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Toward Understanding the Von Neumann Stability Condition

Yu Zhuang
Department of Mathematics and
Department of Computer Science
Louisiana State University
Baton Rouge, LA 70803

1 Introduction

Temporal discretization method

$$u^{n+1} = V(\Delta t; A)u^n \tag{1.1}$$

for well-posed (see [10, 16]) linear time dependent equation

$$\begin{cases} u'(t) = Au(t), & t \geq 0 \\ u(0) = f \in D(A) \subset E, \end{cases} \tag{1.2}$$

approximates the solution at time $(n+1)\Delta t$ by applying a closed linear operator $V(\Delta t; A)$ to the solution at time $n\,\Delta t$. It is well known that the convergence of the approximated solution by method (1.1) depends very much on its stability — the uniform norm-boundedness of the power of the operator V. As revealed by the Lax equivalence theorem [16], the convergence of the approximated solution for all initial values in the space E is equivalent to the stability of the method if the method is consistent (see definition in [16]).

When the initial value f is not in the domain of A, the meaning of the solution of problem (1.2) has to be interpreted in the temporally non-differential sense of mild solution (see [10, 26]). Thus, if we are interested only in solutions in the original sense implied by the equation (1.2), then we need convergence only for initial values in $D(A)$, which may result in a stability requirement possibly weaker than that in the Lax equivalence theorem. Butzer and Weis [4], and Butzer, Dickmeis and Nessel [5] have considered stability and convergence in this direction. They extended the Lax equivalence theorem to some stability-with-orders conditions which imply convergence for initial values in a subset of the space if the consistency of the approximation method is of a higher order than its stability for initial values in that subset.

Besides the aforementioned mathematical significance in approximation theory, weakening of stability requirements also has practical computation implications. A weaker stability requirement could mean more flexible methods which might be advantageous in aspects other than stability. For example, the alternating direction implicit (ADI) method

$$V_{ADI}(\Delta t; A) = (I - \tfrac{\Delta t}{2} A_1)^{-1}(I - \tfrac{\Delta t}{2} A_2)^{-1}(I + \tfrac{\Delta t}{2} A_2)(I + \tfrac{\Delta t}{2} A_1) \tag{1.3}$$

Peaceman and Rachford [27] proposed for two dimensional parabolic problems when A is split into directional components $A = A_1 + A_2$ is a factorization of the Crank-Nicolson method $V_{CN}(\Delta t; A) = (I - \frac{\Delta t}{2} A)^{-1}(I + \frac{\Delta t}{2} A)$. While retains the same accuracy order of the Crank-Nicolson method, the ADI method reduces the computation of the 2-D operator $(I - \frac{\Delta t}{2} A)^{-1}$ into that of two 1-D operators $(I - \frac{\Delta t}{2} A_1)^{-1}$ and $(I - \frac{\Delta t}{2} A_2)^{-1}$, resulting in great computation reduction especially when A is not self-adjoint. However when the two components A_1 and A_2 do not commute and are not dissipative, it is unknown[1] in general if the ADI method is stable. It is even unknown if it satisfies the stability-with-orders condition. What is verifiable (see Section 5) when each component A_i satisfies $\langle A_i f, f \rangle < \omega \|f\|^2$ for some positive constant ω and for all $f \in D(A_i)$ is that the ADI method satisfies the Von Neumann stability condition

$$\rho(V(\Delta t; A_h)) \leq e^{\omega' \Delta t}, \quad \text{for } \Delta t \in [0, \delta] \tag{1.4}$$

for some constant $\omega' > \omega$ and an ω-dependent constant $\delta > 0$ for spatially consistently discretized problems of (1.2) with the spatially discretized operator (in fact a matrix) A_h dependent on spatial mesh size h, where $\rho(V)$ denotes the spectral radius of the matrix V.

The Von Neumann condition is a necessary condition for stability. However as pointed out by Lax and Richtmyer in [16], it is usually surmised also as sufficient for stability for practical numerical computations. Intensive research has been carried out to find conditions under which the Von Neumann condition becomes sufficient for stability, e.g. by F. John [11], Lax and Richtmyer [16], Lax [15], Lax and Wendroff [17], and Strang [29].

[1]Douglas and Gunn [8] gave a stability analysis by re-norming the problem with a time-step-size dependent norm $\|\cdot\|_{\Delta t} := \|I + \Delta t A\|$ for the cases that A is symmetric and positive semi-definite. However, such a norm does not explain the convergence or the small numerical errors of the ADI method.

Recently, we have found a factorized approximation method, and a well-posed problem (see Section 2) for which the approximation method satisfies the Von Neumann stability condition, but is unstable either in the classical sense (see Lax and Richtmyer [16]) or in the stability-with-orders' sense, thereby showing that the Von Neumann condition is not sufficient for stability. However, the method still exhibits excellent numerical stability as indicated by measured errors, and the measured errors are smaller than those of approximation solution computed using the "most" stable implicit method even for large time step sizes .

To understand this phenomenon, we established a convergence theorem (Section 4) for a general class of approximation methods satisfying the Von Neumann condition, which unifies stable methods with many numerically appealing, (possibly) unstable methods including the unstable example method we give in Section 2. In our convergence theorem, we use a consistency condition similar to the one Chernoff used in his product formula [7], which is possibly weaker, and more importantly much easier to check for factorized approximation methods, than that in the Lax equivalence theorem. Thus, our result is a generalization of the Chernoff product formula. Chernoff proved his formula by constructing a stable sequence of C_0-semigroups from the approximation method and then bridging this semigroup sequence to the true solution with the Trotter-Kato [12, 24, 32] theorem. To establish our convergence result, however we do not have the luxury of C_0-semigroups. We turn to the Banach space valued Laplace transform [1, 2] for recourse. We bridge the approximation method to the true solution by a sequence of operator families (where the operators are possibly unbounded and/or the families are possibly unstable), and then employ the approximation theorem of the Laplace transform [2] to establish the convergence of the sequence to the true solution. Under a further assumption which is practically realistic, we obtain convergence of the approximated solution even with the existence of initial value errors. This error absorbing convergence result is applicable to our unstable example method (Section 4.1), and the ADI for quasi-dissipative problems with non-commutative operator splitting (Section4.2).

Finally, we list some notations used in this paper. The domain and range of an operator A is denoted by $D(A)$ and $R(A)$, and $D(A^\infty) = \cap_{n=1}^{\infty} D(A^n)$. The map $\lambda \mapsto R(\lambda,A)$ denotes the resolvent of a closed linear operator A. The letters $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ respectively denote the set of natural numbers, the set of real numbers and the set of complex numbers, and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. We denote by $\mathbf{L}(E)$ the set of bounded linear operators on a Banach space E.

2 An exmaple approximation method

In this section, we present an approximation method and a well-posed problem for which the approximation method is unstable but satisfies the Von Neumann stability condition. Before giving the example, we first would like to make precise what we mean by stability.

Definition 2.1 *Temporal approximation scheme* $V(t;A)$ *for problem (1.2) is called* stable *if there exist a constant* $\delta>0$ *and a continuous positive function* $M(t)$ *such that*

$$\|V^n(t;A)\| \le M(nt) \tag{2.1}$$

for all $t\in[0,\delta]$ *and* $n\in\mathbf{N}$.

Definition 2.2 *Approximation method* $V(t;A)$ *is said to be* stable of order $\beta \ge 0$ *if there exist a constant* $\delta > 0$ *and a continuous positive function* $M(t)$ *such that*

$$\|V^n(t;A)\| \le t^{-\beta} M(nt)$$

for all $t\in(0,\delta]$ *and* $n\in\mathbf{N}$.

The Von Neumann stability condition is originally stated for methods for spatially discretized problems. Though it is generalizable to temporal approximation methods $V(t;A)$ in a purely operator theoretic setting when $V^n(t;A)$ is a bounded operator for each $t\in[0,\delta]$ when n is sufficiently large, the spectral radius based Von Neumann stability condition is more appropriate when stated only for methods for spatially discretized problems, since we are also to remove the boundedness restriction on the operator $V^n(t;A)$ for all $n\in\mathbf{N}$.

Definition 2.3 *Temporal discretization method* $V(t;A_h)$ *for the spatially discretized problems*

$$\begin{cases} u_h'(t) = A_h u_h(t),\ \ t\ge 0 \\ u_h(0) = f_h, \end{cases} \qquad h\in(0,\varepsilon) \tag{2.2}$$

is said to satisfy the Von Neumann stability condition *if there exist two positive constants* δ,ω *such that the spectral radius of* $V(t;A_h)$ *is bounded by* $e^{\omega t}$ *for all* $t\in[0,\delta]$ *and* $h\in(0,\varepsilon)$, *namely,* $\rho(V(t;A_h)) \le e^{\omega t}$ *for all* $(t,h)\in[0,\delta]\times(0,\varepsilon)$.

Similarly, we give a (unconditional) stability definition for spatially discrete problems.

Definition 2.4 *Approximation scheme* $V(t;A_h)$ *for spatially discretized problems (2.2) is called* stable *if there exist a constant* $\delta>0$ *and a continuous positive function* $M(t)$ *such that* $\|V^n(t;A_h)\|_h \le M(nt)$ *for all* $(t,h)\in[0,\delta]\times(0,\varepsilon)$ *and* $n\in\mathbf{N}$.

2.1 The example problem

Denote $\Omega = [0, \pi]$. Let E denote the Hilbert space

$$E = \left\{ f \in L^2(\Omega) \ : \ f(0) = f(\pi) = 0 \right\}.$$

For $k \in \mathbf{N}$, let $e_k = \sqrt{\frac{2}{\pi}} \sin(kx)$. It is known that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of E. Let A_1 denote the differential operator $A_1 f = f''(x)$. It is known that A_1 generates a contraction semigroup and

$$A_1 e_k = -k^2 e_k \quad \text{for all } k \in \mathbf{N}. \tag{2.3}$$

Define an operator A_2 by

$$A_2 f = \sum_{k=1}^{\infty} \langle f, e_{k^2} \rangle \, e_k. \tag{2.4}$$

Obviously, A_2 is bounded and $\|A_2\| \leq 1$. Thus the operator $A := A_1 + A_2$ generates a C_0-semigroup since A_1 generates a contraction semigroup and A_2 is bounded (see Theorem 6.4, [10]).

For $n \in \mathbf{N}$, we partition the spatial domain Ω into n intervals of equal length, and the $n+1$ grid points are denoted by Ω_n, namely, $\Omega_n = \left\{ \frac{k\pi}{n} : k = 0, 1, \cdots, n \right\}$. And the corresponding function space on this discrete domain is

$$L^2(\Omega_n) = \left\{ f(x) \ : \ x \in \Omega_n, f(0) = f(\pi) = 0, \text{ with } \langle f, g \rangle_n = \tfrac{\pi}{n} \textstyle\sum_{k=1}^{n-1} f(\frac{k\pi}{n}) \, g(\frac{k\pi}{n}) \right\}.$$

Define a projection operator $P_n : L^2(\Omega) \to L^2(\Omega_n)$ by

$$(P_n f)(x) = f(x) \quad \text{for } f \in L^2(\Omega) \text{ and } x \in \Omega_n.$$

For notational simplicity, we will use f_n to denote $P_n f$. With the discrete domain and function space ready, we are to give the discretization of the two operators A_1 and A_2. For $k = 1, 2, \cdots, n-1$, let $e_{n,k} = P_n e_k$. It is easily verifiable that $\{e_{n,k}\}_{k=1}^{n-1}$ forms an orthonormal basis of $L^2(\Omega_n)$. On $L(\Omega_n)$, we use the commonly used central finite difference to discretize A_1, yielding

$$A_{n,1} = \left(\frac{n}{\pi}\right)^2 \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

A straightforward calculation shows that

$$A_{n,1}e_{n,k} = -a_{n,k}\; e_{n,k}, \tag{2.5}$$

where $a_{n,k} = \frac{2-2\cos\frac{k\pi}{n}}{(\pi/n)^2}$. For operator A_2, we use the following discretization

$$A_{n,2}\, f_n = \sum_{k=1}^{k^2<n} \langle f_n, e_{n,k^2}\rangle_n\, e_{n,k}. \tag{2.6}$$

Then it is easily verifiable that for $t\in[0,\infty)$,

$$\begin{cases} \|(I-\frac{t}{2}tA_{n,1})^{-1}\| \leq 1, \\ \|(I-\frac{t}{2}A_{n,1})^{-1}(I+\frac{t}{2}A_{n,1})\| \leq 1, \\ \|I+tA_{n,2}\| \leq 1+t. \end{cases} \tag{2.7}$$

2.2 The temporal discretization method

Define temporal approximation methods $V(t;A)$ and $V(t;A_n)$ for $n\in\mathbf{N}$ by

$$\begin{cases} V(t;A)f = (I-\frac{t}{2}A_1)^{-1}(I+tA_2)(I+\frac{t}{2}A_1)f, \\ V(t;A_n)f_n = (I-\frac{t}{2}A_{n,1})^{-1}(I+tA_{n,2})(I+\frac{t}{2}A_{n,1})f_n. \end{cases} \tag{2.8}$$

for $f\in D(A)$ and $t\in[0,1]$. We shall show below that

(a) $V(t;A)$ is unbounded for $t>0$;

(b) $V(t;A)$ is not stable of any order $\beta\geq 0$;

(c) $V(t;A_n)$ is unstable;

(d) $V(t;A_n)$ satisfies the Von Neumann stability condition.

Proof: (a). For $k\in\mathbf{N}$, $V(t;A)e_{k^2} = (I-\frac{t}{2}A_1)^{-1}(I+tA_2)(I+\frac{t}{2}A_1)e_{k^2}$ by definition. Then it follows from (2.3) that

$$\begin{aligned} V(t;A)e_{k^2} &= (I-\tfrac{t}{2}A_1)^{-1}(I+tA_2)(1-\tfrac{t}{2}k^4)e_{k^2} \\ &= (1-\tfrac{t}{2}k^4)(I-\tfrac{t}{2}A_1)^{-1}(I+tA_2)e_{k^2}. \end{aligned}$$

Then by the definition of A_2, we obtain from the above equality that

$$\begin{aligned} V(t;A)e_{k^2} &= (1-\tfrac{t}{2}k^4)(I-tA_1)^{-1}(e_{k^2}+te_k) \\ &= (1-\tfrac{t}{2}k^4)[\tfrac{1}{1+\frac{t}{2}k^4}e_{k^2}+\tfrac{t}{1+\frac{t}{2}k^2}e_k] \\ &= \tfrac{1-\frac{t}{2}k^4}{1+\frac{t}{2}k^4}e_{k^2}+\tfrac{t(1-\frac{t}{2}k^4)}{1+\frac{t}{2}k^2}e_k, \end{aligned} \tag{2.9}$$

where the second "=" sign above is due to (2.3). Therefore,

$$\|V(t;A)e_{k^2}\| \;\geq\; \|\tfrac{t(1-\frac{t}{2}k^4)}{1+\frac{t}{2}k^2}e_k\| \;=\; |\tfrac{t(1-\frac{t}{2}k^4)}{1+\frac{t}{2}k^2}|.$$

Then for any $t>0$, $\lim_{k\to\infty}\|V(t;A)e_{k^2}\|=\infty$. Therefore, $V(t;A)$ is unbounded for any $t\in(0,\infty)$.

(b). When k is not a square of an integer, $A_2\,e_k=0$. Then we obtain

$$V(t;A)e_k \;=\; \tfrac{1-\frac{t}{2}k^2}{1+\frac{t}{2}k^2}e_k. \tag{2.10}$$

by repeatedly applying (2.3) and (2.4) For each $t\in(0,\delta]$, let $c_{1,k}=\frac{1-\frac{t}{2}k^2}{1+\frac{t}{2}k^2}$, $c_{2,k}=\frac{1-\frac{t}{2}k^4}{1+\frac{t}{2}k^4}$, and $b_k=\frac{t(1-\frac{t}{2}k^4)}{1+\frac{t}{2}k^2}$. Then (2.10) and (2.9) become

$$\begin{cases} V(t;A)e_k &= c_{1,k}e_k, \\ V(t;A)e_{k^2} &= c_{2,k}e_{k^2}+b_ke_k. \end{cases} \tag{2.11}$$

A calculation using (2.11) shows that $V^n(t;A)e_{k^2}=c_{2,k}^ne_{k^2}+\frac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}b_ke_k$, from which we obtain $\|V^n(t;A)e_{k^2}\|\geq\left|\frac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}b_k\right|$. Thus to show $V(t;A)$ is not stable of any order $\beta\geq 0$, it suffices to show that

$$\lim_{k\to\infty} t^\beta\left|\frac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}b_k\right|=\infty. \tag{2.12}$$

We shall first show that

$$\lim_{k\to\infty}\left|\frac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}\right|=n+1. \tag{2.13}$$

Obviously, $|c_{1,k}|<|c_{2,k}|<1$ and $\lim_{k\to\infty}c_{1,k}=\lim_{k\to\infty}c_{2,k}=-1$. So, $|c_{2,k}|>0$ when k is large enough, and thus

$$\begin{cases} 0<\frac{c_{1,k}}{c_{2,k}}<1, & k \text{ large enough}, \\ \lim_{k\to\infty}\frac{c_{1,k}}{c_{2,k}} \;=\; 1. \end{cases} \tag{2.14}$$

Therefore,

$$\begin{aligned}\left|\tfrac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}\right| &= \tfrac{1-(c_{1,k}/c_{2,k})^{n+1}}{1-(c_{1,k}/c_{2,k})} \\ &= \tfrac{1-x_k^{n+1}}{1-x_k},\end{aligned}$$

where $x_k=c_{1,k}/c_{2,k}$. Then by L'Hopital's rule,

$$\begin{aligned}\lim_{k\to\infty}\left|\tfrac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}\right| &= \lim_{k\to\infty}\tfrac{d(1-x_k^{n+1})}{dx_k}\Big/\tfrac{d(1-x_k)}{dx_k} \\ &= n+1.\end{aligned}$$

which proves (2.13). Obviously, $\lim_{k\to\infty} b_k = -\infty$. Then (2.12) follows immediately from (2.13).

(c). By definition, stability of the spatially discrete method $V(t;A_n)$ implies the existence of constant $\delta > 0$ and positive continuous function $M(t)$ such that

$$\sup_{n\in\mathbf{N}} \left\| V\left(\tfrac{t}{n}\right)^n f_n \right\|_n \leq M(t) \sup_{n\in\mathbf{N}} \|f_n\|_n. \tag{2.15}$$

For $k\in\mathbf{N}$ and k is a prime number, let $n = k^2+1$. Then a calculation using (2.5) and (2.6) shows that

$$\begin{cases} V(t;A_n)e_{n,k^2} &= \frac{1-\frac{t}{2}a_{n,k^2}}{1+\frac{t}{2}a_{n,k^2}} e_{n,k^2} + \frac{t(1-\frac{t}{2}a_{n,k^2})}{1+\frac{t}{2}a_{n,k}} e_{n,k}, \\ V(t;A_n)e_{n,k} &= \frac{1-\frac{t}{2}a_{n,k}}{1+\frac{t}{2}a_{n,k}} e_{n,k}, \end{cases} \tag{2.16}$$

which has the exact the same form as (2.11). And the coefficients before e_{n,k^2} and $e_{n,k}$ have the same properties as those of $c_{1,k}$, $c_{2,k}$ and b_k in (2.11). And these properties of $c_{1,k}$, $c_{2,k}$ and b_k are used in (b) to show the unboundedness of $\{\|V(t;A)^m e_{k^2}\|\}_{k=1}^{\infty}$ for any fixed $m\in\mathbf{N}$ and $t > 0$. Then by exactly the same argument, we can show that the sequence $\{\|V(t;A_n)^m e_{n,k^2}\|_n\}_{k=1}^{\infty}$ is unbounded for any fixed $m\in\mathbf{N}$ and $t > 0$ for $n = k^2+1$, which means that (2.15) does not hold.

(d). Denote $W_n(t) = (I+tA_{n,2})(I+\frac{t}{2}A_{n,1})(I-\frac{t}{2}A_{n,1})^{-1}$. Then $\|W_n(t)\| \leq 1+t$ by (2.7). Then for $m\in\mathbf{N}_0$ and $(t,n)\in[0,\delta]\times\mathbf{N}$,

$$\begin{aligned} \|V^{m+1}(t;A_n)f_n\|_n &= \|(I-\tfrac{t}{2}A_{n,1})^{-1}W_n^m(t)(I+tA_{n,2})(I+\tfrac{t}{2}A_{n,1})f_n\|_n \\ &\leq \|W_n(t)\|^m \|(I-\tfrac{t}{2}A_{n,1})^{-1}\| \cdot \|(I+tA_{n,2})\| \cdot \|(I+\tfrac{t}{2}A_{n,1})f_n\|_n \\ &\leq (1+t)^{m+1}\|(I+\tfrac{t}{2}A_1)f_n\|_n, \end{aligned}$$

where the last inequality is due to (2.7). Again since the discrete matrix $A_{n,1}$ is a bounded linear operator, $\|V^m(t;A_n)\| \leq (1+t)^m\|I+\frac{t}{2}A_{n,1}\|_n$. Therefore,

$$\rho(V(t;A_n)) = \lim_{m\to\infty} \sqrt[m]{(1+t)^m\|I+tA_{n,1}\|_n} = 1+t.$$

Therefore $V(t;A_n)$ satisfies the Von Neumann stability condition.

□

2.3 Numerical testing of the example approximation method

To examine the numerical stability of the unstable spatially discretized example method given in (2.8), we choose an initial value condition for the the problem with a known

Table 1: Stability testing — Small spatial partition sizes

Δx	Δt	Initial Err	Explicit	Implicit	F A C	C-N
		$e_1(x)$	$2.47e-01$	$4.06e-02$	$3.04e-03$	$3.04e-03$
$\pi/16$	$1/5$	$e_2(x)$	$7.13e+01$	$4.06e-02$	$3.03e-03$	$3.03e-03$
		$e_3(x)$	$1.31e+00$	$4.06e-02$	$3.04e-03$	$3.03e-03$
		$e_1(x)$	$6.70e-03$	$9.32e-03$	$7.24e-04$	$7.24e-04$
$\pi/16$	$1/20$	$e_2(x)$	$5.48e+07$	$9.32e-03$	$7.24e-04$	$7.24e-04$
		$e_3(x)$	$1.77e+01$	$9.32e-03$	$7.29e-04$	$7.29e-04$
		$e_1(x)$	$2.01e-03$	$4.18e-03$	$9.45e-04$	$9.45e-04$
$\pi/16$	$1/50$	$e_2(x)$	$2.27e-03$	$4.18e-03$	$9.45e-04$	$9.45e-04$
		$e_3(x)$	$2.00e-03$	$4.18e-03$	$9.49e-04$	$9.49e-04$
Δx	Δt	Initial Err	Explicit	Implicit	F A C	C-N
		$e_1(x)$	$2.46e+01$	$1.77e-02$	$7.81e-04$	$7.81e-04$
$\pi/32$	$1/10$	$e_2(x)$	$2.87e+11$	$1.77e-02$	$7.88e-04$	$7.88e-04$
		$e_3(x)$	$1.07e+10$	$1.77e-02$	$7.79e-04$	$7.78e-04$
		$e_1(x)$	$3.76e+34$	$1.78e-03$	$2.32e-04$	$2.32e-04$
$\pi/32$	$1/100$	$e_2(x)$	$1.26e+45$	$1.78e-03$	$2.32e-04$	$2.32e-04$
		$e_3(x)$	$1.96e+37$	$1.78e-03$	$2.34e-04$	$2.34e-04$
		$e_1(x)$	$5.07e-04$	$1.00e-03$	$2.40e-04$	$2.40e-04$
$\pi/32$	$1/200$	$e_2(x)$	$1.94e+01$	$1.00e-03$	$2.40e-04$	$2.40e-04$
		$e_3(x)$	$5.05e-04$	$1.01e-03$	$2.42e-04$	$2.42e-04$

solution. The simulation time interval, the initial value condition, and the true solution of the example problem for the initial value condition are:

$$\begin{cases} \text{Time interval:} & [0,1]; \\ \text{Initial value:} & u(0,x) = \sin(2x)+\sin(3x); \\ \text{True solution:} & u(t,x) = e^{-4t}\sin(2x)+e^{-9t}\sin(3x). \end{cases}$$

We also choose three initial value errors for the testing problem, and they are

$$\begin{cases} e_1(x) &=& 0, \\ e_2(x) &=& \frac{x(\pi-x)}{10^5}\cos(nx), \\ e_3(x) &=& \sum_{k^2=1}^{k^2<n} \frac{\sin(k^2x)}{10^4 n}, \end{cases}$$

where n is the spatial partition size, i.e. $n = \pi/\Delta x$. The second and third initial value errors contain very high frequency components, and high frequency errors usually tend

Table 2: Stability testing — Large spatial partition sizes

Δx	Δt	Initial Err	Explicit	Implicit	F A C	C-N
$\pi/4096$	1/40	$e_1(x)$	∞	3.91e-03	6.45e-05	6.45e-05
		$e_2(x)$	∞	3.91e-03	8.28e-05	8.28e-05
		$e_3(x)$	∞	3.91e-03	4.33e-04	6.47e-05
$\pi/4096$	1/400	$e_1(x)$	∞	3.77e-04	6.31e-07	6.31e-07
		$e_2(x)$	∞	3.77e-04	2.25e-05	2.25e-05
		$e_3(x)$	∞	3.77e-04	4.04e-06	8.69e-07
$\pi/4096$	1/4000	$e_1(x)$	∞	3.76e-05	8.27e-09	8.27e-09
		$e_2(x)$	∞	3.76e-05	9.81e-09	9.81e-09
		$e_3(x)$	∞	3.76e-05	2.85e-08	2.85e-08
Δx	Δt	Initial Err	Explicit	Implicit	F A C	C-N
$\pi/8192$	1/80	$e_1(x)$	∞	1.92e-03	1.61e-05	1.61e-05
		$e_2(x)$	∞	1.92e-03	3.56e-05	3.56e-05
		$e_3(x)$	∞	1.92e-03	3.46e-04	1.63e-05
$\pi/8192$	1/800	$e_1(x)$	∞	1.88e-04	1.58e-07	1.58e-07
		$e_2(x)$	∞	1.88e-04	2.25e-05	2.25e-05
		$e_3(x)$	∞	1.88e-04	2.96e-06	3.66e-07
$\pi/8192$	1/8000	$e_1(x)$	∞	1.88e-05	2.10e-09	2.10e-09
		$e_2(x)$	∞	1.88e-05	3.69e-09	3.69e-09
		$e_3(x)$	∞	1.88e-05	1.29e-08	1.29e-08

to be enlarged considerably for unstable methods like the explicit method. And we choose these two errors in order to see how they will affect the simulation errors for our unstable example method.

We solved the example problem with the three perturbed initial value conditions by the explicit method, the implicit method, the unstable example method (listed as FAC in the tables) and the Crank-Nicolson method (listed as C-N in the tables). We have tested these methods with different spatial mesh sizes Δx and time step size Δt, and the computed approximated solutions are compared with the true solution and the maximal errors are listed in the two tables.

For the four methods chosen, the explicit method is unstable and only conditionally stable [6, 17], the FAC is proven unstable but satisfies the Von Neumann stability condition. The implicit method is stable [2] for all well-posed problems, and the Crank-

Nicolson is stable for quasi-dissipative problems (see Section 5) including our example problem.

From the testing results, the explicit methods is obviously unstable. It has huge numerical errors when time step size Δt are large reletive to the spatial mesh size Δx, the unstable method FAC exhibits much smaller numerical errors than does the explicit method, even smaller than the errors produced by the always-stable implicit method. Only when compared with the stable Crank-Nicolson method, the FAC method shows larger numerical errors for the cases of simultaneously large Δt and small Δx when the intial value has high frequency errors. The Crank-Nicolson is a second order method while the FAC method is only first order. The first order accuracy of the FAC method is due to the first order approximation of the semigroup e^{tA_2} by the explicit method $(I+tA_2)$ — the second factor in the FAC method (2.8). And even in the worst tested case for the FAC method when compared with the Crank-Nicolson, the errors are still close to those of the C-N method, so it is unclear if these larger errors of the FAC method is caused by the unstability or the first order accuracy.

Experimental tests were conducted on a SUN Ultra 10 workstation running SunOS 5.7. 64-bit arithmetic operation was chosen.

3 Preliminaries

In this section, we accumulate some preliminary results which are needed for establishing the major convergence theorem in Section 4. One important tool we use is the Laplace-Stieljes transform and its approximation theory. Here are some definitions and results for the Laplace-Stieljes transform.

Definition 3.1 *Let Lip_ω denote the Banach space of functions:*

$$Lip_\omega := \{v : [0,\infty) \to E \ : \ v(0) = 0, \text{ and } \|v\|_{Lip_\omega} < \infty\},$$

where $\|v\|_{Lip_\omega} = \sup\{\frac{\|v(t)-v(t')\|}{|\int_{t'}^{t} e^{\omega\tau}d\tau|} \ : \ t, t' \in [0,\infty)\}$. For $v \in Lip_\omega$, the **Laplace-Stieljes transform** *of v is defined as $\mathcal{L}_s(\lambda)v := \int_0^\infty e^{-\lambda t}dv(t)$ for $\lambda > \omega$.*

Remark 3.1 *It is not difficult to verify [1] that for any locally integrable function $u : [0,\infty) \to E$ satisfying $\|u(t)\| \leq Me^{\omega t}$ for some constants $M, \omega > 0$, $v(t) = \int_0^t u(s)ds \in Lip_\omega$ and $\mathcal{L}_s v = \int_0^\infty e^{-\lambda t}u(t)dt$.*

The following approximation theorem for Laplace-Stieljes transform is taken from Bäumer and Neubrander's paper [2].

Theorem 3.1 *Let $M > 0$, $f_n \in Lip_\omega$ with $\|f_n\|_{Lip_\omega} \le M$ and $r_n = \mathcal{L}_s f_n$ for all $n \in \mathbf{N}_0$. The following are equivalent.*

(i) $\lim_{n\to\infty} r_n(\lambda) = r_0(\lambda)$ *on* (ω, ∞).

(i i) $\lim_{n\to\infty} f_n(t) = f_0(t)$ *uniformly on compact subsects of* $[0, \infty)$.

We now generalize an exponential formula for semigroups generated by bounded linear operators [22, 23, 33] to one-parameter families of closed, linear (possibly unbounded) operators.

Lemma 3.1 *Let A be a closed linear operator with $D(A^\infty) \ne \emptyset$. Suppose that there exists a constant $\omega \ge 1$ such that for each $f \in D(A^\infty)$,*

$$\|A^n f\| \le M_f \omega^n \quad \textit{for all } n \in \mathbf{N}, \tag{3.1}$$

for some f-dependent constant $M_f > 0$. Then for $f \in D(A^\infty)$, the map $t \mapsto e^{tA} f : [0, \infty) \to E$ given by $e^{tA} f = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} f$

(i) *is well-defined, and differentiable with $\frac{d}{dt} e^{tA} f = e^{tA} A f$;*

(i i) *has Laplace-Stieljes transform and $\mathcal{L}_s(\lambda) e^{tA} f = \int_0^\infty e^{-\lambda t} e^{tA} A f \; dt$; and*

(iii) *satisfies that $e^{tA} f = \lim\limits_{n\to\infty} (I + \frac{t}{n} A)^n f$ for uniformly for $t \in [0, T]$ for any $T > 0$.*

Proof: (i). It is obvious that the series $\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} f$ is uniformly convergent and thus $e^{tA} f$ is well-defined.

Since each item in the series $\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} f$ is differentiable in t and $\frac{d}{dt} \frac{t^n A^n}{n!} f = \frac{t^{n-1} A^n}{n!} f$ for $n \in \mathbf{N}$. Hence, $\sum_{n=0}^{\infty} \frac{d}{dt} \frac{t^n A^n}{n!} f = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} A f$, which converges uniformly in compact intervals of t. It then follows from Theorem 11.4.6 in [25] that

$$\frac{d}{dt} e^{tA} f = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} A f, \tag{3.2}$$

which is exactly $\frac{d}{dt} e^{tA} f = e^{tA} A f$ by definition.

(ii). For $f \in D(A^\infty)$, it is easily verifiable that condition (3.1) implies that $\|e^{tA} f\| \le M_f e^{\omega t}$ for $t \ge 0$, which immediately leads to

$$\|e^{tA} A f\| \le M_{Af} e^{\omega t}, \quad t \ge 0.$$

Then by statement (i) of this lemma and Remark 3.1, the map $t \mapsto e^{tA} f$ has Laplace-Stieljes transform for $\lambda > \omega$ and the Laplace-Stieljes transform satisfies

$$\mathcal{L}_s(\lambda) e^{tA} f = \int_0^\infty e^{-\lambda t} e^{tA} A f \; dt.$$

□

To prove statement (iii) of the lemma above, we need the following lemma which extends an estimate due to Chernoff [7].

Lemma 3.2 *Let L be an operator defined on $D(L) \subset E$. Suppose that there exists a constant $l \geq 1$ such that for each $f \in D(L^\infty)$,*

$$\|L^{n+1} f - L^n f\| \leq M_f\, l^n$$

for all $n \geq 0$ and an f-dependent constant M_f. Then $e^{t(L-I)}$ is well-defined on $D(L)$ for all $t \geq 0$, and $\|e^{n(L-I)} f - L^n f\| \leq M_f \sqrt{n}\, l^n e^{(l^2-1)n/2}$, for each $f \in D(L^\infty)$ and $n \in \mathbf{N}$.

Proof: For $f \in D(L^\infty)$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|(L-I)^n f\| &= \|(L-I)^{n-1}(L-I)f\| \\ &= \| \textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{n-i} L^i (L-I) f\| \\ &\leq \textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{n-i} \|L^i (L-I) f\| \\ &\leq \textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} M_f\, l^i \\ &= M_f (1+l)^{n-1}. \end{aligned}$$

Then by Lemma 3.1, $e^{t(L-I)} f$ is well-defined for all $t > 0$ and $f \in D(A)$.

It is well known that the series $\sum_{k=0}^\infty \frac{t^k}{k!}$ is absolutely convergent to e^t, and therefore the series $\sum_{k=0}^\infty \frac{t^k}{k!} L^k f$ converges to $e^{tL} f$ by the convergence theorem for the Cauchy product $e^t e^{t(L-I)} f$. Thus, we have that $e^{n(L-I)} f = e^{-n} \sum_{k=0}^\infty \frac{n^k}{k!} L^k f$ for all $f \in D(L)$, which implies that

$$\left\| e^{n(L-I)} f - L^n f \right\| = e^{-n} \left\| \sum_{k=0}^\infty \frac{n^k}{k!} (L^k - L^n) f \right\|. \tag{3.3}$$

Now for $k \geq n$ and $f \in D(L)$,

$$\begin{aligned} \|L^k f - L^n f\| &\leq \textstyle\sum_{i=n+1}^k \|L^i f - L^{i-1} f\| \\ &\leq \textstyle\sum_{i=n+1}^k M_f l^i \\ &\leq M_f l^{k+n} |k-n|. \end{aligned}$$

Similarly we can prove that $\|L^k f - L^n f\| \leq M_f l^{k+n} |k-n|$ for $k < n$. It then follows from (3.3) that

$$\begin{aligned} \|e^{n(L-I)} f - L^n f\| &\leq M_f e^{-n} l^n \textstyle\sum_{k=0}^\infty \frac{n^k}{k!} |n-k| l^k \\ &\leq \frac{M_f l^n}{e^n} \left\{ \textstyle\sum_{k=0}^\infty \frac{(nl^2)^k}{k!} \right\}^{1/2} \left\{ \textstyle\sum_{k=0}^\infty \frac{n^k}{k!} (n-k)^2 \right\}^{1/2} \\ &\leq M_f \sqrt{n}\, l^n e^{(l^2-1)n/2}. \end{aligned}$$

□

Proof of Lemma 3.1: (iii). For $t=0$, the limit in statement (ii) holds trivially.

For $s>0$, let $L_s = I+sA$. Then for $f\in D(A^\infty)$,

$$\begin{aligned}
\|L_s^{n+1}f - L_s^n f\| &= \|L_s^n(L_s f - f)\| \\
&= s\|(I+sA)^n Af\| \\
&= s\sum_{k=0}^n \binom{n}{k} s^k A^k Af\| \\
&\leq s\sum_{k=0}^n \binom{n}{k} s^k M_f\omega^{k+1} \\
&= s\omega M_f(1+s\omega)^n \ \leq \ G_f e^{n\omega s},
\end{aligned}$$

where $G_f = s\omega M_f$. It then follows from Lemma 3.2 that for $f\in D(L^\infty)$,

$$\|e^{n(L_s-I)}f - L_s^n f\| \ \leq \ G_f\sqrt{n}e^{n\omega s}e^{(e^{2\omega s}-1)n/2}.$$

Replace L_s by $I+sA$ and G_f by $s\omega M_f$, we obtain

$$\|e^{n\,s\,A}f - (I+sA)^n f\| \ \leq \ s\omega M_f\sqrt{n}e^{n\omega s}e^{(e^{2\omega s}-1)n/2}.$$

Take $s=\frac{t}{n}$ in the above inequality, yielding

$$\|e^{tA}f - (I+\frac{t}{n}A)^n f\| \leq \frac{t}{\sqrt{n}}M_f e^{\omega t}e^{(e^{2\omega t/n}-1)n/2}.$$

For $t\in[0,T]$, there exists $N_T>0$ such that $e^{2\omega t/n}\leq 1+4\omega t/n$ for all $n>N_T$. Then

$$\begin{aligned}
\|e^{tA}f - (I+\tfrac{t}{n}A)^n f\| &\leq \tfrac{t}{\sqrt{n}}M_f e^{\omega t}e^{(1+4\omega t/n-1)n/2} \\
&\leq \tfrac{t}{\sqrt{n}}M_f e^{3\omega t},
\end{aligned}$$

which implies that $\lim_{n\to\infty}(I+\frac{t}{n}A)^n f = e^{tA}f$.

□

The following theorem, in some sense, generalizes an approximation theorem of Troter-Kato type for stabilized semigroup sequences established in [34] to sequence of one-parameter family of closed, linear but possibly unbounded operators.

Theorem 3.2 *Let* $\{A_n\}_{n=0}^\infty$ *be a sequence of closed linear operators with* $D := \bigcap_{n=0}^\infty D(A_n^\infty)\neq\emptyset$. *For each* $n\in\mathbf{N}_0$, *let* $\{S_n(t) : t\in[0,\infty)\}_{n=0}^\infty$ *be a one parameter family of closed linear operators on* D. *Suppose that for all* $f\in D$,

(a) $S_n(t)f$ *is differentable in* t *and* $\frac{d\,S_n(t)f}{dt} = S_n(t)A_n f$ *for* $n\in\mathbf{N}_0$; *and*

(b) *there exists a constant* $\omega > 0$ *such that for each* $f \in D$, $\|S_n(t)A_nf\| \le M_f e^{\omega t}$ *for* $t \ge 0$ *and* $n \in \mathbf{N}_0$ *for some positive constant* M_f *dependent only on* f.

Then the following statements are equivalent.

(i) $\lim_{n\to\infty} R_n(\lambda)f = R_0(\lambda)f$ *for* $\lambda > \omega$ *and* $f \in D$, *where* $R_n(\lambda)f = \int_0^\infty e^{-\lambda t} S_n(t)f\,dt$ *for* $\lambda > \omega$ *and* $f \in D$.

(i i) *For* $f \in D$, $\lim_{n\to\infty} S_n(t)f = S_0(t)f$ *uniformly for* t *in compact intervals of* $[0,\infty)$.

Proof: By conditions (a) and (b), and the definition of $R_n(\lambda)$, a calculation using integration by parts shows that $R_n(\lambda)(\lambda - A_n)f = f$ for $f \in D$ and $n \in \mathbf{N}_0$. Then statement (i) is equivalent to

$$\text{(i}'\text{)} \qquad \lim_{n\to\infty} R_n(\lambda)A_nf = R_0(\lambda)A_0f \quad \text{for } \lambda > \omega,\ f \in D.$$

Condition (a), (b) and Remark 3.1 imply that for each $f \in D$, $S_n(t)f$ has Laplace-Stieljes transform and its Laplace-Stieljes transform is $R_n(\lambda)A_nf$ for all $\lambda > \omega$ and $n \in \mathbf{N}_0$.

Using conditions (a) and (b), a straightforward calculation show that the Lip_ω norm of $S_n(\cdot)f$ as defined in Definition 3.1 satisfies

$$\|S_n(\cdot)f\|_{Lip_\omega} \le M_f$$

for all $f \in D$ and $n \in \mathbf{N}_0$. Then the equivalence of statements (i′) and (ii) follows immediately from Theorem 3.1, which proves the theorem.

□

4 Approximation methods

In this section, we extend the Chernoff product approximation formula (see A. Pazy [26], page 90) to possibly unstable approximation methods.

Theorem 4.1 *Suppose that* A *generates a* C_0*-semigroup* $S(t)$. *Let* $\{V(t) : t \in [0,\delta]\}$ *be a family of closed linear operators satisfying* $D(A) \subset D(V(t)^\infty)$ *for all* $t \in [0,\delta]$. *Suppose that the map* $t \mapsto V(t)f$ *is continuous for all* $f \in D(A)$. *If there exists a constant* $\omega > 0$ *such that*

(a) *for all* $f \in D(A)$, $t \in [0,\delta]$ *and* $n \in \mathbf{N}$, $\|V^{n+1}(t)f - V^n(t)f\| \le tM_f e^{n\omega t}$ *for some* f*-dependent positive number* M_f, *and*

(b) *there exists a family* $\{W(t) : t \in [0,\delta]$ *of operators on defined* $D(A^2)$ *such that for each* $f \in D(A^2)$ *the map* $t \mapsto W(t)f$ *is continuous,* $W(0)f = 0$*, and* $\|V^{n+1}(t)f - V^n(t)f - t\,V^n(t)Af\| \le t\,e^{n\omega t}\|W(t)f\|$ *for all* $t \in [0,\delta]$ *and* $n \in \mathbf{N}$*,*

then $\lim_{n\to\infty} V(\frac{t}{n})^n f = S(t)f$ *for each* $f \in D(A)$*, uniformly for* t *in compact interval* $[0,T]$ *for any* $T > 0$*.*

Remark 4.1 *If* V *is stable in the sense that*

$$\|V(t)\| \le Me^{n\omega t}, \quad \text{for all } (t,n) \in [0,\delta] \times \mathbf{N}$$

for some positive constant M, ω *independent of* n *and* t*, and if* V *also satisfies the consistency condition*

$$\lim_{\Delta t \to 0} \frac{V(\Delta t)f - f}{\Delta t} = Af, \quad \text{for all } f \in D(A),$$

then conditions (a) *and* (b) *hold. (And hence, Theorem 4.1 is a generalization of the Chernoff product approximation formula.)*

Proof: The stability of V implies that there exist constants $M, \omega > 0$ such that

$$\|V^n(t)\| \le Me^{n\omega t} \tag{4.1}$$

for $n \in \mathbf{N}$ and $t \in [0,\delta]$. Then for $f \in D(A)$ and $t \in [0,\delta)$,

$$\begin{aligned} \|V^{n+1}(t)f - V^n(t)f\| &\le \|V^n(t)\| \cdot \|V(t)f - f\| \\ &\le Me^{n\omega t}\|t \cdot \tfrac{V(t)f-f}{t}\|. \end{aligned} \tag{4.2}$$

Since V is consistent for all $f \in D(A)$, it follows that the limit $lim_{t\to 0} \frac{V(t)f-f}{t}$ exists and hence $\|\frac{V(t)f-f}{t}\|$ is bounded on $(0,\delta)$. Without loss of generality, we can assume that $\sup_{t\in(0,\delta)} \|\frac{V(t)f-f}{t}\| \le \frac{M_f}{M}$. Then it follows from (4.2) that $\|V^{n+1}(t)f - V^n(t)f\| \le tM_f e^{n\omega t}$, which is condition (a) of Theorem 4.1.

Condition (b) of Theorem 4.1 can be proven similarly.

□

Lemma 4.1 *Let* $V(t)$ *be as in Theorem 4.1. For any* $T > 0$*, and any* $s \in (0,T]$*, let* $A_s f = \frac{V(s)-I}{s} f$ *for* $f \in D(A)$*. Then condition* (a) *of Theorem 4.1 implies that for any* $T > 0$ *and* $f \in D(A)$*,*

$$\lim_{n\to\infty} \|V^n(t/n)f - e^{tA_{t/n}}f\| = 0 \tag{4.3}$$

uniformly for $t \in [0,T]$*.*

Proof: For $t = 0$, (4.3) is obviously true.

For $t \in (0, T]$, condition (a) of Theorem 4.1 implies that

$$\begin{aligned} \|V^n(s)f - V^{n-1}(s)f\| &\leq sM_f e^{ns\omega} \\ &= G_{s,f}\, l^n, \end{aligned}$$

where $G_{s,f} = sM_f$ and $l = e^{s\omega} \geq 1$. Then, by Lemma 3.2,

$$\begin{aligned} \|V^n(s)f - e^{n[V(s)-I]}f\| &\leq G_{s,f}\sqrt{n} l^n e^{(l^2-1)n/2} \\ &= sM_f\sqrt{n} e^{ns\omega} e^{(e^{2s\omega}-1)n/2}. \end{aligned} \tag{4.4}$$

By definition of A_s, we obtain that $e^{n(V(s)-I)}f = e^{nsA_s}f$. Thus taking $s = t/n$, we have that $e^{n(V(t/n)-I)}f = e^{tA_{t/n}}$. Then it follows from (4.4) that

$$\|V^n\left(\tfrac{t}{n}\right) f - e^{tA_{t/n}}f\| \leq M_f \tfrac{t}{\sqrt{n}} e^{\omega t} e^{(e^{2\omega t/n}-1)n/2}.$$

For any $\omega_o > \omega$, there exists n_o (dependent on T) such that $e^{2\omega t/n} - 1 < 2\omega_o t/n$ for all $n > n_o$. So for all $n > n_o$,

$$\begin{aligned} \|V^n(\tfrac{t}{n})f - e^{tA_{t/n}}f\| &\leq M_f \tfrac{t}{\sqrt{n}}\, e^{\omega t} e^{\omega_o t} \\ &= M_f \tfrac{t}{\sqrt{n}} e^{\omega t} e^{\omega_o t}, \end{aligned}$$

from which, the conclusion of the lemma follows.

□

Lemma 4.2 *Let A_s be as defined in Lemma 4.1. Then condition* (a) *of Theorem 4.1 implies that for any $\omega_o > \omega$, there exist $s_o \in (0, T]$ such that for $s \in (0, s_o)$,*

(i) *$e^{tA_s}A_s f$ is well defined and $\|e^{tA_s}A_s f\| < M_f e^{\omega_o t}$ for $f \in D(A)$ and $t \geq 0$;*

(i i) *$e^{tA_s}f$ is well defined, and $\|e^{tA_s}f\| < (\|f\| + \frac{M_f}{\omega_o})e^{\omega_o t}$ for all $f \in D(A)$ and $t \geq 0$; and*

(iii) *for $\lambda > \omega_o$ and $f \in D(A)$, $R_s(\lambda)(\lambda - A_s)f = f$, where $R_s(\lambda)g := \int_0^\infty e^{-\lambda t}e^{tA_s}g\, dt$ for $g \in D(A) \cup A_s(D(A))$.*

Proof: (i) For $f \in D(A)$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|A_s^n A_s f\| &= \| \left(\tfrac{V(s)-I}{s}\right)^n \left(\tfrac{V(s)-I}{s}\right) f\| \\ &= s^{-n}\| \textstyle\sum_{i=0}^n \binom{n}{i} (-1)^{n-i} V^i(s) \frac{V(s)-I}{s} f\| \\ &\leq s^{-n} \textstyle\sum_{i=0}^n \binom{n}{i} M_f e^{i\omega s} \\ &= M_f \left(\tfrac{1+e^{\omega s}}{s}\right)^n, \end{aligned}$$

where the last inequality is due to condition (a) of Theorem 4.1. Then by Lemma 3.1, $e^{tA_s}A_sf$ is well-defined for $f \in D(A)$ and $t \geq 0$. Then statement (ii) of Lemma 3.1 implies that

$$\begin{aligned} \|e^{tA_s}A_sf\| &= \|\lim_{m\to\infty}\left(I+\tfrac{t}{m}A_s\right)^m A_sf\| \\ &= \lim_{m\to\infty}\|[\lambda_m+\mu_m V(s)]^m A_sf\|, \end{aligned} \tag{4.5}$$

where $\lambda_m = \frac{ms-t}{ms}$ and $\mu_m = 1-\lambda_m = \frac{t}{ms}$. Since

$$\begin{aligned} \|\left(\lambda_m+\mu_m V(s)\right)^m A_sf\| &= \|\textstyle\sum_{i=0}^m \binom{m}{i}\lambda_m^{m-i}\mu_m^i V^i(s)\frac{V(s)-I}{s}f\| \\ &\leq \textstyle\sum_{i=0}^m \binom{m}{i}\lambda_m^{m-i}\mu_m^i M_f e^{i\omega s} \\ &= M_f\left(\lambda_m+\mu_m e^{\omega s}\right)^m, \end{aligned}$$

it follows from (4.5) that $\|e^{tA_s}A_sf\| \leq M_f\lim_{m\to\infty}[1+\mu_m(e^{\omega s}-1)]^m$ by noticing that $\lambda_m = 1-\mu_m$. For $\omega_o > \omega$, there exists $s_o > 0$ such that $e^{\omega s}-1 \leq \omega_o s$ for all $s \in (0, s_o]$. Then,

$$\begin{aligned} \|e^{tA_s}A_sf\| &\leq M_f\lim_{m\to\infty}\left(1+\mu_m\omega_o s\right)^m \\ &= M_f\lim_{m\to\infty}\left(1+\tfrac{\omega_o t}{m}\right)^m \\ &\leq M_f e^{\omega_o t}. \end{aligned}$$

(ii) By an argument similar to that in the proof of statement (i), we can show that $e^{tA_s}f$ is well defined for $f \in D(A)$ and $t \geq 0$. Now by Lemma 3.1, $e^{tA_s}f = f + \int_0^t e^{\tau A_s}A_sf\,d\tau$. Then by (i),

$$\begin{aligned} \|e^{tA_s}f\| &\leq \|f\| + \textstyle\int_0^t M_f e^{\omega_o\tau}d\tau \\ &\leq (\|f\| + \tfrac{M_f}{\omega_o})e^{\omega_o\tau}. \end{aligned}$$

(iii). By (ii), the Laplace transform $R_s(\lambda)f = \int_0^\infty e^{-\lambda t}e^{tA_s}f\,dt$ exists for $\lambda > \omega_o$ and $f \in D(A)$. With the estimate on $e^{tA_s}A_sf$ from (i), we integrate by parts and obtain

$$R_s(\lambda)f = \lambda^{-1}\left(f + \int_0^\infty e^{-\lambda t}e^{tA_s}A_sf\,dt\right),$$

from which it follows that

$$R_s(\lambda)(\lambda - A_s)f = f.$$

□

Lemma 4.3 *Let A_s be as defined in Lemma 4.1. Then condition* (b) *of Theorem 4.1 implies that for any $\omega_o > \omega$, there exist $s_o \in (0, T]$ such that for $s \in (0, s_o)$ and $f \in D(A^2)$, $e^{tA_s}(A_s - A)f$ is well-defined and $\|e^{tA_s}(A_s - A)f\| \leq e^{\omega_o t}\|W(s)f\|$ for $t \geq 0$, and $\lim_{s\to 0} R_s(\lambda)(A_s - A)f = 0$.*

Proof: By an argument similar to that in the proof of statement (i) of Lemma 4.2, we can show that condition (b) of Theorem 4.1 implies that for any $\omega_o > \omega$, there exist $s_o \in (0, T]$ such that for all $s \in (0, s_o)$ and $f \in D(A^2)$, $e^{tA_s}(A_s - A)f$ is well-defined, and

$$\|e^{tA_s}(A_s - A)f\| \leq e^{\omega_o t}\|W(s)f\|.$$

Then $R_s(\lambda)(A_s - A)f$ exists for $f \in D(A^2)$ and

$$\begin{aligned} \|R_s(\lambda)(A_s - A)f\| &\leq \textstyle\int_0^\infty e^{-\lambda t}\|e^{tA_s}(A_s - A)f\|dt \\ &\leq \textstyle\int_0^\infty e^{-\lambda t}e^{\omega_o t}\|W(s)f\|dt \\ &= (\lambda - \omega_o)^{-1}\|W(s)f\|. \end{aligned}$$

Since W is strongly continuous and $W(0)g = 0$ for all $g \in D(A)$, it follws that $\lim_{s\to 0} R_s(\lambda)(A_s - A)f = 0$.

□

Proof of Theorem 4.1: For $f \in D(A)$,

$$\begin{aligned} \|V^n(\tfrac{t}{n})f - S(t)f\| &\leq \|V^n(\tfrac{t}{n})f - e^{tA_{t/n}}f\| \\ &+ \|e^{tA_{t/n}}f - S(t)f\|. \end{aligned} \tag{4.6}$$

Since $\lim_{n\to\infty} \|V^n(\frac{t}{n})f - e^{tA_{t/n}}f\| = 0$ by Lemma 4.1, it remains to show that

$$\lim_{n\to\infty} e^{tA_{t/n}}f = S(t)f. \tag{4.7}$$

Let R_s and ω_o be as in Lemma 4.2. For $f \in D(A)$ and $\lambda > \omega_o$, we have $R(\lambda, A)f = R_s(\lambda)(\lambda - A_s)R(\lambda, A)f$ from statement (iii) of Lemma 4.2. Then,

$$\begin{aligned} R_s(\lambda)f - R(\lambda, A)f &= R_s(\lambda)f - R_s(\lambda)(\lambda - A_s)R(\lambda, A)f \\ &= R_s(\lambda)[(\lambda - A) - (\lambda - A_s)]R(\lambda, A)f \\ &= R_s(\lambda)(A_s - A)R(\lambda, A)f, \end{aligned} \tag{4.8}$$

Since $f \in D(A)$ implies that $R(\lambda, A)f \in D(A^2)$, it follows from Lemma 4.3 and (4.8) that $\lim_{n\to\infty} R_s(\lambda)f = R(\lambda, A)f$ for all $\lambda > \omega_o$. Then with the boundedness condition proven in statement (i) of Lemma 4.2, it follows from Theorem 3.2 and statement (i) of Lemma 3.1 that

$$\lim_{s\to 0} e^{tA_s}f = S(t)f$$

uniformly for $t \in [0, T]$ for all $f \in D(A)$, from which statement (4.7) follows.

□

One well-accepted meaning of stability that is also implied by definition (2.1) for convergent methods is that when the initial value has a small error, the approximated solution also has only a small error. However, approximation methods satisfying only conditions (i) and (ii) of Theorem 4.1 may not necessarily be stable in this sense, but are stable in a sense very close to this.

Theorem 4.2 *If condition* (a) *of Theorem 4.1 is strengthened to*

$$\|V^{n+1}(t)f - V^n(t)f\| \le tMe^{n\omega t}(\|f\|+\|Af\|) \tag{4.9}$$

for some $M>0$ *for all* $f\in D(A)$, $t\in[0,\delta]$ *and* $n\in\mathbf{N}$, *viz., the* f*-dependent constant* $M_f = M(\|f\|+\|Af\|)$, *then for each* $f\in D(A)$ *and each sequence* $\{f_n\}\subset D(A)$ *with*

$$\lim_{n\to\infty}\|f_n-f\| + \|A(f_n-f)\| = 0, \tag{4.10}$$

$\lim_{n\to\infty} V(\frac{t}{n})^n f_n = S(t)f$ *uniformly for* t *in compact interval* $[0,T]$ *for any* $T>0$.

Proof: For $f\in D(A)$, Theorem 4.1 implies that $\lim_{n\to\infty} V(\frac{t}{n})^n f = S(t)f$ uniformly for t in compact interval $[0,T]$ for any $T>0$. It then suffices to show that for any sequence $\{f_n\}$ satisfying (4.10),

$$\lim_{n\to\infty} V\left(\tfrac{t}{n}\right)^n (f_n-f) = 0 \tag{4.11}$$

uniformly for t in compact interval $[0,T]$ for any $T>0$.

For $f\in D(A)$ and a sequence $\{f_n\}$ satisfying (4.10), we have that

$$\begin{aligned} \|V(\tfrac{t}{n})^n(f_n-f)\| &= \|\textstyle\sum_{k=1}^n \left[V(\tfrac{t}{n})^k(f_n-f) - V(\tfrac{t}{n})^{k-1}(f_n-f)\right] + (f_n-f)\| \\ &\le \|f_n-f\| + \textstyle\sum_{k=1}^n \|V(\tfrac{t}{n})^k(f_n-f) - V(\tfrac{t}{n})^{k-1}(f_n-f)\|., \end{aligned}$$

It then follows from (4.9) that

$$\begin{aligned} \|V(\tfrac{t}{n})^n(f_n-f)\| &\le \|f_n-f\| + \textstyle\sum_{k=1}^n \tfrac{t}{n}Me^{k\omega t/n}\left[\|f_n-f\| + \|A(f_n-f)\|\right] \\ &\le \|f_n-f\| + \textstyle\sum_{k=1}^n \tfrac{t}{n}Me^{\omega t}\left[\|f_n-f\| + \|A(f_n-f)\|\right] \\ &= \|f_n-f\| + tMe^{\omega t}\left[\|f_n-f\| + \|A(f_n-f)\|\right] \\ &\le (1+tMe^{\omega t})\left[\|f_n-f\| + \|A(f_n-f)\|\right], \end{aligned}$$

Then statement (4.11) follows from (4.10).

□

Remark 4.2 *If condition* (4.9) *of Theorem 4.2 is preserved for the spatially discretized operator* $\{A_h : h\in(0,\varepsilon)\}$ *for all mesh sizes* $h\in(0,\varepsilon)$, *namely,*

$$\|V^{n+1}(t;A_h)f_h - V^n(t;A_h)f_h\| \le tMe^{n\omega t}(\|Af_h\|+\|f_h\|) \tag{4.12}$$

for all $(t,h)\in[0,\delta]\times(0,\varepsilon)$ *and* $n\in\mathbf{N}$. *Then the Von Neumann stability condition holds for the approximation method* $\{V(t;A_h) : (t,h)\in[0,\delta]\times(0,\varepsilon)\}$.

Proof: From the telescoping equality

$$V^n(t;A_h)f_h\| \;=\; \textstyle\sum_{k=0}^{n-1}\left[V^{k+1}(t;A_h)f_h - V^k(t;A_h)f_h\right] + f_h,$$

we obtain

$$\|V^n(t;A_h)f_h\| \;\le\; \textstyle\sum_{k=0}^{n-1}\|V^{k+1}(t;A_h)f_h - V^k(t;A_h)f_h\| + \|f_h\|.$$

Then (4.12) implies

$$\begin{aligned}\|V^n(t;A_h)f_h\| &\le \textstyle\sum_{k=0}^{n-1} tMe^{k\omega t}(\|A_hf_h\|+\|f_h\|) + \|f_h\| \\ &\le ntMe^{n\omega t}(\|A_hf_h\|+\|f_h\|) + \|f_h\| \\ &\le ntM(\|A_hf_h\|+2\|f_h\|)e^{n\omega t},\end{aligned} \tag{4.13}$$

After spatial discretization, operator A_h is a matrix and hence a bounded linear operator. Then it follows from (4.13) that

$$\|V^n(t;A_h)\| \le ntM(\|A_h\|+2)e^{n\omega t}. \tag{4.14}$$

Since the spectral radius of an operator L satisfies that $\rho(L) = \lim_{n\to\infty}\|L^n\|^{1/n}$. It then follows from (4.14) that

$$\begin{aligned}\rho(V(t;A_h)) &\le \lim_{n\to\infty} e^{\omega t}\sqrt[n]{ntM(\|A_h\|+2)} \\ &= e^{\omega t},\end{aligned}$$

which means that $V(t;A_h)$ satisfies the Von Neumann stability condition.

□

5 Applications

In this section, we apply Theorem 4.1 to the unstable example approximation method given in Section 2, and an ADI type factorized temporal discretization method. For discussion simplicity, we restrict the problems in Hilbert space.

5.1 The unstable example approximation method

In this subsection, we shall prove that the unstable temporal discretization method $V(t;A)$ given by (2.8) for the Cauchy problem (1.2) for $A = A_1 + A_2$ where A_1 and A_2 are given in Section 2.1 satisfies the following statement. For notational simplicity, we will use $V(t)$ to denote $V(t;A)$.

Statement 5.1 *For $f \in D(A_1)$, and sequence $\{f_n\}_{n=1}^{\infty} \subset D(A_1)$ with*

$$\lim_{n\to\infty} \|f_n - f\| + \|f_n'' - f''\| = 0,$$

$\lim_{n\to\infty} V(\frac{t}{n})^n f_n = S(t)f$ *uniformly for $t \in [0,T]$ for any $T \geq 0$, where S is the semigroup generated by A.*

Proof: Since A_2 is bounded, $D(A) = D(A_1)$. And thus $D(A) \subset D(V(t)^\infty)$. Since A_1 generates a contraction semigroup and A_2 is bounded, it follows immediately that the map $t \mapsto V(t)f$ is continuous for all $f \in D(A)$.

Obviously, we also have the following counterpart of (2.7).

$$\begin{cases} \|(I - \frac{t}{2}A_1)^{-1}\| \leq 1, \\ \|(I - \frac{t}{2}A_1)^{-1}(I + \frac{t}{2}A_1)\| \leq 1, \\ \|A_2\| \leq 1. \end{cases} \tag{5.1}$$

For $t \geq 0$, denote $Y(t) = (I + tA_2)(I + \frac{t}{2}A_1)(I - \frac{t}{2}A_1)^{-1}$. Then (5.1) implies that $\|Y(t)\| \leq 1 + t$. Now for $f \in D(A)$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|V(t)^{n+1}f - V(t)^n f\| &= \|V(t)^n [V(t) - I] f\| \\ &= \|(I - \tfrac{t}{2}A_1)^{-1} Y(t)^n \left[(I + tA_2)(I + \tfrac{t}{2}A_1) - (I - \tfrac{t}{2}A_1)\right] f\| \\ &= t\|(I - \tfrac{t}{2}A_1)^{-1} Y(t)^n \left[(I + \tfrac{t}{2}A_2)A - \tfrac{t}{2}A_2^2\right] f\|. \end{aligned}$$

Then by (5.1) and $\|Y(t) \leq 1 + t$, we have that

$$\begin{aligned} \|V(t)^{n+1}f - V(t)^n f\| &\leq t(1+t)^n \| \left[(I + \tfrac{t}{2}A_2)A - \tfrac{t}{2}A_2^2\right] f\| \\ &\leq t\, e^{nt} \left[\|(I + \tfrac{t}{2}A_2)Af\| + \|\tfrac{t}{2}A_2^2 f\|\right] \\ &\leq t\, e^{nt} \left[(1 + \tfrac{t}{2})\|Af\| + \tfrac{t}{2}\|f\|\right] \\ &\leq t\, e^{nt} \left[e^t\|Af\| + e^t\|f\|\right] \\ &\leq t\, e^{2nt} \left[\|Af\| + \|f\|\right]. \end{aligned}$$

Now for $f \in D(A^2)$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|V(t)^{n+1}f - V(t)^n f - tV(t)^n Af\| &= \|V(t)^n [V(t) - I - tA] f\| \\ &= \|(I - \tfrac{t}{2}A_1)^{-1} Y(t)^n [(I + tA_2)(I + \tfrac{t}{2}A_1) - (I - \tfrac{t}{2}A_1)(I - tA)] f\| \\ &= t\|(I - \tfrac{t}{2}A_1)^{-1} Y(t)^n [\tfrac{t}{2}A^2 - \tfrac{t}{2}A_2^2] f\| \\ &\leq t(1+t)^n \|\tfrac{t}{2}(A^2 - A_2^2) f\| \\ &\leq t\, e^{2nt} \|\tfrac{t}{2}(A^2 - A_2^2) f\|. \end{aligned}$$

Set $W(t) = \frac{t}{2}(A^2 - A_2^2)$. Obviously the map $t \mapsto W(t)f$ is continuous and $W(0)f = 0$ for all $f \in D(A^2)$. Then by Theorem 4.2, $\lim_{n\to\infty} V\left(\frac{t}{n}\right)^n f_n = S(t)f$ uniformly for $t \in [0,T]$ for $T \geq 0$.

□

5.2 ADI type factorized approximation method

Proposition 5.1 *Suppose that A satisfies the* quasi-dissipative *condition*

$$\mathrm{Re}\langle Af, f\rangle < \omega\|f\|^2 \quad \textit{for all } f\in D(A) \tag{5.2}$$

for some constant $\omega > 0$. Then the Crank-Nicolson method $V(t) = \Big(I-\frac{t}{2}A\Big)^{-1}\Big(I+\frac{t}{2}A\Big)$ satisfies the stability condition

$$\|V(t)\| \le e^{2\omega t} \quad \textit{for all } t\in[0, \tfrac{1}{\omega}]. \tag{5.3}$$

Proof: By the Hille-Yosida theorem (Lumer-Phillips' type), the quasi-dissipativity condition (5.2) implies that $(I-tA)^{-1}\in\mathbf{L}(E)$ for $t\in[0, \frac{1}{\omega})$. Then, it is easily verifiable that $(I-\frac{t}{2}A)^{-1}$ and $(I+\frac{t}{2}A)$ commute for $t\in[0, \frac{1}{\omega}]$. It then follows that stability conclusion (5.3) is equivalent to that $\|(I+\frac{t}{2}A)(I-\frac{t}{2}A)^{-1}g\| \le e^{2\omega t}\|g\|$ for all $g\in E$, which is again equivalent to the following by replacing $(I-\frac{t}{2}A)^{-1}g$ by f.

$$\|(I+\tfrac{t}{2}A)f\| \le e^{2\omega t}\|(I-\tfrac{t}{2}A)f\| \quad \text{for } t\in[0, \tfrac{1}{\omega}], f\in D(A). \tag{5.4}$$

Square the left hand side of the above inequality, we obtain

$$\begin{aligned}\|(I+\tfrac{t}{2}A)f\|^2 &= \|(1+\tfrac{\omega t}{2})f + \tfrac{t}{2}(A-\omega)f\|^2\\ &= (1+\tfrac{\omega t}{2})^2\|f\|^2 + t\,Re\,\langle(A-\omega)f, f\rangle + \tfrac{t^2}{4}\|(A-\omega)f\|^2.\end{aligned}$$

Square the right hand side of inequality (5.4), we obtain

$$\begin{aligned}e^{4\omega t}\|(I-\tfrac{t}{2}A)f\|^2 &= e^{2\omega t}\|(1-\tfrac{\omega t}{2})I-\tfrac{t}{2}(A-\omega)f\|^2\\ &= e^{4\omega t}\left[(1-\tfrac{\omega t}{2})^2\|f\|^2 - t\,Re\,\langle(A-\omega)f, f\rangle + \tfrac{t^2}{4}\|(A-\omega)f\|^2\right].\end{aligned}$$

Therefore, (5.4) holds as long as

$$t[(1+\tfrac{\omega t}{2})+(1-\tfrac{\omega t}{2})e^{4\omega t}]Re\,\langle(A-\omega)f, f\rangle \le \left[(1-\tfrac{\omega t}{2})^2e^{4\omega t}-(1+\tfrac{\omega t}{2})^2\right]\|f\|^2 \tag{5.5}$$

for all $t\in[0, \frac{1}{\omega}]$ and $f\in D(A)$. Since $t[(1+\frac{\omega t}{2}) + (1-\frac{\omega t}{2})e^{4\omega t}] > 0$ for all $t\in[0, \frac{1}{\omega}]$, and $Re\,\langle(A-\omega)f, f\rangle \le 0$ by the quasi-dissipativity assumption, it follows that (5.5) holds if $(1-\frac{\omega t}{2})^2e^{4\omega t} - (1+\frac{\omega t}{2})^2 \ge 0$ for $t\in[0, \frac{1}{\omega}]$, which is ovbiously equivalent to

$$(1-\tfrac{\omega t}{2})e^{2\omega t} - (1+\tfrac{\omega t}{2}) \ge 0 \quad \text{for } t\in[0, \tfrac{1}{\omega}], \tag{5.6}$$

Let $h(t) = (1-\frac{t}{2})e^{2t} - (1+\frac{t}{2})$ for $t\in[0,1]$. Obviously h is differentiable on $[0,1]$ and $h'(t) = (\frac{3}{2}-t)e^{2t} - \frac{1}{2}$. Since $h'(t) \ge (1-t)e^{2t} \ge 0$ for $t\in[0,1]$, and again since $h(0) = 0$, it follows that $h(t) \ge 0$ for $t\in[0,1]$. Therefore, $h(\omega t) \ge 0$ for $t\in[0, \frac{1}{\omega}]$, which is exactly (5.6).

□

Theorem 5.1 *Suppose that* $A = A_1 + A_2$ *with* $D(A) \subset D(A_1) \cap D(A_2)$ *and*

$$\mathrm{Re}\langle A_i f, f\rangle < \omega \|f\|^2 \quad \text{for all } f \in D(A) \tag{5.7}$$

for $i = 1, 2$. *Let* $\{V(t) : t \in [0, \frac{1}{\omega}]$ *be a family of closed linear operators given by*

$$V(t) = \left(I - \tfrac{t}{2}A_1\right)^{-1}\left(I - \tfrac{t}{2}A_2\right)^{-1}\left(I + \tfrac{t}{2}A_2\right)\left(I + \tfrac{t}{2}A_1\right).$$

Then for $f \in D(A)$ *and* $\{f_n\}_{n=1}^{\infty} \subset D(A)$ *with* $\lim_{n\to\infty} \|f - f_n\| + \|A(f - f_n)\| = 0$, *the limit* $\lim_{n\to\infty} V^n(\frac{t}{n}) f_n = S(t) f$ *uniformly for* t *in any compact subset of* $[0, \infty)$, *where* S *is the* C_0*-semigroup generated by* A.

Proof: By the Hille-Yosida theorem (Lumer-Phillips' type), the quasi-dissipativity condition (5.7) implies that $\|(\lambda - \omega)(\lambda - A_i)^{-1}\| \le 1$ for $\lambda > \omega$ for $i = 1, 2$, which is equivalent to that $\|(I - tA_i)^{-1}\| \le (1 - \omega t)^{-1}$ for $t \in [0, \frac{1}{\omega})$ for $i = 1, 2$. It is easily provable that $(1 - \omega t)^{-1} \le e^{2\omega t}$ for $t \in [0, \frac{1}{2\omega}]$, from which we obtain that

$$\|(I - \tfrac{t}{2}A_i)^{-1}\| \le e^{\omega t} \quad \text{for } t \in [0, \tfrac{1}{\omega}] \tag{5.8}$$

for $i = 1, 2$. For $t \in [0, \frac{1}{\omega}]$, denote

$$Y(t) = (I - \tfrac{t}{2}A_2)^{-1}(I + \tfrac{t}{2}A_2)(I + \tfrac{t}{2}A_1)(I - \tfrac{t}{2}A_1)^{-1}.$$

The quasi-dissipativity condition (5.7) implies

$$\|Y(t)\| \le e^{4\omega t} \quad \text{for } t \in [0, \tfrac{1}{\omega}] \tag{5.9}$$

for $i = 1, 2$ by Proposition 5.1.

Obviously for $t \in [0, \frac{1}{\omega}]$, $D(A) \subset D(V(t)^{\infty})$ and the map $t \mapsto V(t) f$ is continuous for $f \in D(A)$.

For $f \in D(A)$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|V(t)^{n+1} f - V(t)^n f\| &= t\|V^n(t)\left(I - \tfrac{t}{2}A_1\right)^{-1}\left(I - \tfrac{t}{2}A_2\right)^{-1} Af\| \\ &= t\|\left(I - \tfrac{t}{2}A_1\right)^{-1} Y(t)^n \left(I - \tfrac{t}{2}A_2\right)^{-1} Af\|. \end{aligned}$$

Then by (5.8) and (5.9),

$$\begin{aligned} \|V(t)^{n+1} f - V(t)^n f\| &\le t\|Af\|\, e^{\omega t}(e^{4\omega t})^n e^{\omega t} \\ &\le t\|Af\|\, e^{6\omega n t}. \end{aligned}$$

Now for $f \in D(A^2)$ and $n \in \mathbf{N}$,

$$\begin{aligned}
\|V(t)^{n+1}f - V(t)^n f - tV(t)^n Af\| &= t\|V^n(t)\left[(I-\tfrac{t}{2}A_1)^{-1}(I-\tfrac{t}{2}A_2)^{-1} - I\right]Af\| \\
&= t\|(I-\tfrac{t}{2}A_1)^{-1}Y(t)^n\left[(I-\tfrac{t}{2}A_2)^{-1} - (I-\tfrac{t}{2}A_1)\right]Af\| \\
&\leq t\, e^{\omega t}(e^{4\omega t})^n\|\left[(I-\tfrac{t}{2}A_2)^{-1} - (I-\tfrac{t}{2}A_1)\right]Af\| \\
&\leq t\, e^{6\omega n t}\|\left[(I-\tfrac{t}{2}A_2)^{-1} - (I-\tfrac{t}{2}A_1)\right]Af\|.
\end{aligned}$$

Let $W(t) = (I-\frac{t}{2}A_2)^{-1} - (I-\frac{t}{2}A_1)$ for $t \in [0, \frac{1}{\omega}]$. The quasi-dissipativity of A_i implies that the map $t \mapsto W(t)f$ is continuous and $W(0)f = 0$ for all $f \in D(A_1)$. Since $D(A) \subset D(A_1) \cap D(A_2)$, then $Af \in D(A_1)$ for all $f \in D(A^2)$, and hence $t \mapsto W(t)Af$ is continuous and $W(0)Af = 0$ for $f \in D(A^2)$. Then by Theorem 4.2, $\lim_{n\to\infty} V\left(\frac{t}{n}\right)^n f_n = S(t)f$ uniformly for t in compact subsets of $[0,\infty)$.

□

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