Integral Representation of the Error and Asymptotic Error Bounds for Generalized Padé Type Approximants

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Abstract

In this paper we will give an integral representation of the error for the generalized Padé type approximants defined in [2]. We will deduce some asymptotic upper bounds on the error of sequences of these approximants. As applications, we will consider functions defined by their expansions in some families of classical orthogonal polynomials and obtain for the corresponding approximants some results on the speed of convergence. Finally we obtain some results on the asymptotic behaviour of the error of these approximants for generalized Stieljes functions.

1 Introduction.

1.1 Definition of the Generalized Padé-Type Approximants.

Let f be an analytic function defined on a set $A \subseteq \mathbb{C}$ by the series of functions

$$f(t) = \sum_{i=0}^{\infty} c_i g_i(t), \qquad t \in A.$$

Let $G(x,t)$ be a generating function of the family $\{g_i(t)\}_i$, that is, $G(x,t) = \sum_{i=0}^{\infty} x^i g_i(t)$. We define the linear functional c by its moments in the following way:

$$c(x^i) = c_i \qquad i \in \mathbb{N}\,.$$

Then formally we have $f(t) = c(G(x,t))$, $t \in A$, where the linear form c acts on the variable x (this will be the case along all the paper).

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Definition 1 *The generalized Padé type approximant of order n of the series f, $(0/n)_f^G$ – or in short GPTA – is defined in the following way [2]:*

- *we fix $t \in A$ and consider the polynomial $q_n(x,t)$ of degree less or equal n in x which satisfies the following interpolation conditions:*

$$L_i(q_n(x,t)) = L_i(G(x,t)) \quad i = 0, \cdots, n,$$

where the L_i are linear functionals acting on the variable x;

- *we replace G by its approximation q_n and we construct the approximant*

$$(0/n)_f^G(t) = c(q_n(x,t)) \quad n \in N\,.$$

From the definition of the interpolation polynomial we can write

$$q_n(x,t) = -\begin{vmatrix} 0 & 1 & x & \cdots & x^n \\ L_0(G(x,t)) & L_0(1) & L_0(x) & \cdots & L_0(x^n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_n(G(x,t)) & L_n(1) & L_n(x) & \cdots & L_n(x^n) \end{vmatrix} / D_n^{(0)}$$

with

$$D_n^{(0)} = \begin{vmatrix} L_0(1) & L_0(x) & \cdots & L_0(x^n) \\ L_1(1) & L_1(x) & \cdots & L_1(x^n) \\ \cdots & \cdots & \cdots & \cdots \\ L_n(1) & L_n(x) & \cdots & L_n(x^n) \end{vmatrix}$$

This enables us to represent the generalized Padé type approximant as a quotient of determinants in the following way:

$$c(q_n(x,t)) = -\begin{vmatrix} 0 & c_0 & c_1 & \cdots & c_n \\ L_0(G(x,t)) & L_0(1) & L_0(x) & \cdots & L_0(x^n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_n(G(x,t)) & L_n(1) & L_n(x) & \cdots & L_n(x^n) \end{vmatrix} / D_n^{(0)} =$$

$$\left(-\begin{vmatrix} 0 & c_0 & \cdots & c_n \\ L_0(\sum_{i=0}^n x^i g_i(t)) & L_0(1) & \cdots & L_0(x^n) \\ \cdots & \cdots & \cdots & \cdots \\ L_n(\sum_{i=0}^n x^i g_i(t)) & L_n(1) & \cdots & L_n(x^n) \end{vmatrix} - \begin{vmatrix} 0 & c_0 & \cdots & c_n \\ L_0(x^{n+1}G_n(x,t)) & L_0(1) & \cdots & L_0(x^n) \\ \cdots & \cdots & \cdots & \cdots \\ L_n(x^{n+1}G_n(x,t)) & L_n(1) & \cdots & L_n(x^n) \end{vmatrix} \right) / D_n^{(0)}$$

where $G_n(x,t) = \sum_{i=0}^\infty x^i g_{n+1+i}(t)$. So

$$c(q_n(x,t)) = c_0 g_0(t) + c_1 g_1(t) + \cdots + c_n g_n(t) + e_n(t) \text{ with}$$

$$e_n(t) = -\begin{vmatrix} 0 & c_0 & c_1 & \cdots & c_n \\ L_0(x^{n+1}G_n(x,t)) & L_0(1) & L_0(x) & \cdots & L_0(x^n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_n(x^{n+1}G_n(x,t)) & L_n(1) & L_n(x) & \cdots & L_n(x^n) \end{vmatrix} / D_n^{(0)},$$

which shows the main property of the GPTA, namely that the first $n+1$ terms of its expansion in the family $\{g_i(t)\}$ coincide with those of $f(t)$. We can write the approximant in the form

$$c(q_n(x,t)) = a_0 L_0(G(x,t)) + a_1 L_1(G(x,t)) + \cdots + L_n(G(x,t))$$

and we have

$$a_0 L_0(x^i) + a_1 L_1(x^i) + \cdots + a_n L_n(x^i) = c(x^i) \quad i = 0, \cdots, n.$$

If the linear functionals are defined by: $L_i(g) = g(x_i)$, $i \geq 0$, and the generating function is $G(x,t) = (1-xt)^{-1}$, then the approximants constructed are the Padé type approximants.

As $f(t) = c(G(x,t))$, we can write the error of the approximants in the form

$$r_n(t) = f(t) - (0/n)_f^G(t) = \begin{vmatrix} c(G(x,t)) & c_0 & c_1 & \cdots & c_n \\ L_0(G(x,t)) & L_0(1) & L_0(x) & \cdots & L_0(x^n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_n(G(x,t)) & L_n(1) & L_n(x) & \cdots & L_n(x^n) \end{vmatrix} / D_n^{(0)} =$$

$$\begin{vmatrix} c(x^{n+1}G_n(x,t)) & c_0 & c_1 & \cdots & c_n \\ L_0(x^{n+1}G_n(x,t)) & L_0(1) & L_0(x) & \cdots & L_0(x^n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_n(x^{n+1}G_n(x,t)) & L_n(1) & L_n(x) & \cdots & L_n(x^n) \end{vmatrix} / D_n^{(0)} = \sum_{i=0}^{\infty} d_i g_{n+1+i}(t),$$

with

$$d_i = \begin{vmatrix} c_{n+1+i} & c_0 & c_1 & \cdots & c_n \\ L_0(x^{n+1+i}) & L_0(1) & L_0(x) & \cdots & L_0(x^n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_n(x^{n+1+i}) & L_n(1) & L_n(x) & \cdots & L_n(x^n) \end{vmatrix} / D_n^{(0)}$$

In order to have an approximation of higher order (which corresponds to a generalization of Padé approximants or Padé type approximants of higher order) we need that

$$d_i = 0, \quad i = 0, \cdots, k \Leftrightarrow \left(\sum_{j=0}^{n} a_j L_j \right) (x^{n+1+i}) = c(x^{n+1+i}) = c_{n+1+i} \quad i = 0, \cdots, k.$$

We remark that these conditions are independent of the generating function $G(x,t)$.

The existence and unicity conditions for the GPTA have been studied in [2] and a program in Mathematica for the formal recursive computation of these approximants has been given in [5].

Some convergence results for sequences of GPTA corresponding to the two following types of linear functionals L_i were given in [4]:

1. $L_i(f) = f(x_i)$ (if the point is repeated, we consider the derivatives);

   or

2. $L_i(f) = \int_C f(z)\overline{p_i(z)}w(z)\,|dz|$, where $\{p_i(z)\}$ is the family of orthonormal polynomials on C with respect to the weight function $w(z)$.

Conditions on the generating function and on the linear functional c were proposed there in order to obtain convergence.

1.2 Construction of the generalized Padé type table.

Let us generalize the definition given in the previous section to the following case. Let f be given by

$$f(t) = \sum_{i=0}^{k} c_i g_i(t) + \sum_{i=k+1}^{\infty} c_i g_i(t).$$

We want to compute an approximant of the form

$$(k/n)_f^G(t) = \sum_{i=0}^{k} b_i g_i(t) + a_0 L_0(G(x,t)) + \cdots + a_n L_n(G(x,t))$$

for which the expansion in the series of $\{g_i(t)\}$ coincides with the one of $f(t)$ as far as possible. As

$$(k/n)_f^G(t) = \sum_{i=0}^{k} b_i g_i(t) + a_0 \sum_{i=0}^{\infty} L_0(x^i) g_i(t) + \cdots + a_n \sum_{i=0}^{\infty} L_n(x^i) g_i(t)$$

the order condition writes

$$\begin{cases} a_0 L_0(x^{k+1}) + a_1 L_1(x^{k+1}) + \cdots + a_n L_n(x^{k+1}) & = & c_{k+1} \\ \cdots & \cdots & \cdots \\ a_0 L_0(x^{k+1+n}) + a_1 L_1(x^{k+1+n}) + \cdots + a_n L_n(x^{k+1+n}) & = & c_{k+1+n} \end{cases}$$

a system of $(n+1)$ equations and $(n+1)$ unknowns which gives the a_i's. The b_i's follow immediately from

$$b_i = c_i - \sum_{j=0}^{n} a_j L_j(x^i) \quad i = 0, \cdots, k,$$

and so the expansion of $(k/n)_f^G(t)$ coincides with the one of $f(t)$ up to the order $k+n+1$.

In this way, we can construct a table of approximants $(k/n)_f^G$, $k \geq 0$, $n \geq 0$. Let us see now to which interpolation problem correspond these approximants. We set as before

$$x^{k+1} G_{k+1}(x,t) = G(x,t) - \sum_{i=0}^{k} x^i g_i(t) = \sum_{i=k+1}^{\infty} x^i g_i(t),$$

and let $q_n(x,t)$ be the polynomial of degree n satisfying the interpolation conditions

$$L_i(q_n(x,t)) = L_i(G_{k+1}(x,t)) \quad i = 0, \cdots, n.$$

We can approach $G(x,t)$ by $\left(\sum_{i=0}^{k} x^i g_i(t) + x^{k+1} q_n(x,t)\right)$ and so $f(t)$ by

$\sum_{i=0}^{k} c_i g_i(t) + c(x^{k+1} q_n(x,t)) =$

$$= \sum_{i=0}^{k} c_i g_i(t) - \begin{vmatrix} 0 & c_{k+1} & \cdots & c_{k+1+n} \\ L_0(x^{k+1} G_k(x,t)) & L_0(x^{k+1}) & \cdots & L_0(x^{k+1+n}) \\ \cdots & \cdots & \cdots & \cdots \\ L_n(x^{k+1} G_k(x,t)) & L_n(x^{k+1}) & \cdots & L_n(x^{k+1+n}) \end{vmatrix} / D_n^{(k+1)} =$$

$$\begin{aligned}
&= \sum_{i=0}^{k} c_i g_i(t) + \sum_{i=0}^{n} a_i L_i(x^{k+1} G_k(x,t)) = \\
&= \sum_{i=0}^{k} c_i g_i(t) + \sum_{i=0}^{n} a_i [L_i(G(x,t)) - \sum_{j=0}^{k} L_i(x^j) g_j(t)] = \\
&= \sum_{i=0}^{k} (c_i - \sum_{j=0}^{n} a_j L_j(x^i)) g_i(t) + \sum_{i=0}^{n} a_i L_i(G(x,t)) = \\
&= \sum_{i=0}^{k} b_i g_i(t) + \sum_{i=0}^{n} a_i L_i(G(x,t)) = (k/n)_f^G(t).
\end{aligned}$$

We can study the convergence properties of two types of sequences:

- n fixed, $k \to \infty$;
- k fixed, $n \to \infty$.

  For this case, the convergence properties are independent of k because for all k, the function $G_k(\bullet, t)$ has the same analytic properties than $G(\bullet, t)$. We can then consider, without loss of generality, $k = 0$.

In this paper we will be interested in this second problem. We will consider one type of linear functionals: $L_i(g) = g(x_i)$ (and the derivatives if some interpolation points coincide). We will begin by obtaining an integral representation of the error and then we will get somme error bounds for the sequences of generalized Padé type approximants $\left((0/n)_f^G(z)\right)_n$. We will consider particular choices for the generating function and obtain the corresponding convergence results.

2 Integral representation of the error.

Let us consider a function $f(z)$ in a domain $\mathcal{D} \subset \mathbb{C}$ given by a series expansion

$$\forall z \in \mathcal{D} \quad f(z) = \sum_{n=0}^{\infty} c_n g_n(z) \text{ with } \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = R \ , \ R > 1 \tag{1}$$

and let

$$G(x,z) = \sum_{i=0}^{\infty} x^i g_i(z) \tag{2}$$

be a generating function function for the family $\{g_n(z)\}$.

We associate to f a function g defined by the following power series

$$g(t) = \sum_{n=0}^{\infty} c_n t^n, \text{ which is analytic for } |t| < R.$$

We define the linear form c in the space $\mathcal{H}_\alpha$ of holomorphic functions in $D_{1/\alpha} = D(0, 1/\alpha)$ by:

$$\forall k \in \mathcal{H}_\alpha \quad c(k) = \frac{1}{2\pi i} \int_{|x|=r} g(x) k(1/x) \frac{dx}{x} \quad \alpha < r < R. \tag{3}$$

We have

$$c(x^n) = \frac{1}{2\pi i}\int_{|x|=r} g(x)\frac{dx}{x^{n+1}} = \frac{g^{(n)}(0)}{n!} = c_n \quad \forall n \geq 0.$$

Along all the paper, the functional c will act on the variable x.

Let $A \subset \mathcal{D}$ be a regular set such that, for $z \in A$, $G(\bullet, z) \in \mathcal{H}_\alpha$. In other words, if $D_{[z]}$ is the domain of analyticity of $G(\bullet, z)$, $\forall z \in A \;\; D_{[z]} \supset D(0, 1/\alpha)$. Then

$$\begin{aligned} c(G(x,z)) &= \frac{1}{2\pi i}\int_{|x|=r} g(x)G(1/x,z)\frac{dx}{x} = \\ &= \frac{1}{2\pi i}\int_{|x|=r}\sum_{n=0}^{\infty} g_n(z)\frac{g(x)}{x^{n+1}}dx = \frac{1}{2\pi i}\sum_{n=0}^{\infty} g_n(z)\int_{|x|=r}\frac{g(x)}{x^{n+1}}dx = \\ &= \sum_{n=0}^{\infty} c_n g_n(z) = f(z) \quad \forall z \in A \end{aligned}$$

by uniform convergence arguments. So, if $q_n(x,z)$ is the polynomial of degree n which interpolates $G(\bullet, z)$ in the set of points $\{z_{ni}\}_{i=0}^n$, we get

$$(0/n)_f^G(z) = c(q_n(x,z)) = \frac{1}{2\pi i}\int_{|x|=r} g(x)q_n(\frac{1}{x}, z)\frac{dx}{x}$$

and the error of the GPTA can be written

$$f(z) - (0/n)_f^G(z) = \frac{1}{2\pi i}\int_{|x|=r} g(x)\left[G(\frac{1}{x}, z) - q_n(\frac{1}{x}, z)\right]\frac{dx}{x} \qquad \forall z \in A.$$

In this case the speed of convergence of the GPTA can be measured by the speed of convergence of the interpolating polynomials:

$$\left|f(z) - (0/n)_f^G(z)\right| \leq M \max_{|x|=\frac{1}{r}} |G(x,z) - q_n(x,z)|.$$

More precisely, as the interpolation error can be written [3]

$$G(x,z) - q_n(x,z) = \frac{1}{2\pi i}\int_\Gamma \frac{(x-z_{n0})\cdots(x-z_{nn})}{(t-z_{n0})\cdots(t-z_{nn})}\frac{G(t,z)}{t-x}dt \qquad x \in \mathrm{Int}(\Gamma)$$

with Γ a simple, closed, rectifiable curve in $D(0, 1/\alpha)$, $z_{ni} \in \mathrm{Int}(\Gamma)$ (the interior of Γ) , $i = 0, \cdots, n$, $\forall n \in \mathrm{N}$, we obtain the following expression for the error of the generalized Padé type approximants:

$$e_n(z) = f(z) - (0/n)_f^G(z) = -\frac{1}{4\pi^2}\int_{|x|=r} g(x)\int_\Gamma \frac{(x^{-1}-z_{n0})\cdots(x^{-1}-z_{nn})}{(t-z_{n0})\cdots(t-z_{nn})}\frac{G(t,z)}{xt-1}dt\,dx \qquad (4)$$

if $D(0, 1/r) \subset \mathrm{Int}(\Gamma)$. So we obtain

$$|e_n(z)| \leq \frac{1}{4\pi^2}M(r)\frac{\max_{|x|=r}\prod_{i=0}^{n}\left|x^{-1}-z_{ni}\right|}{\min_{t\in\Gamma}\prod_{i=0}^{n}|t-z_{ni}|}$$

with $M(r)$ a quantity depending on r. Let us consider now sequences $\{z_{ni}\}_{i=0}^{n}$ of interpolation points satisfying

$$\lim_{n\to\infty} |(z-z_{n0})\cdots(z-z_{nn})|^{1/(n+1)} = \sigma(z) \qquad \forall z \in D(0,1/\alpha). \tag{5}$$

In (4), making the change of variable $y = x^{-1}$, and remarking that we can replace the curve $|y| = 1/r$ by any simple, closed and rectifiable curve γ such that $D(0,1/R) \subset \mathrm{Int}(\gamma)$, we get

$$\lim_{n\to\infty} |e_n(z)|^{1/(n+1)} \le \frac{\max\limits_{y\in\gamma} \sigma(y)}{\min\limits_{t\in\Gamma} \sigma(t)}, \tag{6}$$

for all Γ and γ satisfying the given conditions. We immediately obtain the following theorem:

Theorem 1 *Let $f(z) = \sum_{n=0}^{\infty} c_n g_n(z)$, $z \in \mathcal{D} \subset \mathbb{C}$ with $\lim_{n\to\infty} |c_n/c_{n+1}| = R$. Let $G(\bullet, z)$ be a generating function for the family $\{g_n(z)\}$. Let us suppose that for $z \in A \subset \mathcal{D}$ (a regular set), $G(\bullet, z)$ is analytic in $D_{1/\alpha}$. We consider sequences of interpolation points $\{z_{ni},\ i = 0,\cdots,n\}_{n\in\mathbb{N}} \subset D_{1/\alpha}$ satisfying (5). We define $C_\rho = \{z : \sigma(z) = \rho\}$, $Int(C_\rho)$ the interior of C_ρ and $Ext(C_\rho)$ the exterior of C_ρ. We suppose that C_ρ is a simple, closed, rectifiable curve and we set*

$$\begin{aligned} \rho_M &= \sup\left\{\rho : C_\rho \subset D_{1/\alpha};\quad z_{ni} \in Int(C_\rho) \forall i, n\right\} \\ \rho_m &= \inf\left\{\rho : C_\rho \subset Int(C_{\rho_M}),\quad C_\rho \subset Ext(D_{1/R})\right\}. \end{aligned}$$

Then

$$\forall z \in A \qquad \lim_{n\to\infty} \left|f(z) - (0/n)_f^G(z)\right|^{1/(n+1)} \le \frac{\rho_m}{\rho_M}.$$

Let us consider two different choices of sequences of interpolation points satisfying (5).

(a) Let $(z_i)_i$ be a sequence of points verifying

$$\lim_{n\to\infty} z_{nk+1} = \xi_1,\quad \lim_{n\to\infty} z_{nk+2} = \xi_2, \cdots, \lim_{n\to\infty} z_{nk+k} = \xi_k.$$

and let us set: $\forall n \in \mathrm{N} \quad z_{ni} = z_i$, $i = 0,\cdots,n$. Then

$$\sigma(z) = \lim_{n\to\infty} |(z-z_1)\cdots(z-z_n)|^{1/n} = |(z-\xi_1)\cdots(z-\xi_k)|^{1/k}$$

and so the set of points $\sigma(z) = r^k$ is a lemniscate.

Corollary 1 *Let f and $G(\bullet, z)$ satisfy the conditions of Theorem 1. We construct the sequence of GPTA corresponding to the following choice of interpolation points:*

$$z_{ni} = z_i \quad i = 0,\cdots,n \text{ with } \lim_{n\to\infty} z_n = 0.$$

Then we obtain

$$\lim_{n\to\infty} \left|f(z) - (0/n)_f^G(z)\right|^{1/(n+1)} \le \frac{\alpha}{R}.$$

Proof: in this case we easily obtain $\sigma(z) = |z|$, $C_\rho = D(0,\rho)$, $\rho_M = 1/\alpha$, $\rho_m = 1/R$ and the result follows from Theorem 1.

$\triangle$

(b) Let us consider the following definitions:

- μ a Borel finite measure on $\mathbb{C}$ with compact support $S(\mu)$= supp(μ);
- $\Omega = \Omega(\mu)$ the unbounded component of $\mathbb{C} \setminus S(\mu)$;
- $g_\Omega(z,\infty)$ the Green function with pole at infinity;
- $p_n(\mu, z)$ the sequence of orthonormal polynomials with respect to the measure μ.

Let us suppose that $\mu \in$ Reg [6], that is

$$\lim_{n\to\infty} |p_n(\mu,z)|^{1/n} = e^{g_\Omega(z,\infty)} \tag{7}$$

locally uniformly for $z \in \mathbb{C} \setminus \mathrm{Co}(S(\mu))$. We also say that the sequence $\{p_n(\mu,z)\}$ has *regular exterior asymptotic behaviour.*

If we choose the sequences of interpolation points $\{z_{ni}\}_{i=0}^n$ as the zeros of $p_{n+1}(\mu, z)$, for $n \in \mathbb{N}$, with $\{p_n(z)\}$ verifying (7) then condition (5) is satisfied. For some particular cases, we can easily compute the form of C_ρ. In fact, we have the following well-known result (see, for instance, [8]):

Proposition 1 *If $S(\mu) = [-1,1]$ then $g_\Omega(z,\infty) = \log\left|z + \sqrt{z^2-1}\right|$ and then $C_\rho = \mathcal{E}_\rho$ where $\mathcal{E}_\rho$ ($\rho > 1$) is the ellipse of foci $-1, 1$ and semi-axes $a = \frac{1}{2}(\rho + \rho^{-1})$, $b = \frac{1}{2}(\rho - \rho^{-1})$.*

So, if we choose $\{z_{ni}\}_{i=0}^n$ as the zeroes of Jacobi, Legendre or Tchebyshev polynomials, we can compute the values of ρ_m and ρ_M. That is what we are going to do in the following subsections, where we will consider some particular choices for the generating function.

2.1 Expansion in a Legendre Series.

If $\{P_n(z)\}$ are the Legendre polynomials then the generating function is

$$G(x,z) = \frac{1}{\sqrt{1-2xz+x^2}} = \sum_{n=0}^{\infty} x^n P_n(z).$$

If we fix $z \in \mathcal{E}_\rho$ ($\rho > 1$) defined as above, $G(\bullet, z)$ is analytic for $x \in D(0, 1/\rho)$, and so if we set $A = \mathcal{E}_\rho$, $G(\bullet, z) \in \mathcal{H}_{1/\rho}$.

Proposition 2 *Let f be given by its expansion in a Legendre series*

$$f(z) = \sum_{n=0}^{\infty} a_n P_n(z) \ \textit{with} \ \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} \quad (R > 1). \tag{8}$$

Let us fix ρ and ρ_* such that $\rho < R$, $\rho_* > \rho$ and choose the sequence of interpolation points in the following way

$$z_{ni} = y_{ni}/\rho_* \quad i = 0, \cdots, n, \tag{9}$$

where y_{ni} are the zeroes of the Tchebyshev polynomial of degree $n+1$, $T_{n+1}(z)$. Then the error of the corresponding sequence of GPTA has the following asymptotic behaviour

$$\lim_{n\to\infty} \left|f(z) - (0/n)_f^G(z)\right|^{1/n} \leq \frac{\rho}{R}\left(1+\sqrt{1+\left(\frac{R}{\rho_*}\right)^2}\right) \Big/ \left(1+\sqrt{1-\left(\frac{\rho}{\rho_*}\right)^2}\right) \quad \forall z \in \mathcal{E}_\rho. \tag{10}$$

Remark: a similar result is obtained if we choose $\{y_{ni}\}$ as the zeroes of the Legendre polynomials.

Proof:

The expansion (8) converges for $z \in \mathrm{Int}(\mathcal{E}_R)$. For $z \in \mathcal{E}_\rho$ with $\rho < R$ we have the integral representation

$$\forall z \in \mathcal{E}_\rho \quad c(G(x,z)) = \frac{1}{2\pi i}\int_{|x|=r} g(x)G(1/x,z)\frac{dx}{x}, \quad \rho < r < R$$

and, as for $\rho_* < \rho$, $1/\rho_* > 1/\rho > 1/r$, this representation is still true for $z \in \mathrm{Int}(\mathcal{E}_\rho)$. It is well-known that the zeroes of the Tchebyshev polynomials satisfy:

$$\sigma_T(y) = \lim_{n\to\infty} |T_n(y)|^{1/n} = \lim_{n\to\infty} |(y-y_{n0})\cdots(y-y_{nn})|^{1/(n+1)} = \delta/2 \text{ for } y \in \mathcal{E}_\delta.$$

So

$$\sigma(z) = \lim_{n\to\infty} |(z-z_{n0})(z-z_{n2})\cdots(z-z_{nn})|^{1/(n+1)} = \frac{1}{\rho_*}\sigma_T(\rho_* z) = \frac{1}{\rho_*}\frac{\delta}{2} \text{ for } \rho_* z \in \mathcal{E}_\delta.$$

In this case, C_α is given by

$$\begin{aligned} C_\alpha &= \{z : \quad \sigma(z) = \alpha\} = \{z : \rho_* z \in \mathcal{E}_{2\rho_*\alpha}\} = \\ &= \left\{z = x+iy : \quad x = \frac{1}{2\rho_*}[2\rho_*\alpha + \frac{1}{2\rho_*\alpha}]\cos\theta, \quad y = \frac{1}{2\rho_*}[2\rho_*\alpha - \frac{1}{2\rho_*\alpha}]\sin\theta\right\} \end{aligned}$$

We obtain that for $z \in C_\alpha$

$$\left(\alpha - \frac{1}{4\rho_*^2\alpha}\right)^2 \leq |z|^2 \leq \left(\alpha + \frac{1}{4\rho_*^2\alpha}\right)^2$$

Then the quantities ρ_M and ρ_m defined in Theorem 1 can be chosen satisfying

$$\begin{cases} \rho_M + \frac{1}{4\rho_*^2\rho_M} &= 1/\rho \\ \rho_m - \frac{1}{4\rho_*^2\rho_m} &= 1/R \end{cases}$$

So

$$\rho_M = \frac{1}{2\rho} + \frac{1}{2}\sqrt{\frac{1}{\rho^2} - \frac{1}{\rho_*^2}}, \quad \rho_m = \frac{1}{2R} + \frac{1}{2}\sqrt{\frac{1}{R^2} + \frac{1}{\rho_*^2}}. \tag{11}$$

The result follows from the application of Theorem 1.

$\triangle$

By the definition of the generalized Padé type approximants, we easily see that the general form of these approximants for a function given by its expansion in Legendre polynomials is

$$\begin{aligned}(0/n)_f^G(z) &= \sum_{i=0}^{n} A_i(\alpha_i + \beta_i z)^{-1/2} \text{ if the interpolation points are distinct} \\ &= \sum_{i=0}^{p} A_i \sum_{j=0}^{n_i} (\alpha_i + \beta_i z)^{-(2j+1)/2} P_{ij}(z) \text{ with } P_{ij} \text{ polynomial of degree } j\end{aligned}$$

where for $i = 0, \cdots, p$, z_{ni} is repeated n_i times.

2.2 Expansion in a Tchebychev series.

If $\{T_n(z)\}$ are the Tchebyshev polynomials, then the generating function will be

$$G(x,z) = \frac{1 - xz}{1 - 2xz + x^2} = \sum_{n=0}^{\infty} x^n T_n(z).$$

Fixing $z \in \mathcal{E}_\rho$, $G(\bullet, z)$ is analytic for $x \in D(0, 1/\rho)$ and so, for the speed of convergence of the sequence of GPTA of a function given by its Tchebyshev expansion we can obtain results equivalent to those of the previous section. We remark that in this case the general form of these approximants is a rational function.

2.3 Expansion in a Laguerre series.

Let us consider the function f given by

$$f(z) = \sum_{n=0}^{\infty} c_n L_n^{(\alpha)}(z) \quad \text{with } \Gamma(\alpha+1) C_n^{n+\alpha} c_n = \int_0^{\infty} e^{-x} f(x) L_n^{(\alpha)}(x) dx, \tag{12}$$

where $\left\{L_n^{(\alpha)}(z)\right\}$ are the Laguerre polynomials. We suppose that $\lim_{n\to\infty} \sup |c_n|^{1/n} = 1/R < 1$

For $z \in \mathbb{C} \setminus]0, +\infty[$ we have $\lim_{n\to\infty} n^{-1/2} \log \left|L_n^{(\alpha)}(z)\right| = 2\mathrm{Re}((-z)^{1/2})$ [7], which gives for the domain of convergence $\mathcal{D}$

$$\mathcal{D} = \left\{z : \quad \mathrm{Re}((-z)^{1/2}) \leq K \text{ (for some constant } K)\right\}$$

the interior of a parabola with focus in the origin. K depends on the value of R and an analog of the Cauchy-Hadamard formula holds.

The generating function of the Laguerre polynomials is

$$G(x,z) = \sum_{n=0}^{\infty} x^n L_n^{(\alpha)}(z) = (1-x)^{-1-\alpha} \exp\{\frac{-xz}{1-x}\},$$

which, for all $z \in \mathbb{C}$, is analytic in $D(0,1)$.

In this case, the general form of these approximants is a linear combination of exponential functions and functions of the form $x^i \exp(\beta_i x)$ (if some interpolation points are repeated), that is

$$(0/n)_f^G(z) = \sum_{i=1}^{q} p_i(z) e^{\beta_i x} \quad \text{with } p_i(z) \text{ polynomial of degree } n_i, \quad \sum_{i=1}^{q} n_i = n. \tag{13}$$

Let us now apply Theorem 1 with simple choices for the interpolation points $\{z_{ni}\}_{i=0}^n \subset D(0,1)$.

Proposition 3 *Let f be given by (12) satisfying $\lim_{n\to\infty} |c_n|^{1/n} = 1/R$ and let us consider the sequences of GPTA $(0/n)_f^G(z)$ corresponding to the following choices of interpolation points:*

1. $\lim_{n\to\infty} z_n = 0$. *Then*

$$\forall z \in \mathcal{D} \quad \lim_{n\to\infty} \left| f(z) - (0/n)_f^G(z) \right|^{1/(n+1)} \leq \frac{1}{R}.$$

2. *Let $\{z_{ni}\}_{i=0}^n$ be given by (9) with $\rho_* > 1$.*

$$\forall z \in \mathcal{D} \quad \lim_{n\to\infty} \left| f(z) - (0/n)_f^G(z) \right|^{1/n} \leq \frac{1}{R} \left(1 + \sqrt{1 + \left(\frac{R}{\rho_*}\right)^2} \right) / \left(1 + \sqrt{1 - \left(\frac{1}{\rho_*}\right)^2} \right).$$

Proof:
1) In this case, $\sigma(z) = |z|$ and $C_\rho = D(0,\rho)$. So $\rho_M = 1$ and $\rho_m = 1/R$, which gives the result by applying Theorem 1.
2) It is sufficient to see that in this case the quantities ρ_M and ρ_m are given by (11) with $\rho = 1$. $\triangle$

2.4 Expansion in a Hermite series.

Let us consider the function f given by its expansion in Hermite polynomials:

$$f(z) = \sum_{n=0}^{\infty} c_n H_n(z) \text{ with } \pi^{1/2} 2^n n! c_n = \int_{-\infty}^{+\infty} e^{-x^2} f(x) H_n(x) dx. \tag{14}$$

We suppose that $\lim_{n\to\infty} |c_n|^{1/n} = 1/R$.

For $z \in \mathbb{C} \setminus]-\infty, +\infty[$ we have $\lim_{n\to\infty} (2n)^{-1/2} \log \left\{ \frac{\Gamma(n/2+1)}{\Gamma(n+1)} |H_n(z)| \right\} = \mathrm{Im}(z)$ [7], which gives for the domain of convergence

$$\mathcal{D} = \{ z : \quad |\mathrm{Im}(z)| \leq K \text{ (for some constant } K) \}.$$

K depends on R and an analog of Cauchy-Hadamard formula holds.

The generating function for the Hermite polynomials is

$$G(x,z) = \sum_{n=0}^{\infty} x^n \frac{H_n(z)}{n!} = e^{2zx - x^2},$$

and so the general form of the generalized Padé type approximants is also of type (13).

As for all $z \in \mathbb{C}$, $G(\bullet, z)$ in an entire function of x, we have the integral representation

$$\begin{aligned} \forall z \in \mathbb{C} \quad c(G(x,z)) &= \frac{1}{2\pi i}\int_{|x|=r} g(x)G(\frac{1}{x}, z)\frac{dx}{x}, \quad r < R \\ &= f(z) \quad \forall z \in \mathcal{D}. \end{aligned}$$

Conditions of theorem 1 are satisfied for all $\alpha \geq 0$. So let us fix $\alpha > 0$ and consider the same choices for the $\{z_{ni}\}_{i=0}^{n}$ as in the previous section.

- $\lim_{n\to\infty} z_n = 0$, $z_n \in D(0, 1/\alpha) \quad \forall n$. Applying theorem 1 we get for the error bound

$$\lim_{n\to\infty} |e_n(z)|^{1/(n+1)} \leq \frac{\alpha}{R}.$$

- **Proposition 4** *Let $f(z)$ be a function given by (14) with $\lim_{n\to\infty} |c_n|^{1/n} = 1/R$. We consider the sequence of generalized Padé type approximants $\left((0/n)_f^G\right)_n$ corresponding to the sequences of interpolation points $\{z_{ni}\}_0^n$ satisfying*

  - *$\{z_{ni}\}_0^n$ are the zeroes of a family of orthogonal polynomials in $[-1, 1]$ with respect to a regular measure (for instance, Tchebychev, Legendre or Jacobi polynomials).*

  Then the sequence $((0/n)_f^G(z))_n$ converges to $f(z)$. More precisely, if we fix $\rho > 1$ satisfying $\frac{1}{2}(\rho - \rho^{-1}) > \frac{1}{R}$ we obtain

$$\lim_{n\to\infty} \left|f(z) - (0/n)_f^G(z)\right|^{1/(n+1)} \leq \frac{1}{2R\rho} + \frac{1}{\rho}\sqrt{\frac{1}{4R^2} + 1} \quad \forall z \in \mathcal{D}. \tag{15}$$

Proof:

By propositon 1 we know that for this choice of interpolation points we obtain $\sigma(z) = \rho$, $\forall z \in \mathcal{E}_\rho$. Applying Theorem 1 we get

$$\begin{cases} \rho_M = \rho \\ \rho_m - \frac{1}{\rho_m} = \frac{1}{R} \end{cases}$$

Taking $\rho_m = \dfrac{1}{2R} + \sqrt{\dfrac{1}{4R^2} + 1}$, (15) follows.

△

3 GPTA for generalized Stieltjes functions. Connection with the Baker-Gammel approximants.

Let us consider now the following class of functions

$$f(z) = \int_\Delta G(t, z)d\alpha(t), \tag{16}$$

where α is a bounded, nondecreasing function taking infinitely many different values on Δ, Δ is a compact interval of $\mathbf{R}^+$. For the special case $G(t,z) = (1+tz)^{-1}$, f becomes a Stieltjes series. If $G(t,z)$ is the generating function of the family $\{g_i(z)\}_{i=0}^{\infty}$ we formally have

$$f(z) = \int_\Delta \left(\sum_{i=0}^{\infty} t^i g_i(z) \right) d\alpha(t) = \sum_{i=0}^{\infty} \left(\int_\Delta t^i d\alpha(t) \right) g_i(z) = \sum_{i=0}^{\infty} c_i g_i(z).$$

Let us suppose that

$$\text{for } z \in C \text{ a compact region of } \mathbb{C}, G(\bullet, z) \text{ is analytic in } A = \mathbf{R}^+ \times]-a, a[. \tag{17}$$

Then by the Cauchy representation we can write

$$f(z) = \int_\Delta \frac{1}{2\pi i} \int_\Gamma \frac{G(x,z)}{x-t} dx d\alpha(t),$$

where Γ is a simple closed contour in A such that Δ lies in its interior. Then

$$f(z) = \frac{1}{2\pi i} \int_\Gamma G(x,z) \int_\Delta \frac{d\alpha(t)}{x-t} dx = \frac{1}{2\pi i} \int_\Gamma G(x,z) g(-1/x) \frac{dx}{x}, \tag{18}$$

where $g(x) = \displaystyle\int_\Delta \frac{d\alpha(t)}{1+xt}$ is a Stieltjes function.

A way of constructing approximants for functions of the form (16) which are a linear combination of $G(u_i, z)$ $(u_i \in A)$ and $g_i^*(z) = \dfrac{\partial^i}{\partial u^i} G(u,z) \mid_{u=0}$ has been proposed by Baker and Gammel. The Baker-Gammel approximants of f [1] consist of replacing g by their Padé approximants. We will obtain an approximant of the form

$$G^{[m+j/m]}(z) = \sum_{i=0}^{j} \beta_i g_i^*(z) + \sum_{i=1}^{m} \alpha_i G(u_i, z). \tag{19}$$

It is easily shown (see [1] for the details) that their expansion in terms of the $\{g_i(z)\}$ coincides with the one of f up to $2m+j$, and a convergence result can be deduced from the convergence results for the Padé approximants of a Stieltjes series. The construction of these approximants needs the computation of the poles and residues of the $[m+j/m]_f(z)$, which implies a rather amount of computations.

The approach studied in this paper - the generalized Padé type approximants - corresponds to, in (18), instead of replacing g by its Padé approximant, replacing $G(\bullet, z)$ by its interpolation polynomial. As we have seen in the previous sections, the general form of these approximants contains (19) for the following choice of interpolation points: u_i, $i = 1, \cdots, m$ and $u_0 = 0$ repeated $j+1$ times. The order of approximation in the expansion in terms of the $\{g_i(z)\}$ is $m+j$ and so less than the one corresponding to the Baker-Gammel approximants, but we get an important reduction in computations. We will now obtain some upper bounds on the asymptotic behaviour of the error of the GPTA for generalized Stieljes functions for particular choices of the interpolation points.

Theorem 2 *Let f be a function of the form (16) satisfying (17). Let us consider sequences of interpolation points $\{z_{ni}\}_{i=0}^{n} \subset A$ satisfying*

$$\lim_{n\to\infty}\left|\prod_{i=0}^{n}(z-z_{ni})\right|^{1/(n+1)} = \sigma(z) \quad \forall z \in A \tag{20}$$

Then the corresponding sequence of GPTA satisfies

$$\lim_{n\to\infty}\left|f(z)-(0/n)_f^G(z)\right|^{1/(n+1)} \leq \frac{\max_{t\in\Delta}\sigma(t)}{\min_{x\in\Gamma}\sigma(x)}$$

for all Γ a contour in A such that $\forall n \in N \quad \{z_{ni}\}_{i=0}^{n} \subset Int(\Gamma)$ and $\Gamma \cap \Delta = \emptyset$.

Proof:

Let $q_n(t,z)$ be the interpolation polynomial for $G(t,z)$ (z fixed) ; then we have

$$\begin{aligned} f(z)-(0/n)_f^G(z) &= \int_\Delta (G(t,z)-q_n(t,z))\,d\alpha(t) = \\ &= \frac{1}{2\pi i}\int_\Delta\int_\Gamma \frac{G(x,t)}{x-t}\frac{\prod_{i=0}^{n}(t-z_{ni})}{\prod_{i=0}^{n}(x-z_{ni})}dx\,d\alpha(t) \end{aligned}$$

by the Cauchy integral representation of the interpolation error. By applying Fubbini's theorem we get

$$f(z)-(0/n)_f^G(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G(x,z)}{\prod_{i=0}^{n}(x-z_{ni})}\left(\int_\Delta \frac{\prod_{i=0}^{n}(t-z_{ni})}{x-t}d\alpha(t)\right)dx. \tag{21}$$

Taking upper bounds we obtain

$$\left|f(z)-(0/n)_f^G(z)\right| \leq \frac{1}{2\pi}C_1C_2(z)\frac{\max_{t\in\Delta}\left|\prod_{i=0}^{n}(t-z_{ni})\right|}{\min_{x\in\Gamma}\left|\prod_{i=0}^{n}(x-z_{ni})\right|}$$

with $C_1 = |\int_\Delta d\alpha(t)|\max_{x\in\Gamma}(\mathrm{dist}(x,\Delta))^{-1}$ and $C_2(z) = |\int_\Gamma G(x,z)dx|$. The result follows from the property (20) on the $\{z_{ni}\}_{i=0}^{n}$.

$\triangle$

Theorem 3 *Let f be a function of the form (16) satisfying (17). Let $\{\pi_n(z)\}_{n\in N}$ be the sequence of orthonormal polynomials on Δ with respect to the measure $d\alpha(t)$. Let us construct the sequence of GPTA corresponding to the following sequences of interpolation points: $\forall n \in N \quad \{z_{ni}\}_{i=0}^{n}$ are the zeroes of $\pi_{n+1}(z)$. Then, if we set $\Omega = \mathbb{C}\backslash\Delta$ and $g_\Omega(z,\infty)$ the corresponding Green function, the error has the following asymptotic behaviour*

$$\forall z \in C \quad \lim_{n\to\infty}\left|f(z)-(0/n)_f^G(z)\right|^{1/2n} \leq \frac{1}{\displaystyle\min_{x\in\Gamma} e^{g_\Omega(x,\infty)}}$$

for all Γ such that

$$\Delta \subset Int(\Gamma), \quad \Gamma \subset A, \quad \Gamma \cap \Delta = \emptyset.$$

Proof:

The sequence $\{\pi_n(z)\}$ satisfies

$$\int_\Delta \pi_n(t)\pi_m(t)d\alpha(t) = \delta_{mn} \quad \forall n, m \in \mathrm{N}\ .$$

Using the orthogonality properties, (21) can be written

$$\begin{aligned} f(z) - (0/n)_f^G(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{G(x,z)}{\pi_n(x)}\left(\int_\Delta \frac{\pi_n(t)}{x-t}d\alpha(t)\right)dx = \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{G(x,z)}{\pi_n(x)^2}\left(\int_\Delta \frac{\pi_n(x)-\pi_n(t)}{x-t}\pi_n(t)d\alpha(t) + \int_\Delta \frac{\pi_n(t)^2}{x-t}d\alpha(t)\right)dx = \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{G(x,z)}{\pi_n(x)^2}\left(\int_\Delta \frac{\pi_n(t)^2}{x-t}d\alpha(t)\right)dx. \end{aligned}$$

We have $\int_\Delta \pi_n(t)^2 d\alpha(t) = 1$ and from the properties on the measure $d\alpha(t)$ and its support, it is well-known that [6]

$$\liminf_{n\to\infty} |\pi_n(z)|^{1/n} \geq e^{g_\Omega(z,\infty)} \qquad \forall z \in \Omega.$$

Taking upper bounds, the result follows.

$\triangle$

4 Conclusion.

As Padé and Padé type approximants provide good approximations for functions given by its in power series expansion, the generalized Padé type approximants studied in this paper enable us to construct approximations for functions given by their expansion in some family $\{g_i(z)\}_{i=0}^\infty$ for which we know a generating function $G(x,t)$. As recalled in the first section, these approximants are a linear combination of the functions $\{G(x_i,t)\}$ and $\left\{\frac{\partial^j}{\partial x^j}G(x_i,t)\right\}$, and the coefficients are very easily computed.

Based on the analytical properties of the generating function and choosing the sequences of interpolation points satisfying some conditions, we obtained a general result on the speed of convergence of the corresponding sequences of approximants. For the most common expansions - the expansion on the classical orthogonal polynomials - we got some upper bounds on the error of the GPTA corresponding to particular choices of the interpolation points. The results show that these approximants can have good convergence properties.

There are some interesting open problems under study:

- a family of functions $\{g_i(z)\}$ being given, compare for different choices of interpolation points the speed of convergence of the corresponding sequence of approximants and determine what is the best choice for obtaining the tastest convergence;
- for a function given by the series $f(z) = \sum_{i=0}^\infty c_i g_i(z)$, compare the speed of convergence of the sequence of partial sums $s_n(z) = \sum_{i=0}^\infty$ with the one of a sequence of GPTA and give sufficient conditions in order to obtain acceleration of convergence;

- study the convergence and acceleration properties of sequences of the form $\left((k/n)_f^G(z)\right)_k$ (with n fixed) defined in section **1.2**.

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