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ON THE LOCAL SOLVABILITY OF PSEUDO-DIFFERENTIAL EQUATIONS

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The construction of the complete theory of the boundary-value problems for elliptic equations is one of the most important achievments of the theory of partial differential equations for the last twenty years. This achievment would be of course impossible without the theory of distributions, and it can be considered rightfully as an attainment of this last theory. But when we pass to some wider classes of equations, we collide at once with the following dilemma: we must either extend the class of the generalized solutions, oversteping the limits of the theory of distributions, or study the conditions when the considered equations can be solved in the class of distributions.

In regard to the first way it has been followed only in particular cases, for example, in oblique derivative problem for the second order elliptic equation. Here we shall talk about the investigations concerning the second approach. We shall limit ourselves to the case of the scalar linear pseudo-differential equations of principal type. This class of equations is apparently next in simplicity to the class of elliptic equations. While the problem of finding solutions of an elliptic equation is reduced to the solving of an algebraic equation, the solving of a principal type equation can be reduced to the integration of a first order differential equation.

1. – The typical situation arising at the investigation of the solvability of pseudo-differential equations of principal type, can be seen from the following result of L. Hörmander.

THEOREM (L. Hörmander [2]). – *If in each point* $(x, \xi) \in T^*(\Omega)$, *where* $p^0(x, \xi) = 0$, *we have*

$$C_1^0(x, \xi) \equiv \frac{1}{2i} \{\overline{p^0}(x, \xi), p^0(x, \xi)\} < 0$$

(*here* $\{\ \}$ – *the Poisson brackets* : $\{f, g\} \equiv \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}\right)$), *then the equation*

$$P(x, \mathcal{D})u = f \tag{1}$$

can be solved always for $f \in \mathcal{D}'(\Omega)$ (Ω *is a domain in* $\mathbf{R}^n$). *Moreover, if* $f \in \mathcal{H}_s(\Omega)$, *then there exists a solution* $u \in \mathcal{H}_{s+m-1/2}(\Omega)$. *But if we have* : $c_1^0(x, \xi) > 0$ *at a point* $(x, \xi) \in T^*(\Omega)$, *where* $p^0(x, \xi) = 0$, *then there exist such smooth functions* $f \in \mathcal{D}(\Omega)$, *that the equation (1) has no solution in the class* $\mathcal{D}'(\Omega)$.

2. – The first example of such an equation has been constructed by H. Lewy (1957) (see [1]).

The results I shall talk about develop the famous results of L. Hörmander and L. Nirenberg - F. Treves (See [2], [5]).

As an example of such results I state the following.

THEOREM. – *Let $P(x, \mathcal{D})$ be a differential operator of order m with smooth (C^∞) coefficients. Let $p^0(x, \xi)$ be its principal symbol, Im $p^0(x, \xi) = a_1(x, \xi)$, Re $p^0(x, \xi) = a_2(x, \xi)$. Let $I = (i_1, \dots i_k)$ be a finite sequence of integers i_j which are equal either to 1 or to 2, and $C_I(x, \xi) = \{\dots\{a_{i_1}, a_{i_2}\}, \dots, a_{i_k}\}$. We put $|I| = k - 1$ and $k(x, \xi) = |I_0|$, if $p^0(x, \xi) = 0$, $C_I(x, \xi) = 0$ for $|I| < |I_0|$, but $C_{I_0}(x, \xi) \neq 0$. If $p^0(x, \xi) \neq 0$, we set $k(x, \xi) = 0$. Suppose that*

$$\sup k(x, \xi) = k < \infty.$$

Then the equation (1) can be solved for all $f \in \mathcal{D}'(\Omega)$ if and only if the function $k(x, \xi)$ has only even values. If it is so and $f \in \mathcal{H}_s(\Omega)$, then (1) has a solution $u \in \mathcal{H}^{\mathrm{loc}}_{s+m-\frac{k}{k+1}}(\Omega)$.

This theorem can be generalized for general pseudo-differential operators (see [12]).

3. – The following statement plays very important role in the proof of the above results.

LOCALISATION THEOREM. – *The estimate*

$$\|u\|_s \leqslant C_{K,s} \left(\|Pu\|_{s-m+\frac{k}{k+1}} + \|u\|_{s-1}\right), \quad u \in C_0^\infty(K).$$

(K is a compact in Ω), holds if and only if for all $x \in K$, $\xi \in S^{n-1}$, $\lambda \geqslant 1$ and $\psi \in C_0^\infty(\mathbf{R}^n)$ the estimate

$$\|\psi\|_{L_2} \leqslant C \left\{ \|T^k_{(x,\xi)} p^0(x + y\lambda^{-\frac{1}{k+1}}, \xi + \mathcal{D}\lambda^{-\frac{k}{k+1}})\, \psi\|_{L_2} \lambda^{-m+\frac{k}{k+1}} + \lambda^{-\epsilon} \sum_{|\alpha+\beta| \leqslant N} \|y^\beta \mathcal{D}^\alpha \psi\|_{L_2} \lambda^{-|\alpha|\frac{k-1}{k+1}} \right\} \tag{2}$$

is valid. Here $\epsilon > 0$, $N \geqslant 0$ are some numbers $\left(\text{in the statement about necessity } N \geqslant k + 1, \epsilon \leqslant \frac{1}{k+1}\right)$, and $T^k_{(x,\xi)} f(x + y, \xi + \eta)$ is a segment of the Taylor expansion for the function $f(x + y, \xi + \eta)$ in a point (x, ξ) with respect to (y, η).

This theorem has been proved by L. Hörmander for the case $k = 1$ (See [2]) and later it has been generalized for any k by L. Hörmander (in some different form ; see [3]) and by myself [8].

4. – The following two theorems about canonical transformations are very important for our theory. The first theorem states that the conditions (2) are invariant relative to any canonical transformations of cotangent bundle $T^*(\Omega)$. We would remind that canonical transformation is a mapping : $(x, \xi) \to (x', \xi')$ preserving the values of Poisson brackets $\{f, g\}$ for all pairs of functions f, g : $T^*(\Omega) \to \mathbf{C}$. This theorem permits to make the imaginary part of principal symbol of our operator equal to ξ_1 in a neighbourhood of a considered point (x^0, ξ^0), where $p^0(x^0, \xi^0) = 0$. It allows to simplify the investigation of such operators essentially.

The second theorem about canonical transformations states that for any pseudo-differential operator P and for any real function $S(x, \xi)$ such that

$$S(x, \lambda\xi) = \lambda S(x, \xi)$$

for $\lambda > 0$ and det $\left\|\dfrac{\partial^2 S}{\partial x_i\, \partial \xi_j}\right\| \neq 0$, there exists such a pseudo-differential operator Q that $P\phi = \phi Q$, where

$$\phi u = \int \widetilde{u}(\xi)\, e^{iS(x,\xi)}\, d\xi$$

and $q^0(x', \xi') = p^0(x, \xi)$, if (x', ξ'), is the image of a the point (x, ξ) at the canonical transformation :

$$\xi = \operatorname{grad}_x S(x, \xi'),\, x' = \operatorname{grad}_{\xi'} S(x, \xi').$$

(G. Eskin had proved that ϕ is bounded and has a bounded inverse operator).

5. – The investigation of the estimates (2) can be reduced to the question of the validity of the inequalities of the following type

$$\|u\|_{L_2} \leqslant C \left\| \sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j} + Q(x, \mu)\, u \right\|_{L_2}, \ \forall\, u \in C_0^\infty(\mathbf{R}^n),$$

where $a = (a_1, \ldots, a_a)$ is a constant complex vector and $Q(x, \mu)$ is a polynomial of degree k. Precise necessary and sufficient conditions are obtained here for the homogeneous polynomials $Q(x, \mu)$. These conditions are sufficient too and if Q is not homogeneous. I shall not state these conditions here (see [9]). Note only that its forms are different for real a and for a, which is not proportional to a real vector.

6. – Our main theorem about necessary conditions of solvability is proved for operator of principal type : $\operatorname{grad}_{x,\xi} p^0(x, \xi) \neq 0$ if $p^0(x, \xi) = 0$. Let us note that such a theorem has been proved independently by L. Nirenberg and F. Treves [6], but at somewhat stronger conditions : $\operatorname{grad}_\xi p^0(x, \xi) \neq 0$, if

$$p^0(x, \xi) = 0.$$

This theorem states that if $k(x^0, \xi^0)$ is odd and (after due normalisation) the value of the first non-vanishing Poisson bracket $C_1(x^0, \xi^0) = (\operatorname{ad} a_1)^k\, C_2\, (x^0, \xi^0)$

is negative, then the equation (1) is unsolvable in the class of the distributions even locally.

This theorem can be formulated in the terms of the null bicharacteristics. We would remind that the bicharacteristic corresponding to the function $f(x, \xi)$ is the curve $x = x(t)$, $\xi = \xi(t)$ such that

$$\frac{dx_j}{dt} = \frac{\partial f(x, \xi)}{\partial \xi_j}, \quad \frac{d\xi_j}{dt} = -\frac{\partial f(x, \xi)}{\partial x_j}$$

The bicharacteristic is named null-bicharacteristic of $f(x, \xi)$, if $f(x(t_0), \xi(t_0)) = 0$ (and hence $f(x(t), \xi(t)) \equiv 0$). The above formulated condition of solvability means that

(A) along the null bicharacteristics of $a_1(x, \xi)$ the function $a_2(x, \xi)$ cannot change its sign from plus to minus.

7. – This condition (A) is close to a sufficient one. It becomes a sufficient condition if the following conditions B or B′ and C (or C′) are fullfilled.

THEOREM. – *The condition* A + B *or* A + B′ + C (*or* C′) *are sufficient for solvability.*

We suppose that either

(B) $$\sup k(x, \xi) = k < \infty$$

or

(B′) $$\operatorname{grad}_\xi p^0(x, \xi) \neq 0, \quad \text{if } p^0(x, \xi) = 0.$$

The conditions (C) and (C′) concern those points $(x, \xi) \in T^*(\Omega)$ only, in which $p^0(x, \xi) = 0$ and the vector grad $p^0(x, \xi)$ is proportional to a real vector. Let (x^0, ξ^0) be such a point and $\varphi = \arg \operatorname{grad} p^0(x^0, \xi^0)$, $q = p^0 e^{-i\varphi}$ so that the vector grad $q(x^0, \xi^0)$ is real.

(C) If the function Im $q(x, \xi)$ changes its sign on the manifold $Re\, q(x, \xi) = 0$ in any neighbourhood of the point (x^0, ξ^0), then the null-bicharacteristics of $Re\, q(x, \xi)$ are transversal to the manifold S, on which this changement of sign is realized. The manifold S is smooth.

The condition (C) is not necessary and can be replaced for example by a following one :

(C′) If (x^0, ξ^0) is such a point as above and the function Im $q(x, \xi)$ changes its sign on the manifold $Re\, q(x, \xi) = 0$ in any neighbourhood of the point (x^0, ξ^0), then there exists a smooth function $r(x, \xi)$ in some neighbourhood ω of the point (x^0, ξ^0) such that $r(x, \lambda\xi) = \lambda^{m-1'} r(x, \xi)$ for $\lambda > 0$, $\xi \neq 0$ and

$$C_1^0(x, \xi) \leqslant Re\, r(x, \xi) p^0(x, \xi) \quad \text{if } Re\, q(x, \xi) = 0$$

If the conditions (A) and (B) are fullfilled, then there exists such a solution of (1), that

$$\|u\|_s \leqslant C\,(\|f\|_{s+\frac{k}{k+1}-m} + \|u\|_{s-1}),$$

if A + B′ + C (or C′) — then the solvability of (1) is proved for a small neighbourhood ω_0 of the point x_0 and

$$\|u\|_s \leqslant C_1\,\|f\|_{s-m+1} + C_2\,\|u\|_{s-1}.$$

where $C_1 \to 0$ if diam $\omega_0 \to 0$. (see [13]).

A close theorem for the differential operators with analytic coefficients has been proved by L. Nirenberg and F. Freves.

At the conclusion of my lecture I should make some remarks about the nearest perspectives in this domain.

(1) The theorem about necessary conditions of solvability is true apparently without the supplementary supposition about the finitness of $k(x, \xi)$.

(2) We can hope to obtain the algebraic conditions for which the estimates

$$\|u\|_s \leqslant C(K, s)\,(\|Pu\|_{s-m+\delta} + \|u\|_{s-1}),\ \forall\ u \in C_0^\infty(K)$$

are valid with $1 \leqslant \delta < 2$. In any case it is possible to foretell these conditions.

(3) The theory of the boundary – value problems for operators of principal type can be constructed in the near future apparently so complete as it has been constructed for elliptic equations.

(4) At last, I think that the full clearness in the theory of solvability will be attained only after the solving of such a problem for the systems of pseudo-differential operators.

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