Extending partial orders on o-minimal structures to definable total orders

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1 Introduction

In this note we answer a question raised by John Truss. He asked if every definable partial ordering in an o-minimal structure can be extended to a definable total order. His question is motivated by analogy with the Order Extension Principle, a weak choice-like axiom of interest to set theorists, which asserts that every partial ordering of a set can be extended to a total order of the set (see [1], for example). We prove the following theorem.

Theorem 1.1 *If $\mathcal{M} = (M, <, \ldots)$ is an o-minimal structure such that $(M, <)$ is a dense total order and $\prec$ is a partial order on M which is definable (with parameters) in $\mathcal{M}$, then $\prec$ has an extension to a definable total order on M.*

This theorem is proved in Section 2. In Section 3 we offer some remarks on definable versions of Ramsey's theorem for o-minimal structures prompted by our proof of 1.1 and we indicate how to adapt our proof of Theorem 1.1 to yield the result for arbitrary o-minimal structures.

We adopt one notational convention, to distinguish between ordered pairs and open intervals. We write $\langle x, y \rangle$ for the ordered pair (x, y) (and extend this notation to ordered n-tuples), and (x, y) for the open interval (x, y). Also, if $\prec$

is any binary relation on a set P, and A, B are subsets of P, we write $A \prec B$ if $a \prec b$ holds for all $a \in A$ and $b \in B$. We extend this notation to write $a \prec B$ if $\{a\} \prec B$. Also, if $(P, <)$ is a partial order and $x, y \in P$, we write $x \| y$ if x and y are incomparable.

We assume that the reader is familiar with the basics of o-minimality as in the papers [3] and [2].

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2 The Proof of Theorem 1.1

We begin with the following two facts which do not require the hypothesis of o-minimality. The first gives us a way of extending a definable partial order on a linearly ordered structure into a definable total order under the assumption that the structure is partitioned into finitely many intervals on which the partial order is total. (Actually, the proof shows that the hypothesis that the structure be totally ordered and partitioned into intervals is inessential; we refrain from stating the lemma in complete generality in order to suggest its use in this paper more clearly). The second lemma provides a sufficient condition guaranteeing that a definable partial order on a linearly ordered structure can be extended definably to a partial order which is total on a given interval.

Lemma 2.1 *Let $\mathcal{N} = (N, <, \ldots)$ and $\prec$ a definable partial order on N. Suppose that N can be partitioned into $N = I_1 \cup \cdots \cup I_t$ such that each I_j is an open interval or a singleton of $(N, <)$ and that each I_j is totally ordered by $\prec$. Then $\prec$ can be extended to a definable total ordering on N.*

Proof. We assume that $I_j < I_k$ for all $j < k \leq t$. By induction, it thus suffices to prove that there is a definable partial ordering $\prec'$ extending $\prec$ which induces a total ordering on $I_1 \cup I_2$. We assume that I_1 and I_2 are open intervals since the argument is trivial otherwise.

For each $x \in I_2$, let $A_x = \{y \in I_1 : y \prec x\}$. Then A_x is an initial segment of $(I_1, \prec)$, and if $x_1, x_2 \in I_2$ satisfy $x_1 \prec x_2$ then $A_{x_1} \subseteq A_{x_2}$. We define

$$\prec^* = \{\langle u, v\rangle \in I_2 \times I_1 : v \notin A_u\},$$

and let $\prec'$ be the transitive closure on N of $\prec \cup \prec^*$. We verify that $\prec'$ is a definable partial order on N

We first show that $\prec'$ is definable. For this we prove that for all $x, y \in N$ if $x = x_0, \ldots, x_p = y \in N$ is a sequence of points such that $x_i \prec x_{i+1}$ or $x_i \prec^* x_{i+1}$ for each $i = 0, \ldots, p-1$ then there is such a sequence with $p \leq 3$. Observe that in a shortest such sequence we have $x_i \prec x_{i+1}$ if and only if $x_{i+1} \prec^* x_{i+2}$ for each

$i=0,\dots,p-2$. Thus our assertion—and with it the definability of $\prec'$—follows immediately from

Claim 2.1.1 *Suppose that $x,y\in N$ with $x\prec' y$. Let p be minimal such that there is a sequence $x=x_0,\dots,x_p=y$ with $x_i\prec x_{i+1}$ or $x_i\prec^* x_{i+1}$ for each $i=0,\dots,p-1$. Then we do not have both $p=3$ and $x_0\prec^* x_1\prec x_2\prec^* x_3$.*

Proof of 2.1.1. For a contradiction, assume not. Then $x_0\in I_2$, $x_1\in I_1$, and $x_1\notin A_{x_0}$. Since $x_2\prec^* x_3$ we also must have that $x_2\in I_2$. It follows that $x_2\prec x_0$, as otherwise we could shorten the length of the sequence by omitting x_1. Because $x_1\in A_{x_2}$ we then have that $x_1\in A_{x_0}$ and this is a contradiction. □

To complete the proof of the lemma—that is, to verify that $\prec'$ is a partial order—we have only to prove that $\prec'$ is irreflexive. Checking that none of the possible ways for there to be a pair (x,x) in $\prec'$ is routine. For example, suppose that we had $x\prec x_1\prec^* x_2\prec x$. Then $x_1\in I_2$, $x_2\in I_1$, and $x_2\notin A_{x_1}$. The transitivity of $\prec$ would however imply in this case that $x_2\prec x_1$, contradicting $x_2\notin A_{x_1}$. □

Lemma 2.2 *Let $\prec$ be a definable partial order on $(N,<,\dots)$ such that I is an interval of $(N,<)$ satisfying $y\not\prec x$ for all $x,y\in I$ with $x<y$. Let*

$$\prec^*=\{\langle x,y\rangle\in I^2:x<y\}$$

and let $\prec'$ be the transitive closure of $\prec\cup\prec^*$. Then $\prec'$ is a definable partial ordering on N. In fact, for all $x,y\in N$ such that $x\prec' y$ there is some $p\le 3$ and a sequence $x=x_0,\dots,x_p=y\in N$ with $x_{j-1}\prec x_j$ or $x_{j-1}\prec^* x_j$ for $j=1,\dots,p$ in which $\prec^*$ occurs at most once.

The same conclusion holds if we assume instead that $x\not\prec y$ for all $x,y\in I$ with $x<y$ and define $\prec^*=\{\langle x,y\rangle:x,y\in I,y<x\}$.

Proof. Just as in the proof of 2.1, to prove definability we just check that a shortest sequence witnessing that $x\prec' y$ has length at most four. So we must check that in a shortest such sequence we do not have $x_i\prec^* x_{i+1}\prec x_{i+2}\prec^* x_{i+3}$. But if this held, then we have $x_i,x_{i+1},x_{i+2},x_{i+3}\in I$ with $x_i<x_{i+1}<x_{i+2}<x_{i+3}$ and hence $x_i\prec^* x_{i+3}$, contradicting that the sequence $x_0,\dots,x_p$ was chosen to have shortest length.

To verify that $\prec'$ is irreflexive, we suppose for a contradiction that there is $x\in N$ with $x\prec' x$. Then there is a sequence $x=x_0,\dots,x_p=x$ witnessing this with $p\le 3$. As in the proof of 2.1 are several cases to consider. They are all easy and are left to the reader.

The proof of the last sentence of the lemma is similar to the argument just given. □

We now work to position ourselves to apply Lemmas 2.1 and 2.2. For the remainder of this section we suppose that $\mathcal{M}=(M,<,\ldots)$ is o-minimal, that $<$ is dense, and that $\prec$ is a definable partial ordering on M. The condition that $(M,\prec)$ has a definable extension to a total ordering is a property of $\mathrm{Th}(\mathcal{M})$. Hence, in the proof of Theorem 1.1 we may assume that M is countably infinite, and in order to make Lemma 2.6 correct we assume this also. We begin by fixing a convenient decomposition of M into 1-cells.

Let $X=\{\langle x,y\rangle\in M^2 : x<y\}$. We partition X into the definable sets

$$X_1=\{\langle x,y\rangle\in X : x\prec y\},$$

$$X_2=\{\langle x,y\rangle\in X : y\prec x\},$$

$$X_3=\{\langle x,y\rangle\in X : x\|y\}.$$

Since $(M,<)$ is dense, by o-minimality we may define a corresponding partition of M by

$$Y_1=\{x : (\exists y>x)\,(\forall z\in(x,y))\,\langle x,z\rangle\in X_1\},$$

$$Y_2=\{x : (\exists y>x)\,(\forall z\in(x,y))\,\langle x,z\rangle\in X_2\},$$

$$Y_3=\{x : (\exists y>x)\,(\forall z\in(x,y))\,\langle x,z\rangle\in X_3\}.$$

Applying the cell decomposition theorem [2], we obtain a partition $\mathcal{P}$ of M^2 into cells such that each of X_1, X_2, and X_3 is the union of some of these cells and such that the projection of these cells onto the first coordinate induces a partition $\mathcal{P}_0$ of M into intervals $W_1,\ldots,W_s$ and a finite set F such that each of Y_1, Y_2, and Y_3 is the union of sets in the induced partition. By the monotonicity theorem [3], we may suppose further that each function arising in the definition of a cell in $\mathcal{P}$ is not only continuous but also either constant or strictly monotonic on its domain.

Lemma 2.3 *Let $i\in\{1,\ldots,s\}$.*

i. *If $W_i\subseteq Y_1$, then $<$ and $\prec$ agree on W_i.*

ii. *If $W_i\subseteq Y_2$, then $\prec$ agrees on W_i with the reverse ordering of $<$.*

Proof. We prove (i); the proof of (ii) is similar. Let $x\in W_i$. By the continuity of the functions used in the definition of the cells in $\mathcal{P}$, there is an open interval $I\subseteq W_i$ containing x such that $\prec$ and $<$ agree on I. By transitivity of $\prec$ we observe that if I_1 and I_2 are two such intervals containing x then so is $I_1\cup I_2$. Hence, for each $x\in W_i$ there exists a greatest such open interval I_x, and it is definable.

We claim for all $x,y\in W_i$ that either $I_x=I_y$ or $I_x\cap I_y=\varnothing$. Indeed if I_x and I_y were distinct such intervals such that $I_x\cap I_y\neq\varnothing$, then, as in the preceding paragraph, we see that $I_x\cup I_y$ is an interval containing x on which $\prec$ and $<$ agree, contradicting the maximality of I_x.

The lemma now follows easily. For suppose that there is some $x \in W_i$ such that $I_x \neq W_i$. Let $y \in W_i$ be a boundary point of I_x. Then $I_x \cap I_y \neq \varnothing$ and $I_x \neq I_y$, a contradiction. □

Lemma 2.4 *For all $i = 1, \ldots, s$ the set $W_i \times W_i$ has non-empty intersection with at most one of X_1 or X_2.*

Proof. For a contradiction assume that there are $x, y, u, v \in W_i$ with $x < u$, $x \prec u$, $y < v$, and $y \succ v$. By Lemma 2.3 we see that $W_i \subseteq Y_3$. Suppose that g is a boundary function of a 2-cell in the partition $\mathcal{P}$ whose domain is W_i and that there are $a, b \in W_i$ such that $a < b$ and $g(a) = b$. Since g is continuous and constant or strictly monotone on W_i, it follows from the fact that $\mathcal{P}$ respects X that g must be strictly increasing on W_i. Hence, we see that we may assume that $x = y$ and also $u \neq v$.

We suppose that $u < v$—the argument in case $v < u$ is similar. We may also choose u and v so that there is a boundary function h of one of the 2-cells in $\mathcal{P}$ such that $h(x) \leq v$, $\langle x, z \rangle \in X_2$ for all $z \in (h(x), v)$ and $\langle x, z \rangle \notin X_2$ for all $z < h(x)$. Since $\langle x, u \rangle \in X_1$ and $\langle x, v \rangle \in X_2$, by the transitivity of $\prec$ we have $v \prec u$. However, since h is continuous and must be strictly increasing, we may choose v such that $\langle x, v \rangle \in X_2$ to satisfy $v < h(u)$. In this case $\langle u, v \rangle \in X_1 \cup X_3$ and so $v \not\prec u$. This contradicts $v \prec u$. □

To extend the given partial order $\prec$ to a definable total order, by Lemmas 2.1 and 2.3 we need only be concerned with extending $\prec$ to a total order on those sets $W_i \subset Y_3$. For a single such W_i we are able to do this by Lemmas 2.4 and 2.2. However, we cannot guarantee that there might not be some other W_j for which the hypothesis of Lemma 2.2 no longer holds with respect to the resulting partial order $\prec'$ on M. Simply refining the partition $\mathcal{P}_0$ is not a solution since *a priori* we might be forced into an endless sequence of refinements. We now work to overcome this seeming obstacle.

Definition 2.5 Let $I \subset M$ be an interval, $f : I \to M$ be a function which is not necessarily definable, $u_1, \ldots, u_n \in I$, and $v_1, \ldots, v_n \in M$ (possibly with $v_i = f(u_i)$ for some values of i). The set

$$\{\langle x, f(x)\rangle : x \in I\} \cup \bigcup_{i \leq n} \{\langle u_i, z\rangle : f(u_i) \leq z \leq v_i \text{ or } v_i \leq z \leq f(u_i)\}$$

is called a *near-path* with *base* I.

So a near-path is the graph of a function possibly with finitely many vertical lines attached to it, the vertical lines being bounded above or below, as the case may be, by the graph of the function. We say that a near-path Φ has the *intermediate value property (IVP)* if for all $x, y \in I$ and $u, v \in M$ such that

$\langle x,u\rangle,\langle y,v\rangle \in \Phi$ and $x<y$, and for all $c\in M$ lying between u and v, there is a point $z\in[x,y]$ with $\langle z,c\rangle\in\Phi$.

Lemma 2.6 *Let I be an open interval of M, and let $f,g{:}I\to M$ be definable functions, each continuous and constant or strictly monotonic on I, such that $f(x)<g(x)$ for all $x\in I$. Also let $a\in I$ and $b\in M$ satisfy $f(a)<b<g(a)$. Then there is a near-path Φ with base I—not necessarily definable—so that:*

i. $\langle a,b\rangle\in\Phi$;

ii. $f(x)<y<g(x)$ *for all* $\langle x,y\rangle\in\Phi$;

iii. Φ *has the (IVP).*

Proof. We construct the near-path Φ as the graph of a function h on I. As M is countable, we can enumerate M by $\{m_i : i<\omega\}$. Let $F{:}\mathbb{N}\to\mathbb{N}^3$ be such that each element of $\mathbb{N}^3$ is the image under F of infinitely many elements of $\mathbb{N}$. We build h as a union of an increasing chain $(h_n : n<\omega)$ of finite partial functions.

We set $h_0=(a,b)$. At step $2n+1$, if $m_n\in I$ and $h_{2n}(m_n)$ is not defined, pick $y\in(f(m_n),g(m_n))$ and extend h_{2n} to h_{2n+1} by putting $h_{2n+1}(m_n)=y$. At step $2n$, we do nothing unless $F(n)=(p,q,r)$, $h_{2n-1}(m_p)$ and $h_{2n-1}(m_q)$ are defined, and m_r lies strictly between $h_{2n-1}(m_p)$ and $h_{2n-1}(m_q)$. If these conditions hold—and, say, $m_p<m_q$—we wish to pick $x\in(m_p,m_q)\setminus\mathrm{Dom}(h_{2n-1})$ so that $f(x)<m_r<g(x)$ and to extend h_{2n-1} to h_{2n} by putting $h_{2n}(x)=m_r$. Using the assumed properties of f and g, we show how to pick such an x in case both f and g are strictly increasing. For this we just note that $c=\inf\{x:g(x)>m_r\}<\sup\{x:f(x)<m_r\}=d$, and so any $x\in(c,d)\setminus\mathrm{Dom}(h_{2n})$ with $m_p<x<m_q$ works. The other cases are left to the reader.

It is clear that the resulting function $h=\cup_{n\in\mathbb{N}}h_n$ has the required properties. □

We now are ready to prove the main result of this section. The need for the generality in the definition of a near-path will become apparent.

Proof of Theorem 1.1. Recall that M is countable, and that we have partitioned M and M^2 by $\mathcal{P}_0$ and $\mathcal{P}$, respectively. By Lemma 2.1 it more than suffices to show that there is a partition of M into finitely many open intervals and singletons which refines $\mathcal{P}_0$, and a definable extension of $\prec$ to a partial order $\prec'$ on M, such that the restriction of $\prec'$ to each of the intervals in the partition is a total order agreeing with the restriction of $<$ or its reverse. For those intervals $W\in\mathcal{P}_0$ such that $W\subset Y_1\cup Y_2$, we have by Lemma 2.3 that the restriction of $\prec$ to W already agrees with $<$ or its reverse. So we confine our attention to those $W\in\mathcal{P}_0$ with $W\subset Y_3$. We enumerate these intervals as $W_1,\ldots,W_r$ so that $W_i<W_j$ whenever $i<j$.

By induction on $p=0,\dots,r$ we construct an increasing chain of definable partial orders $\prec^p$ on M extending $\prec$. Let us denote by X_j^p and Y_j^p for $j=1,2,3$ the sets corresponding to X_j and Y_j for the partial ordering $\prec^p$. Then we shall demand that:

(1^p) for each $i \leq p$, the interval W_i has been partitioned definably into finitely many open subintervals and points such that on each of these subintervals of W_i, $\prec^p$ agrees with $<$, or on each subinterval $\prec^p$ agrees with the reverse of $<$;

(2^p) for each q with $p<q\leq r$, either $(W_q \times W_q)\cap X_1^p=\varnothing$ or $(W_q\times W_q)\cap X_2^p=\varnothing$.

The definable partial ordering $\prec'=\prec^r$ is then as required.

We set $\prec^0=\prec$, which clearly satisfies condition (1^0) and by Lemma 2.4 satisfies (2^0). Assume now that we have constructed $\prec^p$ and we construct $\prec^{p+1}$. We partition W_{p+1} into finitely many open intervals $B_1,\dots,B_t$—listed in increasing order under $<$ with the number t of intervals dependent on $p+1$—and a finite set F_{p+1} so that for each $k\in\{1,\dots,t\}$ we have a definable partition $\mathcal{Q}_k$ of $B_k\times M$ into cells satisfying

a. the projection onto the first coordinate of each cell in $\mathcal{Q}_k$ is equal to B_k and all functions in the definitions of all cells in each $\mathcal{Q}_k$ are monotone or constant;

b. for each $j=1,2,3$ and $C\in\mathcal{Q}_k$ we have that C is disjoint from or contained in X_j^p;

c. for each $p+1<q\leq r$ and $C\in\mathcal{Q}_k$ we have that C is disjoint from or contained in $B_k\times W_q$.

Note that $x=y$ will occur among the boundary functions of these cells, by (b).

By induction hypothesis (2^p) and Lemma 2.2 we observe that $\prec^p$ can be extended (definably) to a partial order $\lhd$ such that $\lhd$ agrees on W_{p+1} with either $<$ or its reverse, and such that if $W_{p+1}\subset Y_1^p\cup Y_3^p$ then $\lhd$ agrees on W_{p+1} with $<$. We now extend $\prec^p$ to $\prec^{p+1}$ by an increasing sequence of partial orders $\prec^{p,k}$ for $k=0,\dots,t$ with each $\prec^{p,k}$ a sub-partial order of $\lhd$. We also require some additional notation: for $j\in\{1,2,3\}$ and $k\in\{0,\dots,t\}$, let $X_j^{p,k}$ and $Y_j^{p,k}$ be the sets corresponding to X_j and Y_j, respectively, for the partial ordering $\prec^{p,k}$.

Let $\prec^{p,0}=\prec^p$. Assuming that we have defined $\prec^{p,k}$ we show how to define $\prec^{p,k+1}$. If $W_{p+1}\subset Y_1^p\cup Y_3^p$, let

$$\prec_*^{p,k+1}=\{<\restriction_{B_{k+1}}\},$$

and otherwise, let

$$\prec_*^{p,k+1}=\{\langle x,y\rangle : x,y\in B_{k+1} \text{ and } y<x\}.$$

We then let $\prec^{p,k+1}$ be the transitive closure of $\prec^{p,k} \cup \prec_*^{p,k+1}$. It follows by induction on k using Lemma 2.2 that each of the $\prec^{p,k}$ is a definable partial ordering on M which is a subordering of $\lhd$. Finally, let $\prec^{p+1}=\prec^{p,t}$. Obviously Condition (1^{p+1}) holds. To prove that (2^{p+1}) is satisfied we verify two claims.

Claim 2.6.1 *Let $k\in\{0,\dots,t\}$ and suppose that $k<m\leq t$ and $p+1<q\leq r$. Suppose that there is some $\langle b,u\rangle\in(B_m\times W_q)\cap X_1^{p,k}$ (respectively, $\langle b,u\rangle\in(B_m\times W_q)\cap X_2^{p,k}$). Then there is a near-path Φ with base B_m having the (IVP) such that*

$$\langle b,u\rangle\in\Phi\subset(B_m\times W_q)\cap X_1^{p,k}$$

(respectively, $\Phi\subset(B_m\times W_q)\cap X_2^{p,k}$).

Proof of Claim 2.6.1. The proof is by induction on k. For $k=0$, the result follows immediately from Lemma 2.6 and how $B_1,\dots,B_t$ were chosen. Assume that the result holds for all $i<k$ where $k\geq 1$. We assume also that

$$\langle b,u\rangle\in(B_m\times W_q)\cap X_1^{p,k};$$

the argument in the case that $\langle b,u\rangle\in(B_m\times W_q)\cap X_2^{p,k}$ is very similar. We may suppose as well that $\langle b,u\rangle\notin X_1^{p,i}$ for all $i<k$, as otherwise the result follows by induction. We have two cases to consider.

Case 1. $\prec_*^{p,k}$ agrees with $<\restriction_{B_k}$.

As $b\prec^{p,k}u$ and $b\not\prec^{p,k-1}u$, by Lemma 2.2 there are $b_1,b_2\in B_k$ such that $b\prec^{p,k-1}b_1\prec_*^{p,k}b_2\prec^{p,k-1}u$. Since $b_1<b_2<b<u$, it follows that $\langle b_1,b\rangle\in X_2^{p,k-1}$ and $\langle b_2,u\rangle\in X_1^{p,k-1}$. By induction hypothesis, there is a near-path Ψ with base B_k having the (IVP) such that

$$\langle b_2,u\rangle\in\Psi\subseteq(B_k\times W_q)\cap X_1^{p,k-1}.$$

Let v be such that $\langle b_1,v\rangle\in\Psi$. Then $\langle b_1,v\rangle\in X_1^{p,k-1}$ and so $b_1\prec^{p,k-1}v$. By transitivity $b\prec^{p,k-1}v$, so $\langle b,v\rangle\in X_1^{p,k-1}$. Hence, again by induction, there is a near-path Θ with base B_m which has the (IVP) such that

$$\langle b,v\rangle\in\Theta\subseteq(B_m\times W_q)\cap X_1^{p,k-1}.$$

Now let $w\in W_q$ with $u\leq w\leq v$ or $v\leq w\leq u$. Since $\langle b_1,v\rangle,\langle b_2,u\rangle\in\Psi$ and Ψ has the (IVP), there is some c satisfying $b_1\leq c\leq b_2$ for which $\langle c,w\rangle\in\Psi$. Hence $c\prec^{p,k-1}w$, and so as $b\prec^{p,k-1}b_1\prec^{p,k}c\prec^{p,k-1}w$ we have $\langle b,w\rangle\in X_1^{p,k}$. Hence

$$\Phi=\Theta\cup\{\langle b,w\rangle : u\leq w<v \text{ or } v<w\leq u\}$$

is a near-path with base B_m having the (IVP) such that $\langle b,u\rangle\in\Phi\subset(B_m\times W_q)\cap X_1^{p,k}$, as required.

Case 2. $\prec_*^{p,k}$ agrees with the reverse of $<\restriction_{B_k}$.

Here there are $b_1, b_2 \in B_k$ such that $b_2 < b_1$, $b \prec^{p,k-1} b_1 \prec_*^{p,k} b_2 \prec^{p,k-1} u$, and also $\langle b_1, b\rangle \in X_2^{p,k-1}$ and $\langle b_2, u\rangle \in X_1^{p,k-1}$. The argument is very similar to that in Case 1 and the details are left to the reader. $\square$

The next claim shows that (2^{p+1}) holds, and hence completes the proof of Theorem 1.1.

Claim 2.6.2 *Let $p+1 < q \le r$. If $(W_q \times W_q) \cap X_1^{p+1} \neq \varnothing$ then $(W_q \times W_q) \cap X_1^{p} \neq \varnothing$, and if $(W_q \times W_q) \cap X_2^{p+1} \neq \varnothing$ then $(W_q \times W_q) \cap X_2^{p} \neq \varnothing$.*

Proof of Claim 2.6.2. We suppose that $(W_q \times W_q) \cap X_1^{p+1} \neq \varnothing$; the other case is similar. Arguing by contradiction, we let $k \in \{0, \ldots, t-1\}$ be least such that $(W_q \times W_q) \cap X_1^{p,k+1} \neq \varnothing$ and $(W_q \times W_q) \cap X_1^{p,k} = \varnothing$. In particular, there are $u, v \in W_q$ with $u < v$ such that $u \prec^{p,k+1} v$ and $u \not\prec^{p,k} v$. Hence by Lemma 2.2 there are $b_1, b_2 \in B_{k+1}$ such that

$$u \prec^{p,k} b_1 \prec_*^{p,k+1} b_2 \prec^{p,k} v.$$

Note that $\langle b_1, u\rangle \in X_2^{p,k}$ and $\langle b_2, v\rangle \in X_1^{p,k}$. We must consider two possibilities.

Case 1. $\prec_*^{p,k+1}$ agrees with $<\restriction_{B_{k+1}}$.

Here $b_1 < b_2$. By Claim 2.6.1 there is a near-path Φ with base B_{k+1} having the (IVP) such that

$$\langle b_1, u\rangle \in \Phi \subset (B_{k+1} \times W_q) \cap X_2^{p,k}.$$

There is no c with $b_1 \le c \le b_2$ and $\langle c, v\rangle \in \Phi$, for otherwise we would have $v \prec^{p,k} c$, which contradicts $\langle b_2, v\rangle \in X_1^{p,k}$ (via $c \prec^{p,k+1} b_2$ in case $c < b_2$) or $b_2 \prec^{p,k} v$ (if $c = b_2$). Hence, as Φ has the (IVP), there is some $w < v$ with $w \in W_q$ such that $\langle b_2, w\rangle \in \Phi$. Then $w \prec^{p,k} b_2$ and $b_2 \prec^{p,k} v$, and so we have $w \prec^{p,k} v$. But this contradicts the minimality of k, which completes the argument in this case.

Case 2. $\prec_*^{p,k+1}$ agrees with the reverse of $<\restriction_{B_{k+1}}$.

Now $b_2 < b_1$. Again by Claim 2.6.1, there is a near-path Φ with base B_{k+1} having the (IVP) and satisfying

$$\langle b_2, v\rangle \in \Phi \subseteq (B_{k+1} \times W_q) \cap X_1^{p,k}.$$

There cannot be a point c with $b_2 \le c \le b_1$ and $\langle c, u\rangle \in \Phi$, for otherwise $c \prec^{p,k} u$ and —as $b_1 \prec^{p,k+1} c$ if $c < b_1$—we have $b_1 \prec^{p,k+1} u$, contradicting $\langle b_1, u\rangle \in X_2^{p,k}$. Hence, as Φ has the (IVP), there is some $w \in W_q$ with $u < w$ and $\langle b_1, w\rangle \in \Phi$. But then $b_1 \prec^{p,k} w$, and as $u \prec^{p,k} b_1$ we have $u \prec^{p,k} w$. This again contradicts the minimality of k, completing the proof of the claim and the theorem. $\square$

3 Further Remarks

We first discuss a definable version of Ramsey's Theorem implicit in our arguments in Section 2. We begin by proving the following easy version for partitions of the two-element subsets of an o-minimal structure.

Theorem 3.1 *Let $(M,<,\ldots)$ be an o-minimal structure whose underlying order is dense, let $X=\{\langle x,y\rangle \in M^2 : x<y\}$, and let $X=X_1\cup\ldots\cup X_r$ be a partition of the two-element subsets of M into finitely many definable sets. Then there is an infinite interval $I\subset M$ and some $i_0\in\{1,\ldots,r\}$ such that $\langle x,y\rangle\in X_{i_0}$ for all $x,y\in I$ with $x<y$.*

Proof. By o-minimality, for each $x\in M$ there is an infinite open interval J with left endpoint x and some $i=i(x)\in\{1,\ldots,r\}$ such that for all $y\in J$, $\langle x,y\rangle\in X_i$. For each $x\in M$ choose J_x to be a maximal such interval. Define a function $f\colon M\to M\cup\{\infty\}$ by $f(x)=\sup(J_x)$. By the monotonicity theorem, there is an infinite interval $K\subset M$ on which f is constant or strictly monotonic (the fact that we are allowing $f(x)=\infty$ makes no essential difference). We may further suppose by o-minimality that $i(x)$ takes a constant value i_0 as x ranges through K. Choose $x_0\in\mathrm{Int}(K)$. By the continuity of f on K, there is some $y>x$ in K such that $f(x)>y$. It is now easy to see that for all u,v with $x<u<v<y$ we have $\langle u,v\rangle\in X_{i_0}$. Putting $I=(x,y)$, we have the result. □

It is natural to conjecture an analogue of Theorem 3.1 for n-element subsets in place of pairs: namely, that for any finite coloring of the set of increasing n-tuples of a dense o-minimal structure $(M,<,\ldots)$, there is an infinite interval all of whose increasing n-tuples have the same color. This, however, turns out to be false even for $n=3$, as the following example demonstrates.

Example 3.2 Let $\mathcal{M}=(\mathbb{Q},<,+,0)$. We define a partition of $\{\langle u,v,w\rangle\in\mathbb{Q}^3 : u<v<w\}$ into sets X_1 and X_2 as follows. If $w-v>v-u$, let $\langle u,v,w\rangle\in X_1$, and otherwise let $\langle u,v,w\rangle\in X_2$. It is easy to see that there is no non-empty interval J of $\mathbb{Q}$ whose increasing triples all lie in X_1 or all lie in X_2.

We conclude the paper with some comments about how our proof of Theorem 1.1 can be extended to handle arbitrary o-minimal structures.

Remark 3.3 Let $\mathcal{M}$ be an arbitrary o-minimal structure. We quickly sketch how to extend Theorem 1.1 to $\mathcal{M}$. By Theorem 3.12 of [3] we have that $(M,<)$ decomposes into an ordered sum of dense and "discretely" ordered intervals. The given partial order $\prec$ can be extended to a definable total order on the union of the dense intervals by Theorem 1.1, and exactly as we have done before it can be checked that the transitive closure of the resulting binary relation $\prec_0$ on M is a definable partial order. By Theorem 3.4 of [4] definable functions on the discrete part of $(M,<)$ are piecewise very simple: either constant functions or "translations" (see [4] for details). With the aid of this, the proof of Theorem 1.1 can be modified to extend the restriction of $\prec_0$ to a total order on the discrete part of $(M,<)$ (Lemma 2.6, whose proof above requires that the underlying order be dense, can be handled via a different more concrete argument in the discrete case using the fact that definable functions are so simple). It is easy to check that the transitive closure of the resulting binary relation on M is a partial order. The result is a definable partial order $\prec_1$ whose restrictions to the dense and discrete parts of $(M,<)$ are (separately) total. By adapting the proof of Lemma 2.1 it is routine to extend $\prec_1$ to a definable total order on M.

References

[1] T. Jech, *The axiom of choice*, Amsterdam: North Holland 1973.

[2] J. Knight, A. Pillay, and C. Steinhorn, Definable sets in ordered structures II, Trans. Amer. Math. Soc. **295** (1986), 593–605.

[3] A.Pillay, C. Steinhorn, Definable sets in ordered structures I, Trans. Amer. Math. Soc. **295** (1986), 565–592.

[4] A.Pillay, C. Steinhorn, Discrete o-minimal structures, Ann. Pure Appl. Logic **34** (1987), 275–289.