ASER of Rectangular MQAM in Noise-Limited and Interference-Limited MIMO MRC Systems

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Abstract—Novel expressions for the average symbol error rate (ASER) of general rectangular quadrature amplitude modulation (QAM) in Rayleigh fading are derived in this work for MIMO MRC systems. The number of antennas is assumed to be arbitrary at both the transmit and receive ends and, unlike previous related work, expressions are given in terms of finite summations where all the terms are explicitly given. Both noise-limited and interference-limited systems are analyzed. Monte Carlo simulations have been carried out to validate the accuracy of the results.

Index Terms—Symbol error rate, rectangular QAM, MIMO MRC.

I. INTRODUCTION

MUltiple-input/multiple-output (MIMO) maximal ratio combining (MRC) has been proposed to maximize the output signal-to-noise ratio (SNR) in wireless systems with multiple transmit and receive antennas [1], [2]. Rectangular quadrature amplitude modulation (QAM) is a general modulation technique which includes important modulation schemes as particular cases, such as binary phase-shift keying (BPSK), pulse amplitude modulation (PAM) or square QAM. Although MIMO MRC systems under different conditions have been investigated in the last few years, no results seem to be available for the average symbol error rate (ASER) of rectangular QAM in noise-limited environments. In [3], an expression for the bit error rate (BER) of rectangular QAM is derived, yet the result is only valid for Gray code mapping and no expression for the ASER is provided. Additionally, the analysis in [3] is actually semi-analytical, as the derived expression rely on coefficients not given in explicit form and which have to be calculated numerically. The use of these coefficients is avoided in [4] for the same scheme, but only for a system with two receive antennas, and again no ASER result is provided. In practice, wireless communication systems are usually limited by co-channel interference (CCI). An analysis of the ASER of MIMO MRC with rectangular QAM in the presence of CCI was presented in [5]. However, the analysis is based on an approximated expression of the instantaneous error rate and the provided results are again based on coefficients which are not given explicitly.

In this work, we present novel and general expressions for the ASER of general rectangular QAM in MIMO MRC with no restriction in the number of transmit and receive antennas. Unlike previous related work, all the terms in the derived expressions are explicitly given, as our results do not rely on coefficients to be computed numerically. For systems without CCI a closed-form expression of the ASER in terms of elementary functions and independent of the code mapping of the constellation symbols is derived for the first time. For systems with CCI the derived expression is given in terms of an integral which can be solved fast and as accurately as desired using the generalized Laguerre polynomials. This work also presents a novel expression for the probability of outage when CCI is present. The analysis presented here has been validated by means of Monte Carlo simulations.

II. SER ANALYSIS WITHOUT CCI

A. System model

We assume a MIMO system with T transmit and R receive antennas. The desired signal undergoes flat Rayleigh fading that is uncorrelated across the antennas. We also assume that the signal at each of the R receive antennas is corrupted by additive white Gaussian noise (AWGN). The received noise vector is assumed to be zero mean with covariance matrix \( \sigma_n^2 \mathbf{I}_R \), where \( \mathbf{I}_R \) denotes the \( R \times R \) identity matrix. Thus, after matched filtering and sampling at the symbol interval, the \( R \times 1 \) received vector \( \mathbf{r} \) can be modeled as

\[
\mathbf{r} = \mathbf{H}_D \mathbf{w}_D s_D + \mathbf{n},
\]

where \( \mathbf{H}_D \) is an \( R \times T \) matrix whose \((i, j)^{th}\) element represents the channel complex gain from transmit antenna \( j \) to receive antenna \( i \); \( \mathbf{n} \) is the \( R \)-dimensional received noise vector; \( s_D \) is the transmitted symbol of the desired user and \( \mathbf{w}_D \) is the \( T \)-dimensional weight vector associated with the transmit antenna array. We assume that the channel matrix \( \mathbf{H}_D \) is perfectly known at both the transmitter and receiver. For simplicity, we also assume \( |s_D| = 1 \), although this assumption is easily relaxed.

In MIMO MRC systems without CCI, the combiner SNR is maximized with appropriate weighting at the transmitter and receiver. These weights, derived in [1] and [2], are given by \( \mathbf{w}_t = \sqrt{\Omega_0} \mathbf{u} \) and \( \mathbf{w}_r = \mathbf{H}_D^H \mathbf{u} \), where \( \sqrt{\Omega_0} \) is the norm of vector \( \mathbf{w}_t \), \( \mathbf{u} \) is a unitary \((||\mathbf{u}|| = 1)\) eigenvector corresponding to the largest eigenvalue \( \lambda \) of the matrix \( \mathbf{F} = \mathbf{H}_D^H \mathbf{H}_D \), and \( (\cdot)^H \) is the Hermitian operator. Thus, the received SNR, which we denote as \( \beta \), is a random variable (RV) given by

\[
\beta = \frac{\Omega_0 \lambda}{\sigma_n^2} = \overline{\beta} \lambda,
\]

where \( \overline{\beta} = \Omega_0/\sigma_n^2 \) is the average SNR per receive antenna.
Defining $z = \min(T, R)$, $t = \max(T, R)$ and $g(i, j) = t - z + i + j - 2$, from [6, eqs. (29)-(33)] the cdf of $\lambda$ can be written in a compact form as

$$F_\lambda(x) = \sum_{\sigma \in S_z} \sum_{k=0}^{t} \Phi_{sk} \sum_{\tau(z, k) \geq l=0} C_{l,z} x^l e^{-kx}, \quad (3)$$

where $\tau(k, z)$ is the set of $z$-tuples such that $\tau(k, z) = \{(k_1, \ldots, k_z) : k_i \in \{0, 1\}, \sum_{i=1}^{z} k_i = k\}$, $\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(z)\}$, $S_z$ denotes the set of the $z$! permutations of the integers $\{1, 2, \ldots, z\}$, $\delta(\sigma)$ denotes the sign of the permutation $\sigma$ and where

$$U_z = \sum_{i=1}^{z} k_i g(i, \sigma(i)), \quad C_{l,z} = \prod_{i=1}^{l} \frac{R}{(l+i)\Gamma(l+i+1)},$$

with $\omega(l, z)$ the set of $z$-tuples such that $\omega(l, z) = \{(l_1, \ldots, l_z) : l_i \in \{0, 1, \ldots, k_i g(i, \sigma(i))\}, \sum_{i=1}^{z} l_i = l\}$ and $\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-yt}dy$ denotes the Gamma function.

### B. Error Probability

The ASER can be calculated averaging the conditional error probability (CEP), i.e., the error rate under AWGN, over the output SNR, that is

$$\overline{P}_e = \int_0^{\infty} P E(x) f_\beta(x) dx, \quad (4)$$

where $P E(x)$ denotes the CEP. It is also possible to calculate the average error rate in terms of the cdf of $\lambda$. Considering the equality

$$F_\beta(x) = \Pr(\beta < x) = \Pr(\lambda < x/\beta) = F_\lambda(x/\beta), \quad (5)$$

and integrating (4) by parts, the ASER can be computed as

$$\overline{P}_e = - \int_0^{\infty} P'_E(x) F_\lambda(x/\beta) dx, \quad (6)$$

where $P'_E(x)$ is the first order derivative of the CEP.

General rectangular ($M_1 \times M_2$) QAM constellations can be generated from two independent $M$-ary PAM signals, and the CEP is given by [7], [8]

$$P_e(x) = 2pQ(\sqrt{ax}) + 2qQ(\sqrt{bx}) - 4pqQ(\sqrt{ax})Q(\sqrt{bx}), \quad (7)$$

Introducing now (3) and (8) into (6), an expression of the ASER is obtained as

$$\overline{P}_e = \frac{1}{\sqrt{2 \pi}} \sum_{\sigma \in S_z} \sum_{k=0}^{t} \Phi_{sk} \sum_{\tau(z, k) \geq l=0} C_{l,z} \beta^{-l} \times \left( p\sqrt{\alpha} \Gamma(l+1/2) \Gamma(l+1/2) \left( 1 - 2q\Gamma(l+1/2) (1 - 2q\Gamma(l+1/2)) \right) \right)$$

Now, (10) can be calculated as [9, eq. (5.1.3)]

$$\overline{I}(m, \alpha, x) = \frac{\sqrt{c(x)}}{2(1+c)m^{1/2}} \frac{\Gamma(m+1/2)}{\Gamma(m+1)}$$

$$\times_2 F_1 \left( 1, m + \frac{1}{2}; m + 1; \frac{1}{1+c} \right),$$

for $c = \beta/2\alpha$ and where $_2 F_1$ is the Gauss hypergeometric function, defined as

$$_2 F_1(a, b; c; x) = \sum_{k=1}^{\infty} \frac{(a)_k(b)_k x^k}{(c)_k k!},$$

### III. SER ANALYSIS WITH CCI

#### A. System model

The model of last section is now generalized to include $L$ interference signals. Thus, the $R \times 1$ received vector $r$ can be modeled as

$$r = H_D w_D s_D + \sum_{i=1}^{L} \sqrt{\Omega_i} h_i s_i + n,$$
where $s_i$ is the transmitted symbol of the $i^{th}$ interferer; $h_i$ is the $R \times 1$ complex Gaussian vector representing the normalized complex gain of the channel from the $i^{th}$ interferer to the receive antenna array; and $\Omega_i$ is the mean power of the $i^{th}$ interferer at each receiving antenna. We also assume $|s_j| = 1 \forall j, 1 \leq j \leq L$. The rest of terms and assumptions in (15) are the same as the ones in Section II.A.

The received signal-to-interference-plus-noise ratio (SINR) of a RV, denoted as $\beta$, which cdf can be calculated as [12]

$$
F_\beta (\beta) = \int_0^\infty F_\lambda \left( \frac{\beta}{\lambda_0} \left( x + \sigma_n^2 \right) \right) \frac{n_i - j}{(n_i - j)!} \Omega_i^{n_i - j + 1} dx,
$$

where the coefficients $A_{ij}$ are computed as in [12, eq. 26]).

Introducing (3) in (16) and solving the integral results, after some manipulation

$$
F_\beta (\beta) = \sum_{\sigma \in S_k} \sum_{\tau(k,z)} U_{\tau(k,z)} \sum_{I=0}^{m=0} \sum_{J=1}^{J=1} \sum_{I=1}^{I=1} \sum_{J=1}^{J=1} C_{I,J} A_{ij}
$$

$$
\times \left( \frac{1}{m} \right) \frac{(n_i - j + m)! \Omega_i^{m+1} \sigma_n^{2(m-1)}}{(n_i - j)! \Omega_0^m \sigma_n^{2j-1}} \beta^m e^{-\frac{\beta}{\sigma_n^2}} \left( 1 + \lambda \frac{\beta}{\lambda_0} \right)^{n_i - j + m + 1}
$$

(17)

The probability of outage is often defined as the probability that the instantaneous received SINR falls below a predefined threshold. Note that, with this definition, (17) is actually a novel expression for the probability of outage of MIMO MRC with an arbitrary number of transmit and receive antennas and in the presence of Rayleigh interferers with arbitrary powers. Previous expressions of the outage probability in similar system models (e.g. [12, eq. 22]) are based on coefficients which must be numerically evaluated. In contrast, in (17) all the terms in the summation are explicitly provided.

B. Error Probability

Applying the central limit to the total interference, the ASER in the presence of interference can be calculated as

$$
\mathcal{T}_E = -\int_0^\infty P'_E(x) F_\beta (x) dx.
$$

(18)

Introducing (8) and (17) into (18), an expression of the ASER is obtained as

$$
\mathcal{T}_E = \sum_{\sigma \in S_k} \sum_{\tau(k,z)} U_{\tau(k,z)} \sum_{I=0}^{m=0} \sum_{J=1}^{J=1} \sum_{I=1}^{I=1} \sum_{J=1}^{J=1} C_{I,J} A_{ij}
$$

$$
\times \left( \frac{1}{m} \right) \frac{(n_i - j + m)! \Omega_i^{m+1} \sigma_n^{2(m-1)}}{(n_i - j)! \Omega_0^m \sigma_n^{2j-1}} \beta^m e^{-\frac{\beta}{\sigma_n^2}} \left( 1 + \lambda \frac{\beta}{\lambda_0} \right)^{n_i - j + m + 1}
$$

$$
\times \int_0^\infty \left( P_1(x) + P_2(x) \right) x^j e^{-\frac{x^2}{2\sigma_n^2}} \left( 1 + \lambda \frac{\beta}{\lambda_0} \right)^{n_i - j + m + 1} dx.
$$

(19)

The integral in (19) can be computed fast and as accurately as desired using the generalized Laguerre-Gauss method, which is based on the generalized Laguerre polynomials [13, ch. 4.6]. Thus, after some manipulation, we can finally write

$$
\mathcal{T}_E \approx P_c (0) \sum_{\sigma \in S_k} \sum_{\tau(k,z)} U_{\tau(k,z)} \sum_{I=0}^{m=0} \sum_{J=1}^{J=1} C_{I,J} A_{ij}
$$

$$
\times \left( \frac{1}{m} \right) \frac{(n_i - j + m)! \Omega_i^{m+1} \sigma_n^{2(m-1)}}{(n_i - j)! \Omega_0^m \sigma_n^{2j-1}} \beta^m e^{-\frac{\beta}{\sigma_n^2}} \left( 1 + \lambda \frac{\beta}{\lambda_0} \right)^{n_i - j + m + 1}
$$

$$
\times \left( P_1(x) + P_2(x) \right) x^j e^{-\frac{x^2}{2\sigma_n^2}} \left( 1 + \lambda \frac{\beta}{\lambda_0} \right)^{n_i - j + m + 1} dx.
$$

(20)

where $c_i = 1 + 1/2$, $d_{ijml} = -n_i + j - m + c_i, \alpha_{kia} = 2k\sigma_i^2 + m\Omega_i, \alpha_{kib} = 2k\sigma_i^2 + m\Omega_i, \gamma_{kib} = 2k\sigma_i^2 + m\Omega_i, N$ is the Gauss-Laguerre approximation order, $\gamma_{kia} = N(N+c_i-1)w_i$ with $y_i$ the $i^{th}$ zero of the generalized Laguerre polynomial $L_N^{c_i-1}$, and where we have used the confluent hypergeometric function of second kind, also known as the Tricomi function, defined as [14, eq. 9.221.4]

$$
U(c, d; s) = \frac{1}{\Gamma(c)} \int_0^\infty e^{-sx} x^{c-1} (1 + x)^{d-c-1} dx
$$

(21)

for positive values of $c$ and $s$.

Note that, in (20), $P_c (0) = 1 - 1/(M_I M_Q)$ from (7), and the symbol ’$\simeq$’ becomes ’$\approx$’ when $N \to \infty$.

IV. Numerical Results

Fig. 1 shows the ASER when CCI is not present for a $8 \times 4$-QAM system with three transmit antennas. Monte Carlo simulations are also presented, showing an excellent agreement with the theoretical results. As expected, this figure shows a decrease in the ASER as the number of diversity channels $(R \times T)$ increases. The figure shows the case $\eta = 1$, when the in-phase and quadrature decision distances are the same and the in-phase signals have $(M_I^2 - 1)/(M_Q^2 - 1)$ times the average energy of the quadrature signals. Results are also shown for $\eta = \sqrt{(M_I^2 - 1)/(M_Q^2 - 1)}$, which represents the case when the in-phase and quadrature signals have the same average energy. Finally, results are depicted for $\eta = (M_I^2 - 1)/(M_Q^2 - 1)$, in which case the quadrature signals have $(M_Q^2 - 1)/(M_Q^2 - 1)$ times the average energy of the in-phase signals. It is shown that the case $\eta = 1$ provides the best performance (minimum error), and the error increases as $\eta$ also increases. Nevertheless, in the high SNR regime, the number of diversity channels becomes dominant.
interferer. A comparison is provided when the system is limited by interference, with a total interference to noise ratio (INR) of 60 dB, and when the system is limited by noise, (INR=40 dB). Three different sizes of the receive array are shown. We can appreciate that at the low SINR regime, the noise-limited system has a worse behavior, just the opposite to the high SINR regime, as the MIMO MRC scheme is not actually able to counteract the effect of interference.

V. CONCLUSIONS

We have presented closed-form expressions for the ASER of a general rectangular QAM modulation system in MIMO MRC with Rayleigh fading and without restriction in the number of antennas. When no CCI is present, an exact result is given as a finite summation. In systems with both CCI and Gaussian noise, we have found an approximated expression as accurate as desired for the ASER, which shows an excellent agreement with Monte Carlo simulation data.

Fig. 2 shows results for a $8 \times 4$-QAM system when CCI and AWGN are present. We have chosen $N = 128$ for the Gauss-Laguerre quadrature, which has shown a good accuracy and fast convergence. Monte Carlo simulation has been carried out showing an excellent match with the Gauss-Laguerre approximation. Results are presented as a function of the average SINR per antenna, defined as $\text{SINR}_{\text{branch}} = \frac{\sum_{i=1}^{L_d} \Omega_i}{\sum_{i=1}^{L_d} \Omega_i + \sigma_i^2}$. We have considered $T = 3$ transmit antennas and unequal power interferers. In particular, we assume that there are $L_d = 3$ dominant interferers out of $L = 6$ total interferers. Every dominant interferer is assumed to be received with an average power $R_d = 40$ dB higher than a non-dominant interferer.

Fig. 1. ASER for $8 \times 4$-QAM, three transmit antennas ($T = 3$), different number of receive antennas and no CCI.

Fig. 2. ASER for $8 \times 4$-QAM with $T = 3$, $\eta = 1$ and different number of receive antennas. Comparison when the system is limited by interferers (6 interferes, three of them dominant, with $R_d=40$ dB, INR=60dB) or by noise (INR=40dB).

REFERENCES