Applications of soft union sets to hemirings via $SU-h$-ideals

Jianming Zhana,∗, Naim Çağmanc and Aslıhan Sezgin Sezerb,c

aDepartment of Mathematics, Hubei University for Nationalities, Enshi, Hubei Province, P.R. China
bDepartment of Mathematics, Gaziosmanpasa University, Tokat, Turkey
cDepartment of Mathematics, Amasya University, Amasya, Turkey

Abstract. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, we introduce the concepts of soft union hemirings ($SU-h$-ideals) of hemirings by soft intersection-union product and obtain some related results. Finally, we investigate some characterizations of $h$-hemiregular hemirings by soft union $h$-ideals.

Keywords: Soft set, soft intersection-union product, soft union hemiring, soft union $h$-ideal, $h$-hemiregular hemiring

1. Introduction

There are three theories: probability theory, fuzzy set theory and rough set theory which we can consider as mathematical tools for dealing with uncertainties. However, all of them have their advantages as well as inherent limitations in dealing with uncertainties. To overcome these difficulties, Molodtsov [23] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Nowadays, works on the soft sets are progressing rapidly. Maji [21] discussed further soft set theory. Ali et al. [4, 5] proposed some new operations on soft sets. Qin [24] investigated soft equality. At the same time, this theory has been proven useful in many different fields such as decision making [7–9, 13, 14, 22], data analysis [29, 33], forecasting and so on. It also is interesting to see that soft sets are closely related to many other soft computing models such as rough sets and fuzzy sets. Feng et al. [14] first considered the combination of soft sets, fuzzy sets and rough sets. Using soft sets as the granulation structures, Feng et al. [10] initiated soft approximation spaces and soft rough sets, which extended Pawlak’s rough set model using soft sets. In some cases Feng’s soft rough set model could provide better approximations than classical rough sets. Recently, the algebraic structures of soft sets have been studied increasingly, such as, soft rings [1, 6], soft groups [3], soft semirings [10], soft hemirings [20], soft $BCK/BCI$-algebras [16, 18], soft ordered semigroups [17], soft $BCH$-algebras [19], soft near-rings [26, 27]. Feng et al. [12] investigated five different types of soft subsets and considered the free soft algebras associated with soft product operations. It has been shown that soft sets have some non-classical algebraic properties which are distinct from those of crisp sets or fuzzy sets.

We note that the ideals of semirings play a crucial role in the structure theory, ideals in semirings do not in general coincide with the ideals of a ring. For this reason, the usage of ideals in semirings is somewhat limited. By a hemiring, we mean a special semiring
with a zero and with a commutative addition. The properties of \( h \)-ideals were thoroughly investigated by Torre [28] and by using \( h \)-ideals, Torre established some analogous ring theorems for hemirings. In particular, the \( h \)-hemiregular hemirings were described by Zhan et al. [32]. Some characterizations of the \( h \)-semisimple and \( h \)-\textit{intra}-hemiregular hemirings were investigated by Yin et al. [2, 30, 31].

In [25], Scrgin made a new approach to the classical ring theory via soft set theory with the concept of soft union rings. By this new idea, in this paper, we introduce the concept of soft union \( h \)-ideals of hemirings and obtain some related results. Finally, we investigate some characteristics of \( h \)-hemiregular hemirings by soft union \( h \)-ideals.

2. Preliminaries

A semiring is an algebraic system \((S, +, \cdot)\) consisting of a non-empty sets \( S \) together with two binary operations on \( S \) called addition and multiplication (denoted in the usual manner) such that \((S, +)\) and \((S, \cdot)\) are semigroups and the following distributive laws:

\[
ab + c = ab + ac \quad \text{and} \quad (a + bc) = ac + bc
\]

are satisfied for all \( a, b, c \in S \).

By zero of a semiring \((S, +, \cdot)\) we mean an element 0 \( \in S \) such that 0 \( \cdot x = x = 0 \) and 0 \( + x = x + 0 = x \) for all \( x \in S \). A semiring with zero and a commutative semigroup \((S, +)\) is called a hemiring. For the sake of simplicity, we shall write \( ab \) for \( a \cdot b, a, b \in S \).

A subhemiring of a hemiring \( S \) is a subset \( A \) of \( S \) closed under addition and multiplication. A subset \( A \) of \( S \) is called a \textit{left ideal} of \( S \) if \( A \) is closed under addition and \( S \cdot A \subseteq A \) (resp., \( A \cdot S \subseteq A \)). Further, \( A \) is called an \textit{ideal} of \( S \) if it is both a left ideal and a right ideal of \( S \).

A subhemiring \((\text{left ideal, right ideal, ideal})\) \( A \) of \( S \) is called a \textit{left} \( h \)-subhemiring \((\text{left right ideal, right \( h \)-ideal, \( h \)-ideal})\) of \( S \), respectively, if for any \( x, z \in S, a, b \in A \), and \( x + a = z = b + z \) implies \( x \in A \).

The \textit{\( h \)-closure} \( \overline{A} \) of a subset \( A \) of \( S \) is defined as

\[
\overline{A} = \{ x \in S | x + a + z = b + z \text{ for some } a, b \in A, \ z \in S \}.
\]

From now on, \( S \) is a hemiring, \( U \) is an initial universe, \( E \) is a set of parameters, \( P(U) \) is the power set of \( U \) and \( A, B, C \subseteq E \).

Definition 2.1. [23] A soft set \( f_A \) over \( U \) is defined as \( f_A : E \rightarrow P(U) \) such that \( f_A(x) = \emptyset \) if \( x \notin A \). Here \( f_A \) is also called an approximate function.

A soft set \( f_A \) over \( U \) can be represented by the set of ordered pairs \( f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\} \).

It is clear to see that a soft set is a parameterized family of subsets of \( U \). Note that the set of all soft sets over \( U \) will be defined by \( S(U) \).

Definition 2.2. [8, 27] (i) Let \( f_A, f_B \in S(U) \). Then, \( f_A \) is called soft subset of \( f_B \) and denoted by \( f_A \subseteq f_B \) if \( f_A(x) \subseteq f_B(x) \) for all \( x \in E \).

(ii) Let \( f_A, f_x \in S(U) \). Then, union of \( f_A \) and \( f_B \), denoted \( f_A \cup f_B \), is defined as \( f_A \cup f_B = \{(x, f_A(x) \cup f_B(x)) | x \in E\} \).

(iii) Let \( f_A \) be a soft set over \( U \) and \( \alpha \subseteq U \). Then, lower \( \alpha \)-inclusion of \( f_A \), denoted by \( L(\alpha, f_A) \), is defined as \( L(\alpha, f_A) = \{(x, A \cap f_A(x)) | x \in \alpha \} \).

Definition 2.3. [25] Let \( X \) be a subset of \( S \). We denote by \( S_X^{C} \) the soft characteristic function of the complement of \( X \) and define as

\[
S_X^{C}(x) = \begin{cases} \emptyset & \text{if } x \in X, \\ U & \text{if } x \in S \setminus X. \end{cases}
\]

Proposition 2.4. [25] Let \( X, Y \subseteq S \). Then the following hold:

1. \( Y \subseteq X \Rightarrow S_X^{C} \subseteq S_Y^{C} \)
2. \( S_X^{C} \cup S_Y^{C} = S_{X \cup Y}^{C} \)

3. SU-hemirings

In this section, we introduce the concept of soft union hemirings and investigate some related properties.

Definition 3.1. A soft set \( f_x \) over \( U \) is called a soft union hemiring (briefly, SU-hemiring) of \( S \) if

\( (SU_1) f_x(x + y) \subseteq f_x(y) \cup f_x(y) \text{ for all } x, y \in S, \)

\( (SU_2) f_x(x) \subseteq f_x(y) \cup f_x(y) \text{ for all } x, y \in S, \)

\( (SU_3) f_x(x) \subseteq f_x(y) \cup f_x(y) \text{ for all } x, a, b, z \in S. \)

It is clear that \( f_x(0) \subseteq f_x(x) \) for all \( x \in S \).

Example 3.2. (i) Let \( S = Z_5 = \{0, 1, 2, 3, 4, 5\} \) be the hemiring of non-negative integers modulo 6. Assume that \( U = Z_5 \) is the universal set.
Define a soft set \( f_S \) over \( U \) by 
\[ f_S(0) = \{1\}, \quad f_S(1) = \{2, 3, 4\}, \quad f_S(2) = f_S(4) = \{1, 2, 3\} \quad \text{and} \quad f_S(3) = \{1, 4\}. \]
Then, one can easily check that \( f_S \) is an SU-hemiring of \( S \) over \( U \).

(ii) Assume that \( U = \{xx, x0 | x \in \mathbb{Z}_5\} \), \( 2 \times 2 \)
matrices with \( \mathbb{Z}_5 \) terms, is the universal set.

Let \( S = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\} \) be the hemiring of non-negative integers module 6. Define a soft set \( f_S \) over \( U \) by
\[ f_S(0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_S(2) = f_S(4) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \quad f_S(5) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \]

Then, one can easily check that \( f_S \) is an SU-hemiring of \( S \) over \( U \).

Remark 3.3. It is easy to see that if \( f_S(x) = \emptyset \) for all \( x \in S \), then \( f_S \) is an SU-hemiring of \( S \) over \( U \). We denote such a kind of SU-hemiring by \( \theta \). It is obvious that \( \theta = S^2 \).

Definition 3.4. Let \( f_S \) and \( g_S \) be two soft sets over \( U \). Then

(1) Soft intersection-union sum \( f_S \oplus g_S \) is defined by
\[ (f_S \oplus g_S)(x) = \bigcup_{x \in \mathbb{Z}_5} (f_S(a_1) \cup f_S(a_2) \cup g_S(b_1) \cup g_S(b_2)) \]
and \( f_S \oplus g_S(x) = U \) if \( x \) cannot be expressed as \( x + a_1 + b_1 + z = a_2 + b_2 + z \) for some \( a_1, a_2, b_1, b_2, z \in S \).

(2) Soft intersection-union product \( f_S \circ g_S \) is defined by
\[ (f_S \circ g_S)(x) = \bigcup_{x \in \mathbb{Z}_5} (f_S(a_1) \cup f_S(a_2) \cup g_S(b_1) \cup g_S(b_2)) \]
and \( f_S \circ g_S(x) = U \) if \( x \) cannot be expressed as \( x + \sum_{j=1}^{n} a_j b_j + z = \sum_{j=1}^{n} a_j' b_j' + z \).

Theorem 3.5. Let \( f_S \) be a soft set over \( U \). Then \( f_S \) is an SU-hemiring of \( S \) over \( U \) if and only if it satisfies (SU1) and
\( (SU_4) f_S \oplus f_S \supseteq f_S \cup f_S; \quad (SU_5) f_S \circ f_S \supseteq f_S. \)

Proof. Assume that \( f_S \) is an SU-hemiring of \( S \) over \( U \). Let \( x \in S \). If \( f_S \oplus f_S(x) = U \), then \( f_S \oplus f_S(x) \supseteq f_S(x) \). Thus, \( f_S \oplus f_S \supseteq f_S \). Otherwise, let \( x + a_1 + b_1 + z = a_2 + b_2 + z \) for some \( a_1, a_2, b_1, b_2, z \in S \).

Thus, \( f_S \oplus f_S(x) \subseteq (f_S(a_1) \cup f_S(a_2) \cup f_S(b_1) \cup f_S(b_2)) \)
\[ \supseteq \bigcup_{x \in \mathbb{Z}_5} f_S(x) = f_S(x), \]
which implies, \( f_S \oplus f_S \supseteq f_S \). Thus, \( SU_4 \) holds.

Let \( x \in S \). If \( f_S \circ f_S(x) = U \), then \( f_S \circ f_S(x) \supseteq f_S(x) \). Thus, \( f_S \circ f_S \supseteq f_S \). Otherwise, let \( x + \sum_{j=1}^{n} a_j b_j + z = \sum_{j=1}^{n} a_j' b_j' + z \) for all \( i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n \).

Thus, \( f_S \circ f_S(x) \subseteq \bigcup_{x \in \mathbb{Z}_5} f_S(x) \), which implies, \( f_S \circ f_S \subseteq f_S \). Thus, \( SU_5 \) holds.

Conversely, assume that the conditions \( (SU_4), (SU_5) \) and \( (SU_3) \) hold. Then

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$f_S(x + y) = f_S(x) \cup f_S(y) = f_S(x) \cup f_S(y) = f_S(x) \cup f_S(y)$

Then, $(SU_1)$ holds.

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Hence, $f_S$ is an $SU$-hemiring of $S$ over $U$.

Proposition 3.6. A non-empty subset $A$ of $S$ is an $h$-subhemiring of $S$ if and only if the soft subset $f_S$ defined by

$f_S(x) = \begin{cases} 
   \alpha & \text{if } x \in S \setminus A, \\
   \beta & \text{if } x \in A,
\end{cases}$

is an $SU$-hemiring of $S$ over $U$, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof. Let $A$ be an $h$-subhemiring of $S$ and $x, y \in S$.

(i) If $x, y \in A$, then $x + y \in A$. Hence $f_S(x + y) = f_S(x) \cup f_S(y) = f_S(x) \cup f_S(y) = f_S(x) \cup f_S(y)$.

Thus, $f_S(y) \subseteq f_S(x) \cup f_S(y)$ and $f_S(x) \subseteq f_S(x) \cup f_S(y)$.

(ii) If either one of $x$ and $y$ does not belong to $A$, then $x + y \notin A$ or $x \notin A$ and $y \notin A$ or $x \notin A$.

In any case, $f_S(x + y) \subseteq f_S(x) \cup f_S(y) = f_S(x) \cup f_S(y)$.

Thus, $f_S(x + y) \subseteq f_S(x) \cup f_S(y) = f_S(x) \cup f_S(y)$.

(iii) Now, let $a, b, c \in S$ be such that $x + a + z = b + z$. (1) If $a, b \in A$, then $x \in A$, and so $f_S(x) = f_S(a) \cup f_S(b) = f_S(x)$. (2) If $a \notin A$ or $b \notin A$, then $f_S(x) \subseteq f_S(a) \cup f_S(b) = f_S(x)$. Thus, $f_S(x)$ is an $SU$-hemiring of $S$ over $U$.

Conversely, assume that $f_S$ is an $SU$-hemiring of $S$ over $U$. (i) Let $x, y \in A$, then $f_S(x + y) \subseteq f_S(x) \cup f_S(y) = \beta$ and $f_S(x) \subseteq f_S(x) \cup f_S(y) = \beta$, which implies, $x + y, y \in A$. (ii) Let $x, z, a, b \in A$ be such that $x + a + z = b + z$. Then $f_S(x) = f_S(a) \cup f_S(b) = \beta$, and so $f_S(x) = \beta$. Thus, $x \in A$.

The following is a consequence of the above proposition.

Corollary 3.7. Let $A$ be a non-empty subset of $S$. Then $A$ is an $h$-subhemiring of $S$ if and only if $S \cup A$ is an $SU$-hemiring of $S$ over $U$.

Theorem 3.3. (i) Let $f_S$ be a soft set over $U$ and $a \subseteq U$ such that $a \in L(f_S(x))$. If $f_S$ is an $SU$-hemiring of $S$ over $U$, then $L(f_S(a))$ is a non-empty subset of $S$.

(ii) Let $f_S$ be a soft set over $U$, $L(f_S(a))$ a lower $h$-subhemiring of $f_S$, and $f_S(a) \subseteq \alpha$, then $L(f_S(a))$ is an $SU$-hemiring of $S$ over $U$.

Proof. (i) Since $f_S(x) = a$ for some $x \in S$, then $f_S(x) \subseteq a$. Thus, $f_S(x) \subseteq a$.

Thus, $f_S(x) \subseteq a$ and $f_S(x) \subseteq a$. Thus, $f_S(x) \subseteq a$ and $f_S(x) \subseteq a$.

(ii) Let $x, y \in S$ be such that $f_S(x) = a_1$ and $f_S(y) = a_2$, where $a_1 \subseteq a_2$. Then $x \in L(f_S(a_1))$ and $y \in L(f_S(a_2))$. Since $f_S(a_1)$ is an $h$-subhemiring of $S$ for all $a \subseteq U$, then $x + y \in L(f_S(a_2))$ and $xy \in L(f_S(a_2))$.

Hence $f_S(x + y) \subseteq a_2 = a_1 \cup a_2 = f_S(x) \cup f_S(y)$ and $f_S(xy) \subseteq a_2 = a_1 \cup a_2 = f_S(x) \cup f_S(y)$.

Note, $x, z, a, b \in S$ with $x + a + z = b + z$. Then $f_S(x)$ is an $SU$-hemiring of $S$. Therefore $f_S$ is an $SU$-hemiring of $S$ over $U$.

4. $SU$-h-ideals

In this section, we investigate the properties of soft union $h$-ideals of hemirings.

Definition 4.1. A soft set $f_S$ over $U$ is called a soft union (left(right) $h$-ideal of $S$.
Example 4.2. Let $S = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be the hemiring of non-negative integers module 6 and $U = \mathbb{Z}_6$. Define a soft set $f_S$ over $U$ by $f_S(0) = \{1\}$, $f_S(1) = f_S(5) = \{1, 2, 3\}$, $f_S(2) = f_S(4) = \{1, 2\}$ and $f_S(3) = \{1, 3\}$. Then, one can easily check that $f_S$ is an $SU$-$h$-ideal of $S$ over $U$.

Theorem 4.3. Let $f_S$ be a soft set over $U$. Then $f_S$ is an $SU$-$h$-ideal of $S$ over $U$ if and only if it satisfies $(SU_\cup)$ and $(SU_\cap)$.

Proof. Let $f_S$ be an $SU$-$h$-ideal of $S$ over $U$. If $(SU_\cup)f_S = U$, then it is clear that $SU_\cup f_S = f_S$. Otherwise, let $x + \sum_{i=1}^n a_i = \sum_{j=1}^m a_j' + z$ be a soft set over $U$. Then

$$(SU_\cup)f_S(x) = \bigcup_{i=1}^n \left( (SU_\cup)(a_i \cup a_i') \cup f_S(b_i) \cup f_S(b_i') \right) = \sum_{i=1}^n a_i + z = \sum_{j=1}^m a_j' + z,$$

which implies, $SU_\cup f_S \supseteq f_S$. This proves that $(SU_\cup)$ holds.

Conversely, assume that the conditions $(SU_\cap), (SU_\cup)$ and $(SU_\cap)$ hold. Then

$$(SU_\cap)f_S(xy) \subseteq f_S(xy) \subseteq f_S(x) \cup f_S(y).$$

A soft set over $U$ is called a $SU$-$h$-ideal of $S$ (briefly, $SU$-$h$-ideal) if it is both an $SU$-$left$-$h$-ideal and an $SU$-$right$-$h$-ideal of $S$.

Proposition 4.4. A non-empty subset $A$ of $S$ is a $left(right)$-$h$-ideal of $S$ if and only if the soft subset $f_S$ defined by

$$f_S(x) = \begin{cases} \alpha & \text{if } x \in S \setminus A, \\ \beta & \text{if } x \in S, \end{cases}$$

is an $SU$-$left(right)$-$h$-ideal of $S$.

Proof. It is similar to the proof of proposition 3.6.

Corollary 4.5. Let $A$ be a non-empty subset of $S$. Then $A$ is a left(right) $h$-ideal of $S$ if and only if $S \setminus A$ is an $SU$-$left(right)$-$h$-ideal of $S$ over $U$.

Theorem 4.6. (i) Let $f_S$ be a soft set over $U$ and $a \subseteq U$ such that $a \subseteq \bigcup_0^\nu f_S(x)$. If $f_S$ is an $SU$-$left(right)$-$h$-ideal of $S$ over $U$, then $L(f_S, a)$ is a left(right) $h$-ideal of $S$.

(ii) Let $f_S$ be a soft set over $U$, $L(f_S, a)$ a lower left(right) $h$-ideal of $f_S$ for each $a \subseteq U$ and $\varepsilon(f_S)$ an ordered set by inclusion. Then $f_S$ is an $SU$-$left(right)$-$h$-ideal of $S$ over $U$.

Proof. It is similar to the proof of Theorem 3.8.

Proposition 4.7. Let $f_S$ and $h_S$ be two $SU$-$left(right)$-$h$-ideals of $S$ over $U$, then so is $f_S \cup h_S$.

Proof. Let $f_S$ and $h_S$ be two $SU$-$left(right)$-$h$-ideals of $S$ over $U$. For any $x, y \in S$, then

$$(f_S \cup h_S)(x + y) = f_S(x + y) \cup h_S(x + y) \subseteq f_S(x) \cup f_S(y) \cup h_S(x) \cup h_S(y) = f_S(x + y) \cup h_S(x \cup y) = (f_S \cup h_S)(x) \cup (f_S \cup h_S)(y).$$
(f_S \cup h_S)(xy) = f_S(xy) \cup h_S(xy) \subseteq f_S(x) \cup h_S(y) = (f_S \cup h_S)(y).

Now, let x, z, a, b \in S with x + a + z = b + z. Then

\[
(f_S \cup h_S)(x) = f_S(x) \cup h_S(x) \subseteq f_S(a) \cup f_S(b) \cup (h_S(a) \cup h_S(b)) = (f_S(a) \cup (h_S(b)) \cup f_S(b) \cup h_S(b)) = (f_S \cup h_S)(a) \cup (f_S \cup h_S)(b).
\]

Thus, \(f_S \cup h_S\) is an \(SU\)-left \(h\)-ideal of \(S\) over \(U\).

Similarly, we can prove \(f_S \cup h_S\) is an \(SU\)-right \(h\)-ideal of \(S\) over \(U\). \(\square\)

**Proposition 4.8.** Let \(f_S\) and \(h_S\) be two \(SU\)-left (right) \(h\)-ideals of \(S\) over \(U\), then so is \(f_S \cup h_S\).

**Proof.** Let \(f_S\) and \(h_S\) be two \(SU\)-left \(h\)-ideals of \(S\) over \(U\) and \(x, y \in S\). Then

\[
\begin{align*}
(i) (f_S \cup h_S)(x + y) & = \bigcup_{a+b+c = x+y} (f_S(a) \cup f_S(b) \cup h_S(c) \cup h_S(d)) \\
& = \bigcup_{a+b+c = x+y} f_S(a) \cup f_S(b) \cup h_S(c) \cup h_S(d) \\
& = \bigcup_{a+b+c = x+y} f_S(a) \cup f_S(b) \cup h_S(c) \cup h_S(d). \\
(ii) (f_S \cup h_S)(xy) & = \bigcup_{a+b+c = x+y} (f_S(a) \cup f_S(b) \cup h_S(c) \cup h_S(d)) \\
& = \bigcup_{a+b+c = x+y} f_S(a) \cup f_S(b) \cup h_S(c) \cup h_S(d). \\
\end{align*}
\]

Theorem 5.4. \([32]\) A hemiring \(S\) is called \(h\)-hemiregular if for each \(a \in S\), there exist \(x_1, x_2 \in S\) such that \(a + xa + z = a + xa + z\).

**Lemma 5.2.** \([32]\) If \(A\) and \(B\) are, respectively, a right \(h\)-ideal and a left \(h\)-ideal of \(S\), then \(\overline{\pi} A \subseteq A \cap B\).

**Lemma 5.3.** \([32]\) A hemiring \(S\) is \(h\)-hemiregular if and only if for any right \(h\)-ideal \(A\) and left \(h\)-ideal \(B\), we have \(\overline{\pi} A = A \cap B\).

**Theorem 5.4.** For any hemiring \(S\), then the following are equivalent:

\[
\begin{align*}
& (i) \text{ for } a \in S, \text{ there exist } x_1, x_2 \in S\text{ such that } a + xa + z = a + xa + z \Rightarrow a + xa + z = a + xa + z; \\
&(ii) \text{ for } a \in S, \text{ there exist } x_1, x_2 \in S\text{ such that } a + xa + z = a + xa + z \Rightarrow a + xa + z = a + xa + z; \\
&(iii) \text{ for } a \in S, \text{ there exist } x_1, x_2 \in S\text{ such that } a + xa + z = a + xa + z \Rightarrow a + xa + z = a + xa + z; \\
&(iv) \text{ for } a \in S, \text{ there exist } x_1, x_2 \in S\text{ such that } a + xa + z = a + xa + z \Rightarrow a + xa + z = a + xa + z.
\end{align*}
\]
(1) $S$ is $h$-hemiregular;
(2) $f_S \circ g_S = f_S \circ g_S$ for any $SU$-$right$ $h$-ideal $f_S$ and any $SU$-$left$ $h$-ideal $g_S$ of $S$ over $U$.

**Proof.** (1) $\Rightarrow$ (2) Let $S$ be an $h$-hemiregular hemiring, $f_S$ and $g_S$ be an $SU$-$right$ $h$-ideal and an $SU$-$left$ $h$-ideal of $S$ over $U$, respectively. By Theorem 4.9, we have $f_S \circ g_S = f_S \circ g_S$.

Let $x \in S$, then there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since $S$ is $h$-hemiregular.

Thus, we have

$$f_S(a) \cup f_S'(a') \cup g_S(h) \cup g_S(h') \subseteq f_S(x) \cup g_S(x) = f_S \circ g_S(x),$$

which implies, $f_S \circ g_S \subseteq f_S \circ g_S$. Thus, $f_S \circ g_S = f_S \circ g_S$.

(2) $\Rightarrow$ (1) Let $R$ and $L$ be any right $h$-ideal and left $h$-ideal of $S$, respectively. Then by Corollary 4.5, $S_{LC}$ and $S_{RC}$ are an $SU$-$right$ $h$-ideal and an $SU$-$left$ $h$-ideal of $S$ over $U$, respectively. Moreover, by Lemma 5.2, we have $\mathbb{T}_U \subseteq R \cap L$.

Let $a \in R \cap L$, then $a \in R$ and $a \in L$, and so $S_{LC} = S_{RC} = \emptyset$. Thus, $(S_{LC} \cup S_{RC}) = (S_{LC} \cup S_{RC})(a) = S_{LC} = S_{RC} = \emptyset$, which implies, $S_{LC} \cup S_{RC} = \emptyset$.

Thus, $S_{LC} \cup S_{RC} = \emptyset$.

Let $a \in R \cap L \subseteq \mathbb{T}_U$. By Lemma 5.3, we know that $S$ is $h$-hemiregular.

**Corollary 5.5.** For any hemiring $S$, the following are equivalent:

(1) $S$ is $h$-hemiregular;
(2) $f_S \circ g_S = f_S \circ g_S$ for any $SU$-$right$ $h$-ideal $f_S$ and any $SU$-$left$ $h$-ideal $g_S$ of $S$ over $U$.

**Theorem 5.6.** A hemiring $S$ is $h$-hemiregular if and only if every $SU$-$h$-ideal of $S$ is idempotent.

**Proof.** Let $S$ be an $h$-hemiregular hemiring and $h_S$ an $SU$-$h$-ideal of $S$, we have $h_S \circ h_S \subseteq h_S \circ h_S$. Let $x \in S$, then there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since $S$ is $h$-hemiregular.

Thus, we have

$$(h_S \circ h_S)(x) = \bigcup_{i=1}^{n} (h_S(x_i) \cup h_S(x_i')) \cup h_S(y_i) \cup h_S(y_i'),$$

which implies, $h_S \circ h_S \subseteq h_S$. Then, $h_S \circ h_S = h_S$.

Conversely, assume that $h_S$ and $k_S$ are $SU$-$h$-ideals of $S$ over $U$. By Theorem 4.9, we have $h_S \circ k_S = h_S \circ k_S$. Moreover, by Proposition 4.7, $h_S \circ k_S$ is an $SU$-$h$-ideal of $S$ over $U$. By the assumption, we have $(h_S \circ k_S)(x) = (h_S \circ k_S)(x')$ for all $x \in S$.

Thus, $h_S \circ k_S = h_S$.

6. Conclusion

In this paper, we discuss soft union hemirings(soft union $h$-ideals) of hemirings and obtain some related...
properties. Finally, we give some characterizations of $h$-hemiregular hemirings by means of $SU$-$h$-ideals.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of hemirings. In the future study of soft hemirings, perhaps the following topics are worth to be considered:

1. To describe some new kinds of soft union $h$-ideals;
2. To establish some new kinds of soft $h$-hemiregular hemirings;
3. To apply soft hemirings to some applied fields, such as decision making, data analysis and forecasting and so on.

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