded. This implies that if $M \gg 0$, then for every $q \geq 1$ and every $\beta \in \mathbb{N}^n$ such that $\ell_i(\beta) < qb_i$ for all i in I , we have $\ell_i(\beta) < qM$ for all $i \notin I$. For such M we have $\nu_{\mathfrak{a}}^J(q) = \nu_{\mathfrak{a}}^{J'}(q)$ for every $q \geq 1$.

From now on we assume that $I = \{1, \dots, n\}$, and we put $\tau(w)$ for $\tau(I; w)$. For every w in \mathbb{Q}_+^n we define also

$$\tau_{\mathbb{Q}}(w) := \max \left\{ \sum_{j=1}^r \alpha_j \mid \alpha \in \mathbb{Q}_+^r, \ell_i(\alpha) \leq w_i \text{ for all } i \right\}.$$

Here we may replace $\alpha \in \mathbb{Q}_+^r$ with $\alpha \in \mathbb{R}_+^r$ because w is in \mathbb{Q}_+^n and the ℓ_i have coefficients in \mathbb{Q} .

Let $P_{\mathfrak{a}}$ be the Newton polyhedron of \mathfrak{a} and let Δ be the fan decomposition of \mathbb{Q}_+^n whose cones are the closed convex cones over the faces of $P_{\mathfrak{a}}$. A maximal such cone corresponds to a facet Q of $P_{\mathfrak{a}}$ that is not contained in a coordinate hyperplane, and we denote by L_Q or by L_{σ} the linear function such that $Q = P_{\mathfrak{a}} \cap L_Q^{-1}(1)$.

Lemma 4.3. *Let $P_{\mathfrak{a}}$ be the Newton polyhedron of \mathfrak{a} . If there is $\lambda > 0$ such that w is in $\lambda P_{\mathfrak{a}}$, then*

$$\tau_{\mathbb{Q}}(w) = \max\{\lambda > 0 \mid w \in \lambda P_{\mathfrak{a}}\},$$

and if there is no such λ , then $\tau_{\mathbb{Q}}(w) = 0$. In particular, $\tau_{\mathbb{Q}}$ is piecewise linear on \mathbb{Q}_+^n : if σ is a maximal cone in Δ , then $\tau_{\mathbb{Q}} = L_{\sigma}$ on σ . Moreover, there is a positive integer N such that $\tau_{\mathbb{Q}}(w) = \tau(w)$ if $w_i/N \in \mathbb{N}$ for all i .

Proof. Since $P_{\mathbf{a}} = \text{conv}(a_1, \dots, a_r) + \mathbb{R}_+^n$, where $\text{conv}(a_1, \dots, a_r)$ denotes the convex hull of the a_i , we see that $\frac{1}{\lambda}w$ is in $P_{\mathbf{a}}$ if and only if there are $\beta_1, \dots, \beta_r \geq 0$ such that $\sum_j \beta_j = 1$ and $w_i/\lambda \geq \ell_i(\beta)$ for all i , where $\beta = (\beta_1, \dots, \beta_r)$. This gives the first assertion.

Let w be in the closed cone σ over a facet $\sigma^{(1)}$ of $P_{\mathbf{a}}$. If there is $\lambda > 0$ such that w is in $\lambda P_{\mathbf{a}}$, then for the largest such λ we have w in $\lambda \sigma^{(1)}$. Therefore $\lambda = L_{\sigma}(w)$. Since both L_{σ} and $\tau_{\mathbb{Q}}$ are continuous, we deduce that $L_{\sigma} = \tau_{\mathbb{Q}}$ on σ .

We denote by e_1, \dots, e_n the standard basis of \mathbb{R}^n . Every maximal cone σ in Δ is generated as a convex cone by some of the a_j (the vertices of $\sigma^{(1)}$) and some of the vectors e_1, \dots, e_n . By Carthéodory's theorem (see Proposition 1.15 in [Zi]) σ can be written as a union of cones, each generated by a subset of the generators of σ that forms a basis of \mathbb{R}^n .

Suppose now that w lies in the maximal cone σ . By the above discussion, we can write

$$w = \sum_{j \in I_1} \alpha_j a_j + \sum_{i \in I_2} \beta_i e_i \quad \text{with } \alpha_j \text{ and } \beta_i \text{ in } \mathbb{R}_+,$$

where the a_j and the e_i lie in σ and give a basis of \mathbb{R}^n . For the last assertion we need to find N such that for every w as above lying in $(\mathbb{N}\mathbb{N})^n$, the α_j and the β_i are integers.

Let N be a positive number which is divisible by the determinant of any square submatrix of $(a_{i,j})$. Using the determinant of the submatrix $(a_{i,j})$, with i in the complement of I_2 and j in I_1 , we see that α_j is an integer for every j in I_1 . This in turn implies that all β_i are integers, and completes the proof of the lemma. \square

With N as in the above lemma, we have a good understanding of τ on $(\mathbb{N}\mathbb{N})^r$. We describe now the behavior of τ on the congruence classes modulo the subgroup $(N\mathbb{Z})^n$. Let Δ be the fan decomposition in Lemma 4.3. Consider a cone σ in Δ which is not contained in any of the coordinate hyperplanes, and a translate $v + \sigma$ of this cone, for some $v \in \mathbb{Z}^n$.

Lemma 4.4. *With the above notation, there is w in $\sigma \cap \mathbb{N}^n$ such that the restriction of τ to the intersection of each congruence class with $v + w + \sigma$ is given by an affine linear function. More precisely, for every $c = (c_1, \dots, c_n)$ in $\{0, \dots, N-1\}^n$ there is A_c in \mathbb{Q}_+ such that for every u in $(v + w + \sigma) \cap \mathbb{N}^n$ with $u_i \equiv c_i \pmod{N}$ for all i , we have $\tau(u) = \tau_{\mathbb{Q}}(u) - A_c$.*

Proof. Note that we may replace at any time v by $v + v'$ for some v' in $\sigma \cap \mathbb{N}^n$. If $\tilde{\sigma}$ is a maximal cone in Δ such that σ is a face of $\tilde{\sigma}$, then by taking v' to be a large enough multiple of an element in the interior of $\tilde{\sigma}$, we may assume that $v + \sigma$ is contained in $\tilde{\sigma}$.

It is clear that τ is concave: if $b, b' \in \mathbb{N}^n$ we have $\tau(b + b') \geq \tau(b) + \tau(b')$. On the other hand, $\tau(b)$ and $\tau_{\mathbb{Q}}(Nb) = N\tau_{\mathbb{Q}}(b)$ are in \mathbb{N} , so

$$0 \leq \tau_{\mathbb{Q}}(b) - \tau(b) \in \frac{1}{N}\mathbb{Z}.$$

It follows that given c , we may choose $w_c \in \sigma \cap \mathbb{N}^n$ such that

$$\tau_{\mathbb{Q}}(v + w_c) - \tau(v + w_c) =$$

$$\min\{\tau_{\mathbb{Q}}(v+w) - \tau(v+w) \mid w \in \sigma \cap \mathbb{N}^n, v_i + w_i \equiv c_i \pmod{N} \text{ for all } i\}.$$

If $w' \in \sigma \cap (N\mathbb{N})^n$, then by concavity we have

$$(6) \quad \tau_{\mathbb{Q}}(v+w_c+w') - \tau(v+w_c+w') \leq \tau_{\mathbb{Q}}(v+w_c) - \tau(v+w_c)$$

(note that $\tau_{\mathbb{Q}}$ is linear on $\tilde{\sigma}$). By minimality, we have equality in (6). If we take $w = \sum_c w_c$, then we can find for every c an A_c as required by the lemma. \square

Remark 4.5. Note that if σ is a maximal cone, then for every c there are infinitely many u in $(v+w+\sigma) \cap \mathbb{N}^n$ such that $u_i \equiv c_i \pmod{N}$ for every i . This is not necessarily the case if σ is not maximal. However, if given c there is one such u , then there are infinitely many with the same property.

We can solve now the monomial case of Problem 2.3.

Proof of Theorem 4.1. We use the notation in Lemmas 4.3 and 4.4. We have seen that we may assume $J = (X_i^{b_i} \mid 1 \leq i \leq n)$. Consider N and the fan Δ in Lemma 4.3. Let σ be the cone in Δ such that $b = (b_i)$ lies in the relative interior of σ (note that since $b_i > 0$ for every i , σ is not contained in any coordinate hyperplane). We put $e = (1, \dots, 1) \in \mathbb{N}^n$ and let $\tilde{\sigma}$ be a maximal cone in Δ such that $qb - e \in \tilde{\sigma}$ for $q \gg 0$. It follows that σ is a face of $\tilde{\sigma}$. Recall that we have a linear function $L_{\tilde{\sigma}}$ whose restriction to $\tilde{\sigma}$ is equal to $\tau_{\mathbb{Q}}$.

Lemma 4.4 implies that we can find v_{σ} in $\sigma \cap \mathbb{N}^n$ and for every $c \in \{0, \dots, N-1\}^n$, a nonnegative rational number A_c^{σ} such that

$$\tau_{\mathbb{Q}}(u) - \tau(u) = A_c^{\sigma}$$

if u is in $(v_{\sigma} - e + \sigma) \cap \mathbb{N}^n$ and $u_i \equiv c_i \pmod{N}$ for all i .

Recall that $\nu_{\mathfrak{a}}^J(q) = \tau(qb - e)$. Moreover, if q is large enough, then $qb - e$ lies in $v_{\sigma} - e + \sigma$. Given $j \in \{0, \dots, N-1\}$, we take $c = (c_i)$ such that $jb_i - 1 \equiv c_i \pmod{N}$ for all i . If we put $\alpha = L_{\tilde{\sigma}}(b) = \tau_{\mathbb{Q}}(b)$ and $\gamma_j = L_{\tilde{\sigma}}(-e) - A_c^{\sigma}$, then the requirement of the theorem is satisfied. \square

Remark 4.6. For future reference, we give explicitly the description of the roots of $b_{\mathfrak{a}}$ that are obtained by our method (Theorem 4.9 below shows that these are, indeed, all the roots of $b_{\mathfrak{a}}$). For every cone σ in our fan Δ , such that σ is not contained in a coordinate hyperplane, let us choose a maximal cone $\tilde{\sigma}$ in Δ with the property that for some v in σ we have $v - e + \sigma \subseteq \tilde{\sigma}$ (hence σ is a face of $\tilde{\sigma}$). Let $L_{\tilde{\sigma}}$ be the linear function whose restriction to $\tilde{\sigma}$ is equal to $\tau_{\mathbb{Q}}$.

We consider now those c in $\{0, \dots, N-1\}^n$ such that there is b in \mathbb{N}^n in the relative interior of σ with $b_i - 1 \equiv c_i \pmod{N}$ for all i (if σ is maximal, then all c satisfy this condition). With A_c^{σ} as in the proof of Theorem 4.1, we deduce from Remark 4.2 that $L_{\tilde{\sigma}}(-e) - A_c^{\sigma}$ is a root of $b_{\mathfrak{a}}$. Indeed, it is enough to consider j with $j \equiv 1 \pmod{N}$ and $J = (X_1^{b_1}, \dots, X_n^{b_n})$. In addition, every root we obtain by our method is of this form. Note that the roots we obtain for $\sigma \in \Delta$ do not depend on the choice of $\tilde{\sigma}$: if $\tilde{\sigma}'$ is another maximal cone that satisfies the same property, then $L_{\tilde{\sigma}}$ and $L_{\tilde{\sigma}'}$ agree on the linear span of σ and e .

Remark 4.7. With the notation in the previous remark, the class of the root $L_{\bar{\sigma}}(-e) - A_c^\sigma$ in \mathbb{Q}/\mathbb{Z} is equal to the class of $L_{\bar{\sigma}}(-e) - L_{\bar{\sigma}}(qw - e) = -qL_{\bar{\sigma}}(w)$, where $w \in \mathbb{N}^n$ in the relative interior of σ and $q \gg 0$ are such that $w_i - 1 \equiv c_i \pmod{N}$ for all i and $q \equiv 1 \pmod{N}$. Recall that $NL_{\bar{\sigma}}(w)$ is an integer for all such w . We see that in order to compute the set of all such classes, when σ and c vary, it is enough to consider only the maximal cones σ . The set we get in this way is the set of classes of

$$\{-\tau_{\mathbb{Q}}(w) \mid w \in (\mathbb{Z}_{>0})^n\}.$$

Remark 4.8. It follows from Remark 4.6 that we get the same roots of the Bernstein-Sato polynomial if we consider only the invariants $\nu_{\mathfrak{a}}^J(p)$ for ideals J of the form $(X_1^{b_1}, \dots, X_n^{b_n})$, and for p prime and large enough with $p \equiv 1 \pmod{N}$.

The following theorem shows that all the roots of the Bernstein-Sato polynomial of a monomial ideal are detected by our method.

Theorem 4.9. *For every nonzero monomial ideal \mathfrak{a} , and for every root λ of $b_{\mathfrak{a}}$ there is a monomial ideal J together with a rational number α and a positive integer N' such that $\nu_{\mathfrak{a}}^J(q) = \alpha q + \lambda$ for q sufficiently large and with $q \equiv 1 \pmod{N'}$.*

Corollary 4.10. *The procedure described in Remark 4.6 gives all the roots of the Bernstein-Sato polynomial $b_{\mathfrak{a}}$.*

Proof. If λ is a root of $b_{\mathfrak{a}}$, let J , α and N' be as in Theorem 4.9. Applying Theorem 4.1 for \mathfrak{a} and J , we see that there are α' and λ' such that $\nu_{\mathfrak{a}}^J(q) = \alpha'q + \lambda'$ if q is large enough and $q \equiv 1 \pmod{N}$. Taking $q \equiv 1 \pmod{NN'}$, we deduce $\alpha' = \alpha$ and $\lambda' = \lambda$, which shows that λ is obtained by the procedure described in Remark 4.6. \square

Proof of Theorem 4.9. We use the description of the roots of $b_{\mathfrak{a}}$ from Remark 3.5. We can find $A \subseteq \{1, \dots, r\}$, $B \subseteq \{1, \dots, n\}$ and $(\alpha_j)_{j \in A}$ in $\mathbb{Z}_{\geq 0}^{|A|}$ and $(\beta_i)_{i \in B}$ in $\mathbb{Z}_{< 0}^{|B|}$ such that

$$\Gamma := \{w = (w_j) \in \mathbb{Q}^r \mid w_j = \alpha_j \text{ for } j \text{ in } A \text{ and } \ell_i(w) = \beta_i \text{ for } i \text{ in } B\}$$

is nonempty, and if $w \in \Gamma$, then $\lambda = \sum_{j=1}^r w_j$. Moreover, we may assume by Lemma 3.2 that the following condition holds: if $u = (u_j)$ in \mathbb{Z}^r is such that $u_j \geq -\alpha_j$ for all j in A and $\ell_i(u) \leq -\beta_i - 1$ for all i in B , then $\sum_{j=1}^r u_j \leq 0$.

Let us fix w in Γ . We deduce from our condition on (α_j) and (β_i) that

$$(7) \quad \lambda = \max_v \sum_{j=1}^r v_j,$$

the maximum being over those v in \mathbb{Q}^r such that $v_j \geq 0$ for j in A , $\ell_i(v) \leq -1$ for i in B and $v_j - w_j \in \mathbb{Z}$ for all j (the maximum is achieved for $v = w$).

We choose now $u = (u_j)$ in \mathbb{Q}_+^r such that $u_j = 0$ if and only if j is in A , and such that $u + w$ is in \mathbb{Z}^r . Moreover, we may choose u such that $\ell_i(u) = \sum_{j=1}^r a_{i,j} u_j > 0$ for all i in B . Indeed, if this is not the case, then there is i in B such that $a_{i,j} = 0$ whenever j is not in A . This contradicts the fact that $\ell_i(w) < 0$. Note also that since $u + w$ lies in \mathbb{Z}^r , $\ell_i(u)$ is an integer for every i in B .

Let N' be a positive integer such that $N'u$ is in $\mathbb{Z}_{\geq 0}^r$. If $q = mN' + 1$ for $m \geq 1$, then $qu + w$ is in \mathbb{Z}^r . With the notation introduced for the proof of Theorem 4.1, we claim that if q as above is large enough, then we have

$$(8) \quad \tau(B; (q\ell_i(u) - 1)_{i \in B}) = q \sum_{j=1}^r u_j + \lambda.$$

This implies the assertion of the theorem: take $J = (X_i^{\ell_i(u)} | i \in B)$ and $\alpha = \sum_{j=1}^r u_j$.

In order to prove the claim, suppose that $v \in \mathbb{Z}_{\geq 0}^r$ is such that $\ell_i(v) \leq q\ell_i(u) - 1$ for all i in B . For j in A we have $v_j - qu_j = v_j \geq 0$, and for i in B we have $\ell_i(v - qu) \leq -1$. As all $v_j - qu_j - w_j$ are integers, we deduce from (7) that $\sum_{j=1}^r v_j \leq q \sum_{j=1}^r u_j + \lambda$.

On the other hand, if $q \gg 0$ then $qu_j + w_j \geq 0$ for all j . Note that $\ell_i(qu + w) \leq q\ell_i(u) - 1$ for i in B and $\sum_{j=1}^r (qu_j + w_j) = q \sum_{j=1}^r u_j + \lambda$. This completes the proof of the claim, and hence that of the theorem. \square

We can prove now the description of the classes mod \mathbb{Z} of the roots of the Bernstein-Sato polynomial.

Proof of Corollary 1.1. We use Theorems 4.1 and 4.9. Recall that we have seen in Remark 4.7 that the classes in \mathbb{Q}/\mathbb{Z} of the roots of $b_{\mathbf{a}}$ are equal to

$$(9) \quad \{-L_{\sigma}(w) + \mathbb{Z} \mid \sigma \text{ maximal cone in } \Delta, w \in \sigma \cap (\mathbb{Z}_{>0})^n\}.$$

Therefore in order to prove the theorem it is enough to show that for every maximal cone σ in Δ , the set of classes $\{-L_{\sigma}(w) + \mathbb{Z} \mid w \in \sigma \cap (\mathbb{Z}_{>0})^n\}$ is the subgroup of \mathbb{Q}/\mathbb{Z} generated by $\frac{1}{m_{\sigma}}$, where if Q is the facet of $P_{\mathbf{a}}$ corresponding to σ , we put m_{σ} for m_Q .

Since $m_{\sigma}L_{\sigma}$ has integer coefficients, for every w in $\sigma \cap (\mathbb{Z}_{>0})^n$ the class of $-L_{\sigma}(w)$ lies in the subgroup generated by $1/m_{\sigma}$. On the other hand, if $\tau \subseteq \sigma$ is a convex cone generated by a basis e'_1, \dots, e'_n for \mathbb{Z}^n , then m_{σ} is the smallest positive integer such that all $m_{\sigma}L_{\sigma}(e'_i)$ are integers. In this case it follows easily that there is w in the interior of τ such that $\frac{1}{m_{\sigma}} + L_{\sigma}(w)$ is an integer. By taking suitable multiples of w we see that the subgroup generated by the class of $\frac{1}{m_{\sigma}}$ is contained in (9), which completes the proof. \square

Example 4.11. As we have already mentioned, it is a general fact that the largest root of the Bernstein-Sato polynomial of \mathbf{a} is $-\text{lc}(\mathbf{a})$, where $\text{lc}(\mathbf{a})$ is the log canonical threshold of \mathbf{a} (see [BMS2]). Let us prove this for monomial ideals using the above results.

It follows from [Ho1] that $\text{lc}(\mathbf{a})$ is the largest positive real number c such that $e = (1, \dots, 1)$ lies in $cP_{\mathbf{a}}$, where $P_{\mathbf{a}}$ is the Newton polyhedron of \mathbf{a} . It follows from Lemma 4.3 that, with our notation, $\text{lc}(\mathbf{a}) = \tau_{\mathbb{Q}}(e)$.

Let σ_0 be a maximal cone in the fan Δ such that e lies in σ_0 . With the notation in Remark 4.6, we get the root $-\text{lc}(\mathbf{a})$ as $L_{\sigma_0}(-e) - A_0^{\sigma_0}$ (corresponding to $c = (0, \dots, 0)$).

On the other hand, if λ is another root of $b_{\mathbf{a}}$, then by Theorem 4.9 and Remark 4.6 we have $\lambda = L_{\tilde{\sigma}}(-e) - A_c^{\tilde{\sigma}}$ for some $\tilde{\sigma}$, $\tilde{\sigma}$ and some c . Since $A_c^{\tilde{\sigma}} \geq 0$ and since concavity of $\tau_{\mathbb{Q}}$ gives $L_{\tilde{\sigma}}(e) \geq \tau_{\mathbb{Q}}(e) = L_{\sigma_0}(e)$, we get $\lambda \leq -\text{lc}(\mathbf{a})$.

We end this section with a description of the F -thresholds of monomial ideals (see §2 for the definition). It follows from Proposition 6.1 in the Appendix that the set of all F -thresholds of the monomial ideal \mathfrak{a} (computed for the reduction mod p of \mathfrak{a} , where $p \gg 0$ is a prime) is equal to the set of F -thresholds $c^J(\mathfrak{a}_p)$ with respect to monomial ideals J . Moreover, we have seen that it is enough to consider the case when $J = (X_1^{b_1}, \dots, X_n^{b_n})$, where b_i are positive integers. With the notation in the proof of Theorem 4.1, we have shown that for this J we have

$$c^J(\mathfrak{a}_p) = \tau_{\mathbb{Q}}(b)$$

for every prime p large enough. We will denote this number simply by $c^J(\mathfrak{a})$, as it does not depend on p .

We make the connection with the multiplier ideals of \mathfrak{a} . Recall that by the description in [Hol], the multiplier ideal $\mathcal{I}(\mathfrak{a}^\alpha)$ of \mathfrak{a} with exponent α can be described as follows

$$\mathcal{I}(\mathfrak{a}^\alpha) = (X^w | u + e \in \text{Int}(\alpha P_{\mathfrak{a}})).$$

The jumping coefficients of the multiplier ideals defined in [ELSV] are those $\alpha > 0$ such that $\mathcal{I}(\mathfrak{a}^\alpha)$ is strictly contained in $\mathcal{I}(\mathfrak{a}^{\alpha-\epsilon})$ for every $\epsilon > 0$. It follows from the above description of the multiplier ideals of \mathfrak{a} , that every $b = (b_i)$ with b_i positive integers gives a jumping coefficient α characterized by the fact that b lies in the boundary of $\alpha P_{\mathfrak{a}}$. Moreover, every jumping coefficient arises in this way. The connection with the F -thresholds is given by

Proposition 4.12. *For every prime p , the jumping coefficient corresponding to $b = (b_i)$ as above is equal to $c^J(\mathfrak{a}_p)$, where $J = (X_1^{b_1}, \dots, X_n^{b_n})$.*

Proof. The assertion follows from Lemma 4.3. □

Remark 4.13. It follows from the above discussion that the F -thresholds in the monomial case are easy to describe. Alternatively, this can be seen as follows: it is shown in [MTW] that in general, the F -thresholds (for various J) are the jumping coefficients for the test ideals introduced in [HY] as an analogue of multiplier ideals. On the other hand, it is proved in [HY] that for monomial ideals the test ideals coincide with the multiplier ideals. This gives a different approach to the description in Proposition 4.12.

Remark 4.14. It is clear that for monomial ideals all F -thresholds are rational numbers. In fact, more is true. If p is a prime, then the series $\sum_e \nu_{\mathfrak{a}}^J(p^e)t^e$ is rational. This follows from the fact that the function $e \rightarrow \nu_{\mathfrak{a}}^J(p^{e+1}) - p\nu_{\mathfrak{a}}^J(p^e)$ is eventually periodic, which is a consequence of Theorem 4.1.

5. EXAMPLES

In this section we give some examples to illustrate how to use our approach to give roots of the Bernstein-Sato polynomial. We use freely the notation introduced for the proof of Theorem 4.1. In all these examples we do not describe the complete picture given in Remark 4.6, but we give enough information to recover all the roots. For more

complicated examples, based on a more explicit combinatorial description of the roots of the Bernstein-Sato polynomial, we refer to [BMS1].

Example 5.1. Let $\mathfrak{a} = (\prod_{j \neq i} X_j | 1 \leq i \leq 4)$ in $\mathbb{Z}[X_1, \dots, X_4]$, so we have $\ell_i(s) = \sum_{j \neq i} s_j$ for $1 \leq i \leq 4$. If $\alpha \in \mathbb{N}^4$ is such that $\ell_i(\alpha) \leq b_i$ for all i , summing these inequalities we get

$$\sum_i \alpha_i \leq \lfloor (\sum_i b_i)/3 \rfloor,$$

where we use the notation $\lfloor x \rfloor$ for the largest integer $\leq x$.

Consider first the cone

$$\sigma = \{w | w_i \geq 0, \sum_j w_j \geq 3w_i \text{ for all } i\}.$$

If $b \in \sigma \cap \mathbb{N}^4$, then $\tau(b) = \lfloor (\sum_i b_i)/3 \rfloor$. Indeed, suppose for example that $\sum_i b_i \equiv 2 \pmod{3}$. If we take $\alpha_i = (\sum_j b_j - 2)/3 - (b_i - 1)$ for $i = 1, 2$ and $\alpha_i = (\sum_j b_j - 2)/3 - b_i$ for $i = 3, 4$, then $\alpha \in \mathbb{N}^4$ and $\sum_i \alpha_i = (\sum_i b_i - 2)/3$. The other two cases are similar. We get L_i for $i = 0, 1, 2$ with $L_i(b) = (\sum_j b_j - i)/3$, such that whenever $b \in \sigma \cap \mathbb{N}^4$, with $\sum_i b_i \equiv i \pmod{3}$ we have $\tau(b) = L_i(b)$. All $L_i(-1, \dots, -1)$ are roots of $b_{\mathfrak{a}}$, which gives the roots $-\frac{4}{3}, -\frac{5}{3}$ and -2 .

Suppose now that $b \in \mathbb{N}^4$ is such that, for example, $2b_4 > b_1 + b_2 + b_3$. If $\ell_i(\alpha) \leq b_i$ for all i , adding these inequalities for $1 \leq i \leq 3$ implies

$$\sum_{i=1}^4 \alpha_i \leq \lfloor (\sum_{i=1}^3 b_i)/2 \rfloor.$$

If we assume, in addition, that $b_1 + b_2 \geq b_3$, $b_1 + b_3 \geq b_2$ and $b_2 + b_3 \geq b_1$, then this maximum can be achieved. Indeed, if $\sum_{i=1}^3 b_i$ is even, take $\alpha_j = (\sum_{i=1}^3 b_i)/2 - b_j$ for $1 \leq j \leq 3$ and $\alpha_4 = 0$. If $\sum_{i=1}^3 b_i$ is odd, take $\alpha_1 = (\sum_{i=1}^3 b_i - 1)/2 - (b_1 - 1)$, $\alpha_j = (\sum_{i=1}^3 b_i - 1)/2 - b_j$ for $j = 2, 3$, and $\alpha_4 = 0$.

If we take L'_i for $i = 0, 1$ given by $L'_i(b) = (\sum_{j=1}^3 b_j - i)/2$, we see that if b is as above and $\sum_{j=1}^3 b_j \equiv i \pmod{2}$, we have $\tau(b) = L'_i(b)$. Therefore both $L'_i(-1, \dots, -1)$ are roots of $b_{\mathfrak{a}}$, which gives the roots $-\frac{3}{2}$ and -2 .

Therefore we have obtained the roots $-\frac{3}{2}, -\frac{4}{3}, -\frac{5}{3}$ and -2 . Note that, in fact, by [BMS2] (4.5) $b_{\mathfrak{a}} = (s + \frac{3}{2})(s + \frac{4}{3})(s + \frac{5}{3})(s + 2)^3$.

Example 5.2. Let $\mathfrak{a} = (X^2YZ, XY^2Z, XYZ^2)$, so $\ell_1(s) = 2s_1 + s_2 + s_3$, $\ell_2(s) = s_1 + 2s_2 + s_3$ and $\ell_3(s) = s_1 + s_2 + 2s_3$.

If $\alpha \in \mathbb{N}^3$ is such that $\ell_i(\alpha) \leq b_i$, by summing these relations we get $\sum_i \alpha_i \leq \lfloor \frac{1}{4} \sum_i b_i \rfloor$. This maximum is achieved if b lies in the cone

$$\sigma = \{w | w_i \geq 0, w_i \geq \frac{\sum_j w_j}{4} \text{ for all } i\}.$$

Indeed, suppose for example that $\sum_j b_j \equiv 1 \pmod{4}$. If $\alpha_1 = b_1 - \frac{\sum_i b_i - 1}{4}$, $\alpha_2 = b_2 - \frac{\sum_i b_i - 1}{4}$, and $\alpha_3 = b_3 - \frac{\sum_i b_i + 3}{4}$, then $\alpha \in \mathbb{N}^3$ and $\sum_i \alpha_i = (b_1 + b_2 + b_3 - 1)/4$. The other three cases are similar.

We get $L_i(b) = (\sum_j b_j - i)/4$ for $i = 0, 1, 2, 3$ such that if $b \in \mathbb{N}^3$ is in σ and $\sum_j b_j \equiv i \pmod{4}$, then $\tau(b) = L_i(b)$. We get roots of the Bernstein-Sato polynomial of \mathfrak{a} given by $L_i(-1, -1, -1)$ for $i = 0, 1, 2, 3$, i.e. $-\frac{3}{4}$, -1 , $-\frac{5}{4}$ and $-\frac{6}{4}$. Note that by [BMS2] (4.5), $b_{\mathfrak{a}}(s) = (s + \frac{3}{4})(s + \frac{5}{4})(s + \frac{6}{4})(s + 1)^3$.

Example 5.3. Let \mathfrak{a} be the ideal generated by $X_i X_j$ for all $1 \leq i < j \leq n$, with $n \geq 3$. We have

$$\ell_i(s) = \sum_{j < i} s_{j,i} + \sum_{j > i} s_{i,j}$$

for $s = (s_{i,j})_{i < j}$. Note that if $\ell_i(s) \leq b_i$ for all i , by taking the sum we get

$$\sum_{i < j} s_{i,j} \leq \lfloor (\sum_i b_i)/2 \rfloor.$$

It is easy to give, as above, a full-dimensional cone σ on which this maximum is achieved. This gives L_1 and L_2 with $L_i(b) = (\sum_j b_j - i)/2$ such that if $b \in \mathbb{N}^n$ lies in σ and $\sum_j b_j \equiv i \pmod{2}$, then $\tau_i(b) = L_i(b)$. Therefore $L_0(-1, \dots, -1) = -\frac{n}{2}$ and $L_1(-1, \dots, -1) = -\frac{n+1}{2}$ are roots of $b_{\mathfrak{a}}$.

Suppose now that $b \in \mathbb{N}^n$ is such that $b_1 > \sum_{j=2}^n b_j$. If $\ell_i(\alpha) \leq b_i$ for all i , by adding these inequalities for $i \geq 2$, we deduce $\sum_{i < j} \alpha_{i,j} \leq \sum_{j \geq 2} b_j$. Moreover, this maximum can be achieved: let $\alpha_{1,i} = b_i$ for $i > 1$ and all other $\alpha_{i,j} = 0$. If $L(b) = \sum_{j \geq 2} b_j$ we see that if b is as above, then $\tau(b) = L(b)$. Hence $L(-1, \dots, -1) = -(n-1)$ is also a root of $b_{\mathfrak{a}}$. Note that by [BMS2] (4.5), the Bernstein-Sato polynomial of \mathfrak{a} is $b_{\mathfrak{a}}(s) = (s + \frac{n}{2})(s + \frac{n+1}{2})(s + n - 1)$.

6. APPENDIX

We show that if \mathfrak{a} is a monomial ideal, then in order to compute the functions $\nu_{\mathfrak{a}}^J(p^e)$ for $p \gg 0$, it is enough to consider the case when J is a monomial ideal.

Proposition 6.1. *Let \mathfrak{a} be a nonzero ideal generated by monomials in $\mathbb{Z}[X_1, \dots, X_n]$ and let $J \subseteq (X_1, \dots, X_n)$ be an ideal such that \mathfrak{a} is contained in the radical of J . If we put*

$$\tilde{J} := (X^u \mid hX^u \in J \text{ for some } h \in \mathbb{Z}[X_1, \dots, X_n] \setminus (X_1, \dots, X_n))$$

and if $p \gg 0$, then $\nu_{\mathfrak{a}}^J(p^e) = \nu_{\mathfrak{a}}^{\tilde{J}}(p^e)$ for every $e \geq 1$.

For the proof of Proposition 6.1 we will need the following lemma.

Lemma 6.2. *With the notation in Proposition 6.1, for $p \gg 0$ we have*

$$\tilde{J}_p = (X^u \mid X^u \in J_p)$$

as ideals in $\mathbb{F}_p[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$.

Proof. Given X^u in \tilde{J} and $h \in \mathbb{Z}[X_1, \dots, X_n]$ such that $h(0) \neq 0$ and $hX^u \in J$, then for p not dividing $h(0)$ we see that the image of X^u in $\mathbb{F}_p[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$ lies in J_p . Since \tilde{J} is finitely generated, for $p \gg 0$ we deduce the inclusion \subseteq in the statement.

We prove now the reverse inclusion. It is clear that if $a = (a_i)$ and $b = (b_i)$ in \mathbb{N}^n are such that $a_i \leq b_i$ for every i , then $(J: X^a) \subseteq (J: X^b)$. For a subset A of $\{1, \dots, n\}$ and $w = (w_i) \in \mathbb{N}^n$, we denote by $w + \mathbb{N}^A$ the set

$$\{u = (u_i) \in \mathbb{N}^n \mid u_i \geq w_i \text{ for } i \in A \text{ and } u_i = w_i \text{ for } i \notin A\}.$$

Using the fact that $\mathbb{Z}[X_1, \dots, X_n]$ is Noetherian, we deduce that there is a decomposition

$$\mathbb{N}^n = \bigsqcup_{j=1}^m (w^{(j)} + \mathbb{N}^{A_j}),$$

for some $w^{(j)}$ in \mathbb{N}^n and $A_j \subseteq \{1, \dots, n\}$ such that for every w in $w^{(j)} + \mathbb{N}^{A_j}$ the ideal $(J: X^w)$ is equal to a fixed ideal I_j . In particular, we deduce that for every i in A_j , we have $(I_j: X_i) = I_j$.

Given a prime p , for every ideal I in $\mathbb{Z}[X_1, \dots, X_n]$ we denote by \bar{I} the reduction of I in $\mathbb{F}_p[X_1, \dots, X_n]$. There is p_0 such that for every prime $p \geq p_0$ the following hold:

$$(10) \quad (\bar{I}_j: X_i) = \bar{I}_j \text{ in } \mathbb{F}_p[X_1, \dots, X_n] \text{ for all } j \leq m \text{ and all } i \in A_j,$$

$$(11) \quad (\bar{J}: X^{w^{(j)}}) = \bar{I}_j \text{ in } \mathbb{F}_p[X_1, \dots, X_n] \text{ for all } j \leq m.$$

In order to show (10), use the exact sequence

$$(12) \quad 0 \rightarrow (I_j: X_i)/I_j \rightarrow \mathbb{Z}[X_1, \dots, X_n]/I_j \xrightarrow{h} \mathbb{Z}[X_1, \dots, X_n]/I_j,$$

where h is multiplication by X_i . Note that given an arbitrary finitely generated $\mathbb{Z}[X_1, \dots, X_n]$ -module M , there is a positive integer ℓ such that $M[1/\ell]$ is flat over $\mathbb{Z}[1/\ell]$. Indeed, the submodule $N \subseteq M$ of elements annihilated by some positive integer is finitely generated over $\mathbb{Z}[X_1, \dots, X_n]$. Therefore we can find ℓ such that $\ell N = 0$, and $M[1/\ell]$ is torsion-free, hence flat over $\mathbb{Z}[1/\ell]$. Applying this observation for the image and for the cokernel of h , we see that if $p \gg 0$, then the sequence obtained from (12) by tensoring with \mathbb{F}_p is again exact, which gives (10). The proof of (11) is similar.

Consider now a prime $p \geq p_0$ and fix $w \in \mathbb{N}^n$. Let j be such that w is in $w^{(j)} + \mathbb{N}^{A_j}$. Suppose that h is in $\mathbb{F}_p[X_1, \dots, X_n]$ such that $h(0) \neq 0$ and hX^w is in \bar{J} . Using (11) we deduce that hX^{w-w_j} lies in \bar{I}_j . Moreover, (10) implies that h is in \bar{I}_j . If g is an element in I_j whose class is h , then $g(0) \neq 0$ and gX^{w_j} is in J . In particular, gX^w is in J , so $X^w \in \tilde{J}$, which completes the proof of the lemma. \square

Proof of Proposition 6.1. We consider $p \gg 0$ so the assertion in Lemma 6.2 applies. In particular, we have $\tilde{J}_p \subseteq J_p$, so $\nu_{\mathfrak{a}}^{\tilde{J}}(p^e) \geq \nu_{\mathfrak{a}}^J(p^e)$. In order to show that we have equality, since \mathfrak{a} is monomial it is enough to prove that if X^u is in $J_p^{[p^e]}$, then X^u is in $\tilde{J}_p^{[p^e]}$. We do induction on $e \geq 0$, the case $e = 0$ being a consequence of Lemma 6.2.

If $e \geq 1$, we write $u = pv + w$ with v and w in \mathbb{N}^n and $0 \leq w_i < p$ for every i . We deduce

$$(13) \quad X^w \in (J_p^{[pe]} : X^{pv}) = (J_p^{[pe-1]} : X^v)^{[p]},$$

where the equality follows from the fact that $\mathbb{F}_p[X_1, \dots, X_n]$ is flat over $\mathbb{F}_p[X_1^p, \dots, X_n^p]$. The assumption on w implies that $w_i = 0$ for every i and $(J_p^{[pe-1]} : X^v)$ is the unit ideal. We deduce that X^v lies in $J_p^{[pe-1]}$, hence in $\tilde{J}_p^{[pe-1]}$ by the induction hypothesis. Therefore X^u is in $J_p^{[pe]}$, which completes the proof of the proposition. \square

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