ASYMPTOTIC HARVESTING OF POPULATIONS IN RANDOM ENVIRONMENTS

ALEXANDRU HENING, DANG H. NGUYEN, SERGIU C. UNGUREANU, AND TAK KWONG WONG

Abstract. We consider the harvesting of a population in a stochastic environment whose dynamics in the absence of harvesting are described by the logistic Verhulst-Pearl model. Using ergodic optimal control, we find the optimal harvesting strategy which maximizes the asymptotic yield of harvested individuals. To our knowledge, ergodic optimal control has not been used before to study harvesting strategies. However, it is a natural framework because the optimal harvesting strategy will never be such that the population is harvested to extinction – instead the harvested population converges to a unique invariant probability measure.

When the yield function is the identity, we show that the optimal strategy has a bang-bang property: there exists a threshold $x^* > 0$ such that whenever the population is under the threshold the harvesting rate must be zero, whereas when the population is above the threshold the harvesting rate must be at the upper limit. We provide upper and lower bounds on the maximal asymptotic yield, and explore via numerical simulations how the harvesting threshold and the maximal asymptotic yield change with the growth rate, maximal harvesting rate or the competition rate. In the case when the yield function is strictly concave, we prove that the optimal harvesting strategy is continuous.

Contents

1. Introduction

Many species of animals like whales, elephant seals, bisons and rhinoceroses, are at risk of being harvested to extinction ([Gul71, RPLB81, LHW93, Pri06]). Excessive harvesting has already led to both local and gobal extinctions of species ([LES95]). In fact, a significant percentage of the endangered birds and mammals of the world are threatened by harvesting, hunting or other types of overexploitation ([LES95]), and there are similar problems for many

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species of fish ([HR04]). Therefore harvesting strategies have to be carefully chosen. After significant harvests, it takes time for the harvested population to get back to the pre-existing level. Moreover, the harvested population fluctuates randomly in time due to *environmental stochasticity*. As a result, an overestimation of the ability of the population to rebound can lead the harvester to overharvest the population to extinction ([LES95]). A less common but nevertheless important problem is an insufficient rate of harvesting. Because of instraspecific competition, the population is bounded in a specific environment, so an extraction rate that is too low would lead to a loss of precious resources. For the same reason, choosing an efficient extraction strategy for valuable species is important ([Kok01]).

We present a stochastic model of population harvesting and find the *optimal harvesting strategy* that mazimizes the *asmptotic yield* of harvested individuals. We consider a novel framework, the one of *optimal ergodic harvesting*. This is based on the theory of ergodic control ([ABG12]). In most stochastic models that exist in the literature, for example [LES95, AS98, LØ97], the population is either assumed to become extinct in finite time, or it can end up being harvested to extinction. In our framework, if the population goes extinct under some harvesting strategy, the asymptotic yield is 0 and therefore this strategy cannot be optimal. If one wants to ensure that harvested species are preserved, this framework is a natural candidate. Our aim is to present a theory of optimal harvesting that includes the risks of extinction from both environmental stochasticity and harvesting. We assume that the population is homogeneous and can be described by a one dimensional diffusion. The harvesting rate is assumed to be bounded, as infinite harvesting rates would imply an unlimited harvesting capacity, something that is clearly not realistic.

In most cases, environmental noise can be introduced in the system by transforming differential equations into stochastic differential equations (SDE). Such techniques require dealing with significant mathematical difficulties, but their use is not just a case of honoring generality. First, there are direct effects of stochasticity on the predictions of the model, and the parameters quantifying it show up in the results. Second, any realistic biological system will depend on environmental variables that are not, or cannot be, accounted for. The role of stochasticity is to ensure that the solutions proposed are robust to such omissions. For example, if avoiding extinction is important, deterministic models can give misleading solutions even when their parameters are corrected for noise ([Smi78]). The transformation to SDE works especially well when the environmental fluctuations are small and there is no chaos ([LES95]). We focus on models with environmental stochasticity and neglect the *demographic stochasticity* which arises from the randomness of birth and death rates of each indiviual of a population. Throughout the paper we assume that environmental stochasticity mainly affects the growth rate of the population (see [Tur77, BM77, MBHS78, Lei81, Bra02, Gar88, EHS15, ERSS13, SBA11, HN17a] for more details). For computational tractability and for clarity of exposition we look at a one-dimensional stochastic version of the logistic Verhulst-Pearl model. Nevertheless, our framework works for any model that can be written as a system of stochastic differential equations (satisfying some mild assumptions - see [ABG12]).

A major limitation of existing models in the literature is the dependence of the optimal solutions on parameters that are hard to quantify. For example, in [LES95] the level at which the population becomes extinct – the minimal viable population – must be assumed; without it the yields become infinite. In [AS98] the yield must be time discounted to avoid maximizing over yield infinities, and this requires providing a time value for resources. The minimal viable population is a difficult scientific question ([Sha81, TBB07]), and the time value of yields is a difficult economics and policy question, because it implies the comparison of the utility of present and future generations ([DS87]). In contrast, our model sidesteps the issue by assuming no time preference – and therefore no bias towards extracting in the present, and resolves the problem of maximizing over infinite yields naturally by looking at asymptotic behavior.

The rest of the paper is organized as follows. In Section 2 we introduce our model and results. We prove that, if the population in the absence of harvesting survives, the yield function is the identity and the harvesting rate is bounded above by some number $M > 0$, then the optimal strategy is always a bang-bang type solution: there exists an $x^* > 0$ such that one does not harvest if the current population size lies in the interval $[0, x^*]$ and harvests at the maximal possible rate, M, if the current population size lies in the interval (x^*, ∞). In Section 3 we offer some numerical simulations that show how the optimal harvesting strategies and optimal asymptotic change with respect to the parameters of the model. We also provide a discussion of our results. The proofs are collected in Appendix A. Finally, in Appendix B we show that if the yield function is strictly concave then the optimal harvesting strategy has to be continuous, in contrast to the bang-bang type optimal strategy we find when the yield function is the identity. This result is a generalization of one of the stochastic models in [Smi78], where the equivalent to our yield function has a specific simple form. This generalization is useful for economic welfare analysis (a more general form of cost-benefit analysis), which typically relies on a concave utility function (well known, but see [MCWG+95, Proposition 6.C.1] for justification), equivalent to the concave yield function herein.

2. Optimal ergodic harvesting

We consider a population whose density $\tilde{X}(t)$ at time $t \geq 0$, in the absence of harvesting, satisfies the stochastic differential equation (SDE)

$$d\tilde{X}(t) = \tilde{X}(t)(\mu - \kappa\tilde{X}(t))\,dt + \sigma\tilde{X}(t)\,dB(t),\ \tilde{X}(0) = x > 0, \tag{2.1}$$

where $(B(t))_{t\geq 0}$ is a standard one dimensional Brownian motion. This describes a population $\tilde{X}$ with per-capita growth rate $\mu > 0$ whose members compete for resources according to the intracompetition rate $\kappa > 0$. The inifnitesimal variance of fluctuations in the per-capita growth rate is given by σ^2.

The behavior of (2.1) is well-known. The process $\tilde{X}$ does not reach 0 or ∞ in finite time and the stochastic growth rate $\mu - \frac{\sigma^2}{2}$ determines the long-term behavior in the following way (see [EHS15, DP84]):

- If $\mu - \frac{\sigma^2}{2} > 0$ and $\tilde{X}(0) = x > 0$, then $(\tilde{X}(t))_{t\geq 0}$ converges weakly to its unique invariant probability measure ν on $(0, \infty)$.
- If $\mu - \frac{\sigma^2}{2} < 0$ and $\tilde{X}(0) = x > 0$, then $\lim_{t\to\infty} \tilde{X}(t) = 0$ almost surely.
- If $\mu - \frac{\sigma^2}{2} = 0$ and $\tilde{X}(0) = x > 0$, then almost surely $\liminf_{t\to\infty} \tilde{X}(t) = 0$, $\limsup_{t\to\infty} \tilde{X}(t) = \infty$ and $\lim_{t\to\infty} \frac{1}{t}\int_0^t \tilde{X}(s)\,ds = 0$.

We let $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_{++} := (0, \infty)$ throughout the paper.

Assume that the population is harvested at time $t \geq 0$ at the rate $h(t) \in U := [0, M]$ for some fixed $M > 0$. Adding the harvesting to (2.1) yields the SDE

$$dX(t) = X(t)(\mu - \kappa X(t) - h(t))\,dt + \sigma X(t)\,dB(t),\ X(0) = x > 0. \tag{2.2}$$

A stochastic process $(h(t))_{t\geq 0}$ taking values in U is said to be an *admissible strategy* if $(h(t))_{t\geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the Brownian motion $(B(t))_{t\geq 0}$. Let $\mathfrak{U}$ be the class of admissible strategies. An important subset of $\mathfrak{U}$ is the class $\mathfrak{U}_{sm}$ of *stationary Markov strategies*, that is, admissible strategies of the form $h(t) = v(X(t))$ where $v : \mathbb{R}_{++} \mapsto U$ is a measurable function. By abuse of terminology, we often refer to the map $v(\cdot)$ as the stationary Markov strategy. Using a stationary Markov strategy $v(\cdot)$, (2.2) becomes

$$dX(t) = X(t)(\mu - \kappa X(t) - v(X(t)))\,dt + \sigma X(t)\,dB(t),\ X(0) = x > 0. \tag{2.3}$$

Remark 2.1. *The sigma algebra $\mathcal{F}_t$ gives one the information available from time 0 to time t. An admissible harvesting strategy is therefore a strategy which can take into account all the information from the start of the harvesting to the present. These strategies are much more general than constant strategies. Stationary Markov strategies are the harvesting strategies which only depend on the present state of the population density.*

We associate with $X(t)$ the family of generators $(\mathcal{L}_u)_{u\in[0,M]}$ defined by their action on C^2 functions with compact support in $\mathbb{R}_{++}$ as

$$\mathcal{L}_u f(x) := x[\mu - \kappa x - u]f_x + \frac{1}{2}\sigma^2 x^2 f_{xx}. \tag{2.4}$$

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function satisfying $\Phi(0) = 0$ and suppose Φ has a subpolynomial growth rate. We call Φ from now on the *yield function*. Our aim is to find the optimal strategy $h(t)$ that almost surely maximizes the *asymptotic yield*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \Phi\Big(X(t)h(t)\Big)\,dt. \tag{2.5}$$

In other words we want to find v such that, for any initial population size $X(0) = x > 0$, we have with probability 1 that

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \Phi\Big(X(t)v(X(t))\Big)\,dt = \sup_{h\in\mathfrak{U}} \liminf_{T\to\infty} \frac{1}{T}\int_0^T \Phi\Big(X(t)h(t)\Big)\,dt =: \rho^*.$$

We note that many of the existing models that look at the optimal harvesting of a population in a stochastic environment ([LØ97, AS98, LES95]) assume that the yield function Φ is the identity i.e. $\Phi(x) = x, x \geq 0$. This assumption is not always justifiable (see [Alv00]) and as such we present in Appendix B results for more general functions Φ.

Remark 2.2. *We note that if X has an invariant probability measure π on $\mathbb{R}_{++}$, then for any $X(0) = x > 0$ almost surely*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi\Big(X(t)v(X(t))\Big)\,dt = \int_{\mathbb{R}_{++}} \Phi(xv(x))\pi(dx).$$

In particular, if X goes extinct, that is, with probabiliy 1

$$\lim_{t\to\infty} X(t) = 0,$$

then the only invariant ergodic measure of X on $\mathbb{R}_+$ is δ_0 the point mass at 0, and hence, we get that with probability 1

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi\Big(X(t)v(X(t))\Big)\,dt = 0.$$

Our method for maximizing the asymptotic yield forces the optimal harvesting to be such that the population persists.

Remark 2.3. *By* [ABG12, Theorems 2.2.2 and 2.2.12], *the controlled systems* (2.2) *and* (2.3) *have unique local solutions on* $\mathbb{R}_{++}$ *for any admissible control* $h(t)$ *and stationary Markov control* v *respectively. Since*

$$\begin{aligned}\mathcal{L}_u\left(x+\frac{1}{x}\right) =&\frac{\sigma^2-\mu+u}{x}+\kappa+x(\mu-\kappa x-u)\\ \leq&\kappa+(\sigma^2+M)\left(x+\frac{1}{x}\right),\, x\in\mathbb{R}_{++}, u\in U,\end{aligned}$$

one can use the arguments from [Kha12, Theorem 3.5] *to obtain the existence of global solutions on* $\mathbb{R}_{++}$ *of* (2.2) *and* (2.3). *In particular we get that*

$$\mathbb{P}_x\left(X(t)\in\mathbb{R}_{++}, t\geq 0\right)=1, x\in\mathbb{R}_{++}.$$

The main result of the paper is the following.

Theorem 2.1. *Assume that* $\Phi(x)=x, x\in(0,\infty)$ *and that the population survives in the absence of harvesting, that is* $\mu-\frac{\sigma^2}{2}>0$. *The optimal control* v *has the bang-bang form*

$$v(x)=\begin{cases}0 & \text{if } 0<x\leq x^*\\ M & \text{if } x>x^*\end{cases} \tag{2.6}$$

for some $x^*\in(0,\infty)$. *Furthermore, we have the following upper bound for the optimal asymptotic yield*

$$\rho^*\leq\frac{\mu^2}{4\kappa}. \tag{2.7}$$

Remark 2.4. *If we harvest according to a constant strategy* $\ell>0$ *the SDE* (2.3) *becomes*

$$dX(t)=X(t)(\mu-\kappa X(t)-\ell)\,dt+\sigma X(t)\,dB(t).$$

It is then easy to see that the asymptotic yield is

$$L(\ell):=\lim_{T\to\infty}\frac{1}{T}\int_0^T \ell X(t)\,dt=\ell\frac{\mu-\ell-\frac{\sigma^2}{2}}{\kappa}.$$

Then $L(\ell)$ *is maximized for*

$$\ell^*=\frac{1}{2}\left(\mu-\frac{\sigma^2}{2}\right)$$

and the maximal asymptotic yield is

$$L(\ell^*)=\frac{\left(\mu-\frac{\sigma^2}{2}\right)^2}{4\kappa}.$$

Note that $L(\ell^*)$ *is also called* ***maximum sustainable yield (MSY)*** *in the literature. Combining this with* (2.7) *we get that the optimal asymptotic yield* ρ^* *satisfies*

$$\frac{\left(\mu-\frac{\sigma^2}{2}\right)^2}{4\kappa}\leq\rho^*\leq\frac{\mu^2}{4\kappa}.$$

Remark 2.5. *If $v(x) = M$ for all $x \le \eta$ for some $\eta > 0$ and $\mu - \frac{\sigma^2}{2} - M < 0$ then one can see that X goes extinct almost surely. Therefore, by Remark 2.2 one gets that this cannot be the optimal harvesting strategy.*

Remark 2.6. *We note that a different approach to proving Theorem 2.1 would be to take the results from* [AS98]*, which are for the α-discounted problem where the goal is to maximize the expected total time-discounted value of the harvested individuals starting with a population of size x:*

$$V(x) = \sup_h \mathbb{E}_x \int_0^\infty e^{-\alpha s} h(X(t))X(t)\,dt$$

and then letting the discount factor $\alpha \to 0$.

Remark 2.7. *One might wonder when the optimal harvesting strategies are continuous. We show in Appendix B that we get continuous strategies when the yield function Φ is C^2, increasing, strictly concave, and has polynomial growth. However, we cannot find the exact form of the harvesting strategies in this case. Note that in Theorem 2.1 we have $\Phi(x) = x$ which is not strictly concave.*

Having the form (2.6) of the optimal harvesting strategy v in hand we next want to say something about the point x^* at which the harvesting strategy becomes nonzero. In order to achieve that we maximize over all the controls $w(\cdot, \eta)$ of bang-bang type

$$w(x;\eta) = \begin{cases} 0 & \text{if } 0 < x \le \eta \\ M & \text{if } x > \eta, \end{cases} \tag{2.8}$$

and find the η which maximizes the asymptotic yield. For a control w our diffusion (2.3) (with $h \equiv w$) is of the form

$$dX(t) = a(X(t))\,dt + b(X(t))\,dB(t) \tag{2.9}$$

for

$$a(x) = x(\mu - w(x,\eta) - \kappa x)$$

and

$$b(x) = \sigma x.$$

Standard diffusion theory shows (see [HK16, BS12]) that the boundary 0 is natural and the boundary ∞ is entrance for the process X from (2.9). As a result, when $\mu - \frac{\sigma^2}{2} > 0$, one can show using [BS12] that the density $\rho : (0,\infty) \to (0,\infty)$ of the invariant measure π is of the form

$$\begin{aligned} \rho(y) &= \frac{C_1}{b^2(y)} \exp\left(2\int_\eta^y \frac{a(z)}{b^2(z)}\,dz\right) \\ &= \frac{C_1}{\sigma^2 y^2} \exp\left(2\int_\eta^y \frac{z(\mu - w(z,\eta) - \kappa z)}{\sigma^2 z^2}\,dz\right) \\ &= \begin{cases} \frac{C_1}{\sigma^2 y^2}\left(\frac{y}{\eta}\right)^{\frac{2\mu}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)} & \text{if } 0 < y \le \eta \\ \frac{C_1}{\sigma^2 y^2}\left(\frac{y}{\eta}\right)^{\frac{2(\mu-M)}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)} & \text{if } y > \eta, \end{cases} \end{aligned} \tag{2.10}$$

where C_1 is a normalizing constant given by

$$\frac{1}{C_1} = \int_0^\eta \frac{1}{\sigma^2 y^2}\left(\frac{y}{\eta}\right)^{\frac{2\mu}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)}\,dy + \int_\eta^\infty \frac{1}{\sigma^2 y^2}\left(\frac{y}{\eta}\right)^{\frac{2(\mu-M)}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)}\,dy.$$

In this case the harvesting yield is

$$\begin{aligned}
H(\eta) &:= \lim_{T\to\infty}\frac{1}{T}\int_0^T \Phi\Big(X(t)w(X(t),\eta)\Big)\,dt \\
&= \int_{\mathbb{R}_{++}} yw(y,\eta)\pi(dy) \\
&= \int_0^\infty yw(y,\eta)\rho(y)dy \\
&= \int_\eta^\infty yM\frac{C_1}{\sigma^2 y^2}\left(\frac{y}{\eta}\right)^{\frac{2(\mu-M)}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)}\,dy \qquad (2.11) \\
&= \frac{\displaystyle\int_\eta^\infty yM\frac{1}{\sigma^2 y^2}\left(\frac{y}{\eta}\right)^{\frac{2(\mu-M)}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)}\,dy}{\displaystyle\int_0^\eta \frac{1}{\sigma^2 y^2}\left(\frac{y}{\eta}\right)^{\frac{2\mu}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)}\,dy + \int_\eta^\infty \frac{1}{\sigma^2 y^2}\left(\frac{y}{\eta}\right)^{\frac{2(\mu-M)}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)}\,dy}
\end{aligned}$$

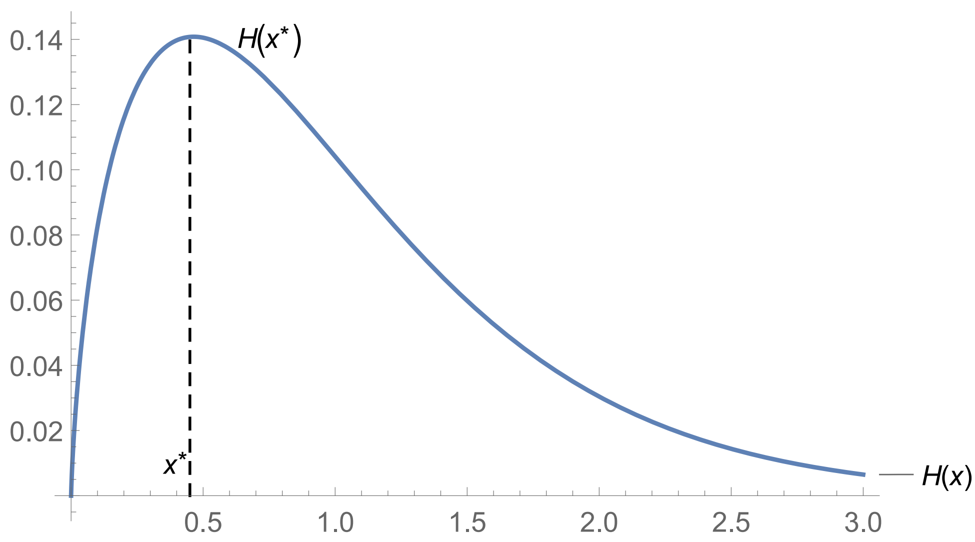

FIGURE 1. Typical shape of $H(x)$, here for $\sigma^2 = 1$ and $M = \mu = \kappa = 1$.

We can find x^* from (2.6) by maximizing the yield:

$$H(x^*) = \max_{\eta\in(0,\infty)} H(\eta).$$

It is clear that H is differentiable, that x^* exists and satisfies $x^* \in (0,\infty)$. As a result x^* is a solution of

$$H'(\eta) = 0. \qquad (2.12)$$

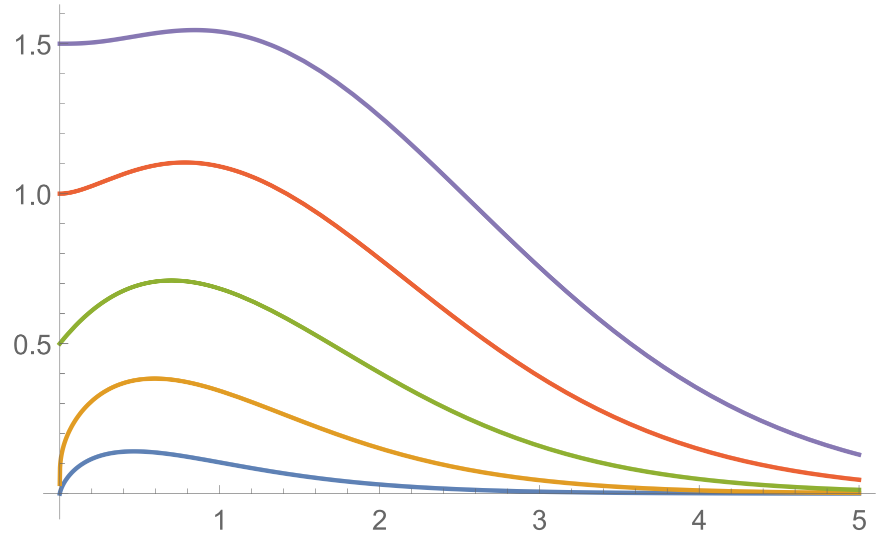

FIGURE 2. $H(\eta)$ for $\sigma^2 = 1$, $M = \kappa = 1$, and $\mu = 1$ (blue), $\mu = 1.5$ (orange), $\mu = 2$ (green), $\mu = 2.5$ (red), and $\mu = 3$ (purple).

The condition above can be restated as an equation involving incomplete gamma functions. We were not able to prove analytically that (2.12) has a unique solution. [BP06, BP08] show possible analytical methods that can be applied to such equations in a simple case. However, numerical experiments that we have done support this conjecture (Figure 1).

Conjecture 2.1. *There exists a unique $x^* \in (0, \infty)$ such that $H'(x^*) = 0$. Furthermore, the optimal harvesting strategy is given by*

$$v(x) = \begin{cases} 0 & \text{if } 0 < x \leq x^* \\ M & \text{if } x > x^*. \end{cases}$$

3. DISCUSSION AND FUTURE RESEARCH

We have analysed a population whose dynamics evolves according to the logistic Verhulst-Pearl model in a stochastic environment, but subjected to strategic harvesting. The rate at which the population gets harvested is bounded above by a constant $M > 0$, and the harvested infinitesimal amount is proportional to the current size of the population. We show that the harvesting strategy v, which describes the harvesting rate and is chosen to maximize the asymptotic harvesting yield

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T X(t)h(t)\, dt,$$

is of bang-bang type, i.e. there exists $x^* > 0$ such that

$$v(x) = \begin{cases} 0 & \text{if } 0 < x \leq x^* \\ M & \text{if } x > x^*. \end{cases}$$

Using this fact, the harvesting yield function $H(\eta)$ is determined, by letting the jump in the bang-bang control be at $x^* = \eta$. The typical behavior of the point x^* where H is

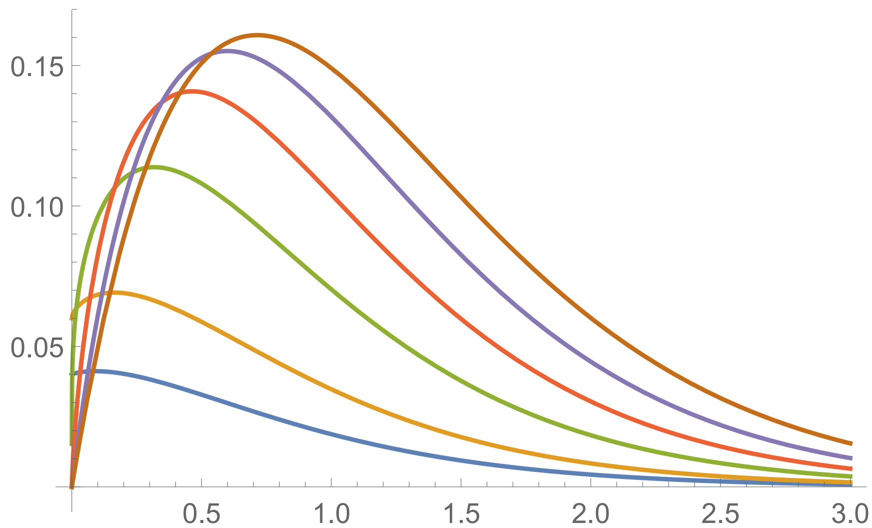

FIGURE 3. $H(\eta)$ for $\sigma^2 = 1$, $\mu = \kappa = 1$, and $M = 0.1$ (blue), $M = 0.2$ (orange), $M = 0.5$ (green), $M = 1$ (red), $M = 2$ (purple), and $M = 5$ (brown).

maximized and of $H(x^*)$ as the parameters μ, κ and M change was analyzed numerically and is presented in Figures 2, 3 and 4, with the normalization $\sigma^2 = 1$.

We note from numerical experiments that increasing the growth rate μ increases the threshold x^* at which one should start optimally harvesting (Figure 2). This is an intuitive result, since an increased growth rate increases the maximal equilibrium value of the population in the equivalent deterministic growth model with competition ([Smi78]). Therefore, it should also increase asymptotic harvesting yield, as well as the point at which harvesting should start. Moreover, higher growth rates make the population get faster to the point x^* where one starts harvesting, reducing the cost of a delay.

If one increases the maximal harvesting rate M then the harvesting threshold x^* is also increased (Figure 3). This also makes sense because if $\mu - \frac{\sigma^2}{2} - M < 0$, then a population with constant harvesting rate M will go extinct almost surely. An increase in the harvesting threshold x^* is necessary to make sure that there is no extinction. Moreover, as M gets larger one can wait longer to start harvesting. With a larger maximal rate available, there is less chance that there will be losses because the population overshoots the optimal extraction point. Similarly, increasing the harvesting rate M also increases the maximal asymptotic harvesting yield, for the obvious reason that there is better control on the population level and therefore extraction can happen closer to the optimal level.

In contrast, if one increases the intraspecific competition rate κ, then the harvesting threshold decreases (Figure 4). The equilibrium value of the population in the equivalent deterministic model ([Smi78]) decreases with κ, and as a result so does the extraction rate. Evidently, even in the stochastic model, if competition is very strong the population cannot spend much time at high densities, and therefore one has to start harvesting early. An increase in κ will also decrease the maximal asymptotic harvesting yield.

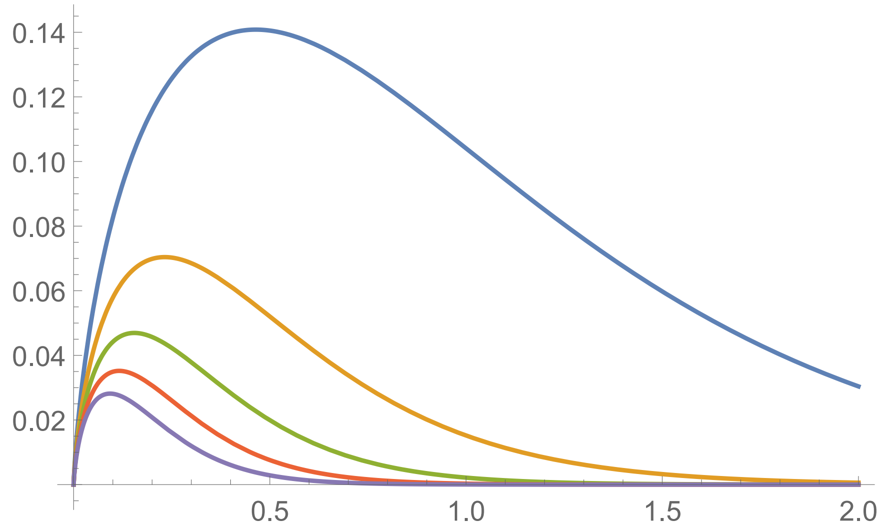

FIGURE 4. $H(\eta)$ for $\sigma^2 = 1$, $M = \mu = 1$, and $\kappa = 1$ (blue), $\kappa = 2$ (orange), $\kappa = 3$ (green), $\kappa = 4$ (red), and $\kappa = 5$ (purple).

We are able to prove that the maximal asymptotic yield ρ^* satisfies the inequality

$$\frac{\left(\mu - \frac{\sigma^2}{2}\right)^2}{4\kappa} \leq \rho^* \leq \frac{\mu^2}{4\kappa}.$$

In particular, the bang-bang optimal strategy has a higher asymptotic yield than the optimal constant harvesting strategy. Moreover, the bang-bang optimal strategy gives a lower asymptotic yield than the optimal constant harvesting strategy in the absence of noise. This means that the analysis of the more complex stochastic model was fruitful, recommending a qualitatively different strategy. Moreover, environmental fluctuations decrease the maximal asymptotic yield and, because the correction is negative, protecting a population from extinction requires a careful measurement of natural fluctuations when designing optimal harvesting. When environmental stochasticity was not taken into account, harvesting often lead populations to extinction ([LES95]).

Real populations do not evolve in isolation. As a result, ecology is concerned with understanding the characteristics that allow species to coexist. Harvesting can disturb the coexistence of species. In future research we intend to tackle multi-dimensional analogues of the setting treated in the current article. Natural models for which one can add harvesting would be predator-prey food chains ([GH79, Gar84, HN17b, HN17c, TL16]), more general Kolmogorov systems ([SBA11, HN17a]) and structured populations where there can be asymmetric harvesting ([ERSS13, EHS15, HNY17, RS14, BS09, SR11]). In the multi-dimensional setting the Hamilton-Jacobi-Bellman (HJB) equation becomes a PDE and the analysis becomes significantly more complex. New tools will have to be developed to tackle these problems.

Above we have imposed a bound on the extraction rate, M. This was because it is a realistic feature, but it was also practical for the analysis. Nevertheless, it is interesting to consider the case when the extraction rate is unbounded. A practical model with no extraction limit corresponds to having unlimited control over a target population, which is sometimes the case. Such a model would have the benefit of not requiring a nuisance parameter that may be hard to determine.

If we allow for general, possibly unbounded harvesting we would have to study the Skorokhod SDE

$$d\tilde{X}(t) = \tilde{X}(t)(\mu - \kappa\tilde{X}(t))\,dt + \sigma\tilde{X}(t)\,dB(t) - dZ_t,\ \ \tilde{X}(0) = x > 0. \tag{3.1}$$

where $(Z_t)_{t\geq 0}$ is supposed to be non-negative, increasing, right-continuous and adapted to $(\mathcal{F}_t)_{t\geq 0}$ - we denote the set of all such strategies by A. Then the problem is to maximize the asymptotic yield, i.e. find

$$V(x) = \sup_{(Z_t)_{t\geq 0}\in A} \liminf_{T\to\infty} \mathbb{E}_x \frac{1}{T}\int_0^T dZ_t = \sup_{(Z_t)_{t\geq 0}\in A} \liminf_{T\to\infty} \frac{\mathbb{E}_x Z_T}{T}$$

We want to find the harvesting strategy $(Z_t^*)_{t\geq 0} \in A$, which we call the optimal harvesting strategy, such that

$$V(x) = \liminf_{T\to\infty} \frac{\mathbb{E}_x Z_T^*}{T}.$$

The analysis above, for the bounded harvesting rate, determined that the optimal strategy has a bang-bang property, where extraction is maximal after some cut-off. This suggests that raising the maximum would not change the bang-bang property, but determining that result required a bounded extraction rate. Thinking of the limiting behavior of the yield function above shows the difficulty: as $M \to \infty$, the density of the distribution above the cut-off x^* goes to 0 (see (2.10)). The conjectured optimal solution is akin to having a reflective boundary at x^*, and the yield is determined by the time spent close to the boundary.

Conjecture 3.1. *Assume that the population survives in the absence of harvesting i.e.* $\mu - \frac{\sigma^2}{2} > 0$. *The optimal extraction strategy* $(Z_t^*)_{t\geq 0}$ *has the form*

$$Z_t^*(x) = \begin{cases} (x - x^*)^+ & \text{if } t = 0 \\ L(t, x^*) & \text{if } t > 0. \end{cases} \tag{3.2}$$

for some $x^* \in (0,\infty)$, *where* $L(t, x^*)$ *is the local time at* x^* *of the process* $\tilde{X}$ *from* (3.1).

This conjecture is supported by the results from [AS98] where the authors study the maximization of the discounted yield

$$V(x) := \sup_{(Z_t)_{t\geq 0}\in A} \mathbb{E}_x \int_0^\tau e^{-rt} dZ_t$$

and $\tau := \inf\{t \geq 0 : \tilde{X}_t = 0\}$ is the extinction time. It is shown in [AS98] that the optimal harvesting strategy is of the form (3.2). One possible approach to prove Conjecture 3.1 would be to use the results from [AS98] and then let the discount factor r go to 0.

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Appendix A. Proofs

In this appendix we present the framework of ergodic optimal control and prove the main results of our paper.

For any $v \in \mathfrak{U}_{sm}$, denote the unique invariant probability measure of $X(t)$ on $\mathbb{R}_{++}$ by π_v if it exists. Define

$$\rho_v = \begin{cases} \int_0^\infty \Phi(xv(x))\pi_v(dx) & \text{if } \pi_v \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

Let $p > 0$. One can see that there exist constants $k_{1p}, k_{2p} > 0$ such that

$$\mathcal{L}_u x^p \le px^p(\mu - \kappa x) + \frac{1}{2}p(p-1)\sigma x^p \le k_{1p} - k_{2p}x^{p+1},\ x \in \mathbb{R}_{++}, u \in [0, M] \tag{A.1}$$

By Dynkin's formula

$$\mathbb{E}_x^v[X(t)]^p \le x^p + k_{1p}t - k_{2p}\mathbb{E}_x^v \int_0^t [X(s)]^{p+1}ds.$$

Thus,

$$\frac{1}{t}\mathbb{E}_x^v \int_0^t [X(s)]^{p+1}ds \le \frac{1}{k_{2p}}\left(\frac{x^p}{t} + k_{1p}\right). \tag{A.2}$$

As a result, the family of occupation measures

$$\Pi_{x,t}^v(\cdot) := \frac{1}{t}\int_0^t \mathbb{P}_x^v\{X(s) \in \cdot\}\, ds,\ \ t \ge 1$$

is tight. If $X(t)$ has an invariant probability measure on $\mathbb{R}_{++}$, then $\left(\Pi^v_{x,t}\right)_{t\geq 0}$ converges weakly to π_v because the diffusion is nondegenerate. This convergence and the uniform integrability (A.2) imply that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi(X(s)v(X(s)))ds = \rho_v.$$

If $X(t)$ has no invariant probability measures on $\mathbb{R}_{++}$, then the Dirac measure with mass at 0 is the only invariant probability measure of $X(t)$ on $\mathbb{R}_+$. Moreover, any weak-limit of $\left(\Pi^v_{x,t}\right)_{t\geq 0}$ as $t\to\infty$ is an invariant probability measure of $X(t)$ ([EK09, Theorem 9.9] or [EHS15, Proposition 8.4]). Thus, $\left(\Pi^v_{x,t}\right)_{t\geq 0}$ converges weakly to the Dirac measure δ_0 as $t\to\infty$. Because of (A.2) and $\Phi(0)=0$, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi(X(s)v(X(s)))ds = \int_0^\infty \Phi(xv(x))\pi_v(dx).$$

Thus, we always have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi(X(s)v(X(s)))ds = \rho_v. \tag{A.3}$$

Define

$$\rho^* := \sup_{v\in\mathfrak{U}_{sm}} \{\rho_v\}. \tag{A.4}$$

It will be shown later that $\rho^*>0$ whenever the population without harvesting persists, i.e. when $\mu-\sigma^2/2>0$.

Theorem A.1. *Suppose $\mu-\sigma^2/2>0$. There exists a stationary Markov strategy $v^*\in\mathfrak{U}_{sm}$ such that π_{v^*} exists and $\rho_{v^*}=\rho^*$. Moreover, for any admissible control $h(t)$, we have*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \Phi\Big(X(t)h(t)\Big)\,dt \leq \rho_{v^*}=\rho^* \ \text{a.s.}$$

Proof. By (A.2) and since Φ has a polynomial growth rate we can conclude that

$$\sup_{v\in\mathfrak{U}_{sm}} \int_0^\infty \Phi(xv(x))\pi_v(dx) < \infty. \tag{A.5}$$

Moreover, since $\mu-\sigma^2/2>0$ we note by Remark 2.4 that $\rho^*>0$. On the other hand, since Φ is continuous and $\Phi(0)=0$ we get that $\Phi(x)<\rho^*$ for x is sufficiently small. This fact combined with (A.5) implies the existence of an optimal Markov strategy v^* according to [ABG12, Theorem 3.4.5, Theorem 3.4.7]. □

Theorem A.2. *The HJB equation*

$$\max_{u\in U}\Big[\mathcal{L}_u V(x)+\Phi(xu)\Big]=\rho \tag{A.6}$$

admits a classical solution $V^*\in C^2(\mathbb{R}_+)$ satisfying $V^*(1)=0$ and $\rho=\rho^*>0$. The solution V^* of (A.6) has the following properties:

a) *For any $p\in(0,1)$*

$$\lim_{x\to\infty}\frac{V^*(x)}{x^p}=0. \tag{A.7}$$

b) *The function V^* is increasing, that is*

$$V^*_x \geq 0,\ x \in \mathbb{R}_{++}. \tag{A.8}$$

A Markov control v is optimal if and only if it satisfies

$$\frac{dV^*}{dx}(x)\Big[x(\mu - v(x) - \kappa x)\Big] + \Phi(xv(x)) = \max_{u\in U}\left(\frac{dV^*}{dx}(x)\Big[x(\mu - u - \kappa x)\Big] + \Phi(xu)\right) \tag{A.9}$$

almost everywhere in $\mathbb{R}_+$.

Proof. Consider the optimal problem with the yield function

$$J_h(x) = \mathbb{E}_x \int_0^\infty e^{-\alpha t} h(t) X(t) dt$$

for some fixed $x \in \mathbb{R}_{++}$ and $h \in \mathfrak{U}$. Note that this is the α-*discounted* optimal problem. Pick any $0 < x_1 < x_2 < \infty$ and let X^{x_1}, X^{x_2} be the solutions to the controlled diffusion

$$dX(t) = X(t)(\mu - \kappa X(t) - h(t))\,dt + \sigma X(t)\,dB(t)$$

with initial values x_1, x_2 respectively. By the Lagrange mean value theorem, for $y_1, y_2 > 0$, there exists y_3 lies between y_1, y_2 such that $y_1 - y_2 = y_3(\ln y_1 - \ln y_2)$. Using Ito's Lemma we have

$$\begin{aligned} d(\ln X^{x_2}(t) - \ln X^{x_1}(t)) =& -\kappa\,(X^{x_2}(t) - X^{x_1}(t))\,dt \\ =& -\kappa\xi(t)(\ln X^{x_2}(t) - \ln X^{x_1}(t))dt \end{aligned}$$

for some $\xi(t)$ lying between $X^{x_2}(t)$ and $X^{x_1}(t)$. As a result

$$\ln X^{x_2}(t) - \ln X^{x_1}(t) = (\ln x_2 - \ln x_1)\exp\left(-\kappa \int_0^t \xi(s)ds\right) > 0.$$

As a result

$$\mathbb{P}(X^{x_2}(t) > X^{x_1}(t), t \geq 0) = 1.$$

This implies that $J_h(\cdot)$ is an increasing function. Therefore, the optimal yield

$$V_\alpha(x) := \sup_{h\in\mathfrak{U}} J_h(x)$$

is also increasing. By [ABG12, Lemma 3.7.8], there is a function $V^* \in C^2(\mathbb{R}_{++})$ satisfying (A.6) for a number ρ such that

$$\rho \geq \rho^*. \tag{A.10}$$

Moreover,

$$V^*(x) = \lim_{n\to\infty}(V_{\alpha_n}(x) - V_{\alpha_n}(1))$$

for some sequence $(\alpha_n)_{n\in\mathbb{N}}$ that satisfies $\alpha_n \to 0$ as $n \to \infty$. This implies that V^* is an increasing function, i.e.

$$V^*_x \geq 0,\ x \in \mathbb{R}_{++}. \tag{A.11}$$

For any continuous function $\psi : \mathbb{R}_{++} \mapsto \mathbb{R}$ satisfying

$$|\psi(x)| \leq c(1 + x^p), x \in \mathbb{R}_{++}, c > 0 \tag{A.12}$$

we have from (A.1) and [ABG12, Lemma 3.7.2] that $\mathbb{E}^v_x|\psi(X(t))|$ exists and satisfies

$$\lim_{t\to\infty}\left(\frac{1}{t}\sup_{v\in\mathfrak{U}_{sm}} \mathbb{E}^v_x|\psi(X(t))|\right) = 0, \tag{A.13}$$

and

$$\lim_{R\to\infty} \mathbb{E}_x^v \psi(X(t\wedge\xi_R)) = \mathbb{E}_x^v\psi(X(t)) < \infty, t \geq 0, \tag{A.14}$$

where $\xi_R = \inf\{t \geq 0 : X(t) > R \text{ or } X(t) < R^{-1}\}$. Moreover, by using [ABG12, Lemma 3.7.2] again we get that

$$\lim_{x\to\infty} \frac{f_R(x)}{x^p} = 0, R \geq 0 \tag{A.15}$$

where

$$f_R(x) := \sup_{v\in\mathfrak{U}_{sm}} \mathbb{E}_x^v \int_0^{\tau_R} \Phi(X(t))dt,$$

and $\tau_R := \inf\{t \geq 0 : X(t) \leq R\}$.

By [ABG12, Formula 3.7.48], we have the estimate

$$V^*(x) \leq \sup_{v\in\mathfrak{U}_{sm}} \mathbb{E}_x^v \int_0^{\tau_R} \left(\Phi(X(t)) + \rho^*\right) dt + \sup_{y\in[0,R]} \{V^*(y)\}$$

which implies

$$V^*(x) \leq c_p(1 + x^p), x \geq R \text{ for some } c_p > 0. \tag{A.16}$$

Now, pick any $\varepsilon > 0$ and divide (A.16) on both sides by $x^{p+\varepsilon}$. We get

$$\frac{V^*(x)}{x^{p+\varepsilon}} \leq c_p \left(\frac{1}{x^{p+\varepsilon}} + x^{-\varepsilon}\right),\ x \geq R$$

and by letting $x \to \infty$

$$\lim_{x\to\infty} \frac{V^*(x)}{x^{p+\varepsilon}} = 0.$$

This implies, since p and $\varepsilon > 0$ are arbitrary, equation (A.7). Let $\chi : \mathbb{R}_{++} \mapsto [0,1]$ be a continuous function satisfying $\chi(x) = 0$ if $x < \frac{1}{2}$ and $\chi(x) = 1$ if $x \geq 1$. Then $\psi(x) := V^*(x)\chi(x)$ satisfies (A.12) because of (A.16). On the other hand, since $V^*(x)$ is increasing and $V^*(1) = 0$, then $V^*(x) \leq 0$ when $x \leq 1$. Thus, we have

$$V^*(x) \leq \chi(x)V^*(x), x \in \mathbb{R}_{++}.$$

Let v^* be the measurable function satisfying (A.9).

$$\rho \geq \rho^* \geq \rho_{v^*}. \tag{A.17}$$

By Dynkin's formula

$$\begin{aligned}\mathbb{E}_x^{v^*} \chi(X(t\wedge\xi_R))V^*(X(t\wedge\xi_R)) - V^*(x) &\geq \mathbb{E}_x^{v^*} V^*(X(t\wedge\xi_R)) - V^*(x)\\ &= \mathbb{E}_x^{v^*} \int_0^{t\wedge\xi_R} \left(\rho - \Phi(X(s)v(X(s)))\right) ds\end{aligned}$$

Letting $R \to \infty$, we obtain from the monotone convergence theorem and (A.14) that

$$\frac{1}{t}\left(\mathbb{E}_x^{v^*}\chi(X(t))V^*(X(t)) - V^*(x)\right) \geq \rho - \frac{1}{t}\mathbb{E}_x^{v^*}\int_0^t \Phi(X(s)v(X(s)))ds, t > 0$$

Letting $t \to \infty$ and using (A.13) and (A.3), we have

$$0 \geq \rho - \rho_{v^*}.$$

This and (A.17) implies that $\rho = \rho^* = \rho_{v^*}$.

By the arguments from [ABG12, Theorem 3.7.12], we can show that v is an optimal control if and only if (A.9) is satisfied. □

Remark A.1. *We mention the following facts connected to our use of* [ABG12]

(i) *Note that we are maximizing the asymptotic yield in our paper while* [ABG12] *are minimizing it. However, we can still apply their results. The only thing we need to do is to convert our maximizing problem to the minimizing one by replace* Φ *by* $-\Phi$*,* V *by* $-V$ *and then applying their results. After that, we can return to* V *and* Φ*.*

(ii) *In* [ABG12] *the authors use* $\mathbb{R}^d$ *as their domain. Nevertheless, their results can be applied to any invariant open domain. If the reader is still not convinced, one can do the change of variable:* $y = \ln x$*. This will allow us to work on* $\mathbb{R}$ *rather than* $(0, \infty)$*.*

When Φ is the identity mapping the equation (A.9) becomes

$$-\frac{dV^*}{dx}v(x) + v(x) = \max_{u\in U}\left(-\frac{dV^*}{dx}u + u\right),$$

which implies

$$v(x) = \begin{cases} 0 & \text{if } V_x^* > 1 \\ M & \text{if } V_x^* < 1. \end{cases} \tag{A.18}$$

Our main result is the following theorem.

Theorem 2.1. *Assume that* $\Phi(x) = x, x \in (0, \infty)$ *and that the population survives in the absence of harvesting, that is* $\mu - \frac{\sigma^2}{2} > 0$*. The optimal control* v *has the bang-bang form*

$$v(x) = \begin{cases} 0 & \textit{if } 0 < x \le x^* \\ M & \textit{if } x > x^* \end{cases} \tag{2.6}$$

for some $x^* \in (0, \infty)$*. Furthermore, we have the following upper bound for the optimal asymptotic yield*

$$\rho^* \le \frac{\mu^2}{4\kappa}. \tag{2.7}$$

Remark A.2. *If* $V_x^*(x) = 1$ *then we note that* (A.18) *does not provide any information about* $v(x)$*. However, in this case we can set the harvesting rate equal to anything since the yield function will not change. This is because our diffusion is non-degenerate and changing the values of the drift on a set of zero Lebesgue measure does not change the distribution of* X*.*

We split up the proof of Theorem 2.1 into a few propositions. It is immediate to see that the HJB equation (A.6) becomes

$$\begin{aligned} \rho &= \max_{u\in U}\left[x(\mu - \kappa x - u)f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} + xu\right] \\ &= x(\mu - \kappa x)f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} + \max_{u\in U}[(1 - f_x)xu] \\ &= \begin{cases} x(\mu - \kappa x)f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} & \text{if } f_x > 1 \\ x(\mu - \kappa x - M)f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} + Mx & \text{if } f_x \le 1. \end{cases} \end{aligned} \tag{A.19}$$

Sketch of proof of Theorem 2.1. Since the optimal control is given by (A.18) we need to analyze the properties of the function V_x^* which by (A.19) satisfies a first order ODE. The analysis of this is split up into several propositions. Note that the ODE governing V_x^* is different, depending on whether $V_x^* > 1$ or $V_x^* \leq 1$.

In Proposition A.1 we analyze the ODE for when $V_x^* \leq 1$ and find its asymptotic behavior close to 0. Using this we can show in Proposition A.2 that one cannot have a $\eta > 0$ such that $V_x^*(x) \leq 0$ for all $x \in (0, \eta]$.

Similarly, in Proposition A.3 we show that there can exist no $\zeta > 0$ such that $V_x^*(x) \geq 1$ for all $x \geq \zeta$.

In Proposition A.4 we explore the possible ways V_x^* can cross the line $y = 1$ and find using soft arguments that there can be at most 3 crossings. Finally, we show that actually there must be exactly one crossing of $y = 1$ by V_x^* and that this crossing has to be from above (Figure 7). This combined with (A.18) completes the proof. □

The Taylor series expansion of the function e^{-t},

$$e^{-t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k,$$

converges uniformly on any bounded set on $\mathbb{R}$. Let $s \in \mathbb{R}$ be any fixed real number. For any $x > 0$, the function t^{s-1} is uniformly bounded by the positive constant $\max\{x^{s-1}, 1\}$ in the interval $[\min\{x, 1\}, \max\{x, 1\}]$, that is,

$$t^{s-1} \leq \max\{x^{s-1}, 1\}$$

for all $t \in [\min\{x, 1\}, \max\{x, 1\}]$. Therefore, the series expansion

$$t^{s-1} e^{-t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{s+k-1}$$

converges uniformly in $[\min\{x, 1\}, \max\{x, 1\}]$. As a result, we can use term-by-term integration to obtain

(A.20)

$$\begin{aligned}
\int_x^1 t^{s-1} e^{-t}\, dt &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_x^1 t^{s+k-1}\, dt \\
&= \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+s)} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+s)} x^{s+k} & \text{if } s \neq 0, -1, -2, \cdots \\ \displaystyle\sum_{\substack{k=0 \\ k \neq -s}}^{\infty} \frac{(-1)^k}{k!(k+s)} - \frac{(-1)^{-s}}{(-s)!} \ln x - \sum_{\substack{k=0 \\ k \neq -s}}^{\infty} \frac{(-1)^k}{k!(k+s)} x^{s+k} & \text{if } s = 0, -1, -2, \cdots \end{cases} \\
&= \begin{cases} \displaystyle A_s - x^s \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+s)} x^k & \text{if } s \neq 0, -1, -2, \cdots \\ \displaystyle A_s - \frac{(-1)^{-s}}{(-s)!} \ln x - x^s \sum_{\substack{k=0 \\ k \neq -s}}^{\infty} \frac{(-1)^k}{k!(k+s)} x^k & \text{if } s = 0, -1, -2, \cdots, \end{cases}
\end{aligned}$$

where

$$A_s := \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+s)} & \text{if } s \neq 0,\, -1,\, -2,\, \cdots \\ \displaystyle\sum_{\substack{k=0 \\ k\neq -s}}^{\infty} \frac{(-1)^k}{k!(k+s)} & \text{if } s = 0,\, -1,\, -2,\, \cdots \end{cases}$$

is a constant depending on the parameter s only.

Proposition A.1. *The ODE*

$$\frac{d\varphi_2}{dx}(x) + \frac{2(\mu - M - \kappa x)}{\sigma^2 x}\varphi_2(x) = \frac{2(\rho - Mx)}{\sigma^2 x^2} \tag{A.21}$$

has the following properties near 0:

i) *If $\mu - \frac{\sigma^2}{2} - M \neq 0$ then*

$$\lim_{x\to 0^+} \varphi_2(x) = \lim_{x\to 0^+} C_1 x^{-\frac{2(\mu-M)}{\sigma^2}} - C_2 + C_3 x^{-1} = \pm\infty$$

for constants $C_1, C_2 \in \mathbb{R}$ and $C_3 \in \mathbb{R}\setminus\{0\}$.

ii) *If $\mu - \frac{\sigma^2}{2} - M = 0$ then*

$$\lim_{x\to 0^+} \varphi_2(x) = \lim_{x\to 0^+} \frac{2\rho}{\sigma^2} x^{-1} \ln\left(\frac{2\kappa}{\sigma^2}x\right) = -\infty.$$

Proof. It follows from the method of integrating factors that the solution to the ODE is

$$\varphi_2(x) = x^{-\frac{2(\mu-M)}{\sigma^2}} e^{\frac{2\kappa x}{\sigma^2}} \left(c - \frac{2}{\sigma^2}\int_x^{\frac{\sigma^2}{2\kappa}} (\rho - My) y^{\frac{2(\mu-M)}{\sigma^2}-2} e^{-\frac{2\kappa y}{\sigma^2}}\, dy\right),$$

where c is an integration constant:

$$c := \frac{1}{e}\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}} \varphi_2\left(\frac{\sigma^2}{2\kappa}\right).$$

Using the change of variable $t := \frac{2\kappa y}{\sigma^2}$, we can rewrite the definite integral above as

$$\frac{2}{\sigma^2}\int_x^{\frac{\sigma^2}{2\kappa}} (\rho - My) y^{\frac{2(\mu-M)}{\sigma^2}-2} e^{-\frac{2\kappa y}{\sigma^2}}\, dy = \left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}-1} \int_{\frac{2\kappa}{\sigma^2}x}^{1} \left(\frac{2\rho}{\sigma^2} - \frac{M}{\kappa}t\right) t^{\frac{2(\mu-M)}{\sigma^2}-2} e^{-t}\, dy$$

$$= \left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}-1} \left\{\frac{2\rho}{\sigma^2} G\left(\frac{2(\mu-M)}{\sigma^2} - 1, \frac{2\kappa}{\sigma^2}x\right) - \frac{M}{\kappa} G\left(\frac{2(\mu-M)}{\sigma^2}, \frac{2\kappa}{\sigma^2}x\right)\right\},$$

where the Gamma type function G is defined as $G(s,z) := \int_z^1 t^{s-1}e^{-t}\, dt$. Hence,

$$\varphi_2(x) = x^{-\frac{2(\mu-M)}{\sigma^2}} e^{\frac{2\kappa x}{\sigma^2}} \left(c + \left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}-1} \left\{\frac{M}{\kappa} G\left(\frac{2(\mu-M)}{\sigma^2}, \frac{2\kappa}{\sigma^2}x\right) - \frac{2\rho}{\sigma^2} G\left(\frac{2(\mu-M)}{\sigma^2} - 1, \frac{2\kappa}{\sigma^2}x\right)\right\}\right).$$

Case 1: $\frac{2(\mu-M)}{\sigma^2} \in \mathbb{R} \setminus \{1, 0, -1, -2, \cdots\}$. In this case, both $\frac{2(\mu-M)}{\sigma^2}$ and $\frac{2(\mu-M)}{\sigma^2} - 1$ do not belong to $\{0, -1, -2, \cdots\}$, so it follows from the expansion (A.20) that

$$
\begin{aligned}
&\lim_{x\to 0^+} \varphi_2(x) \\
&= \lim_{x\to 0^+} x^{-\frac{2(\mu-M)}{\sigma^2}} e^{\frac{2\kappa x}{\sigma^2}} \left(c + \left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}-1} \left\{ \frac{M}{\kappa} G\left(\frac{2(\mu-M)}{\sigma^2}, \frac{2\kappa}{\sigma^2}x\right) - \frac{2\rho}{\sigma^2} G\left(\frac{2(\mu-M)}{\sigma^2} - 1, \frac{2\kappa}{\sigma^2}x\right) \right\} \right) \\
&= \lim_{x\to 0^+} x^{-\frac{2(\mu-M)}{\sigma^2}} \left(c + \left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}-1} \left\{ \frac{M}{\kappa} \left[A_{\frac{2(\mu-M)}{\sigma^2}} - \left(\frac{2\kappa}{\sigma^2}x\right)^{\frac{2(\mu-M)}{\sigma^2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\left(k + \frac{2(\mu-M)}{\sigma^2}\right)} \left(\frac{2\kappa}{\sigma^2}x\right)^k \right] \right.\right. \\
&\qquad \left.\left. - \frac{2\rho}{\sigma^2} \left[A_{\frac{2(\mu-M)}{\sigma^2}-1} - \left(\frac{2\kappa}{\sigma^2}x\right)^{\frac{2(\mu-M)}{\sigma^2}-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\left(k + \frac{2(\mu-M)}{\sigma^2} - 1\right)} \left(\frac{2\kappa}{\sigma^2}x\right)^k \right] \right\} \right) \\
&= \lim_{x\to 0^+} \left\{ c + \frac{M}{\kappa}\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}-1} A_{\frac{2(\mu-M)}{\sigma^2}} - \frac{2\rho}{\sigma^2}\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}-1} A_{\frac{2(\mu-M)}{\sigma^2}-1} \right\} x^{-\frac{2(\mu-M)}{\sigma^2}} \\
&\qquad - \left\{ \left(\frac{M}{\mu-M}\right) + \frac{2\rho\kappa}{\sigma^2(\mu-M)} \right\} + \left(\frac{2\rho}{2(\mu-M)-\sigma^2}\right) x^{-1} \\
&\qquad - \frac{2M}{\sigma^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\left(k + \frac{2(\mu-M)}{\sigma^2}\right)} \left(\frac{2\kappa}{\sigma^2}x\right)^k + \frac{1}{x}\frac{2\rho}{\sigma^2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k!\left(k + \frac{2(\mu-M)}{\sigma^2} - 1\right)} \left(\frac{2\kappa}{\sigma^2}x\right)^k \\
&= \lim_{x\to 0^+} C_1 x^{-\frac{2(\mu-M)}{\sigma^2}} - C_2 + C_3 x^{-1} + F(x),
\end{aligned}
$$

where the constants C_1, C_2, and C_3 are given by

$$
\begin{aligned}
C_1 &:= c + \frac{M}{\kappa}\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}-1} A_{\frac{2(\mu-M)}{\sigma^2}} - \frac{2\rho}{\sigma^2}\left(\frac{\sigma^2}{2\kappa}\right)^{\frac{2(\mu-M)}{\sigma^2}-1} A_{\frac{2(\mu-M)}{\sigma^2}-1} \\
C_2 &:= \left(\frac{M}{\mu-M}\right) + \frac{2\rho\kappa}{\sigma^2(\mu-M)} \\
C_3 &:= \frac{2\rho}{2(\mu-M)-\sigma^2},
\end{aligned}
$$

and the function F is defined by

$$
F(x) := -\sum_{k=1}^{\infty} \frac{(-1)^k}{k!\left(k + \frac{2(\mu-M)}{\sigma^2}\right)} \left\{ \frac{2M}{\sigma^2} + \frac{4\rho\kappa}{\sigma^2(k+1)} \right\} \left(\frac{2\kappa}{\sigma^2}x\right)^k.
$$

Since

$$
\lim_{x\to 0^+} F(x) = 0,
$$

we finally have

$$
\lim_{x\to 0^+} \varphi_2(x) = \lim_{x\to 0^+} C_1 x^{-\frac{2(\mu-M)}{\sigma^2}} - C_2 + C_3 x^{-1}.
$$

It is worth noting that both of the constants C_1, and C_2 may be equal to any real number (including 0), but the constant C_3 must be non-zero. Recall that in this case, $\frac{2(\mu-M)}{\sigma^2} \neq 1$, so no cancellation may appear between the first and last terms, and hence, we must have

$$\lim_{x\to 0^+} \varphi_2(x) = \pm\infty.$$

Case 2: $\frac{2(\mu-M)}{\sigma^2} \in \{0, -1, -2, \cdots\}$. Using (A.20) and straightforward computations one can see that for some constant $C \in \mathbb{R}$

$$\begin{aligned}\lim_{x\to 0^+} \varphi_2(x) &= \lim_{x\to 0^+} \left[\frac{2\rho}{\sigma^2}\frac{1}{\frac{2(\mu-M)}{\sigma^2}-1}x^{-1} + Cx^{-\frac{2(\mu-M)}{\sigma^2}}\ln\left(\frac{2\kappa}{\sigma^2}x\right)\right]\\ &= \lim_{x\to 0^+} \frac{2\rho}{\sigma^2}\frac{1}{\frac{2(\mu-M)}{\sigma^2}-1}x^{-1}.\end{aligned}$$

Case 3: $\frac{2(\mu-M)}{\sigma^2} = 1$. By (A.20) one can note that

$$\lim_{x\to 0^+} \varphi_2(x) = \lim_{x\to 0^+} \frac{2\rho}{\sigma^2}x^{-1}\ln\left(\frac{2\kappa}{\sigma^2}x\right).$$

□

Proposition A.2. *There does not exist any $\eta > 0$ such that $V_x^*(x) \leq 1, x \in (0, \eta]$.*

Proof. We will argue by contradiction. Assume there exists $\eta > 0$ such that $V_x^*(x) \leq 1, x \in (0, \eta]$. Then by (A.19) we get that V_x^* follows the ODE (A.21) for all $x \in (0, \eta)$. Making use of Proposition A.1 we get that

$$\lim_{x\to 0+} V_x^*(x) = \lim_{x\to 0+} \varphi_2(x) = \pm\infty$$

which contradicts that $V_x^* \geq 0$ or that $V_x^*(x) \leq 1, x \in (0, \eta]$. The proof is complete. □

The above Proposition shows that the scenario from Figure 6 cannot happen.

Proposition A.3. *There does not exist any $\zeta > 0$ such that $V_x^*(x) \geq 1$ for all $x \geq \zeta$.*

Proof. Once again we will argue by contradiction. Assume there exists $\zeta > 0$ such that $V_x^*(x) \geq 1$ for all $x \geq \zeta$. First, note that by (A.19) V_x^* will follow the ODE

$$\frac{d\varphi_1}{dx}(x) + \frac{2(\mu-\kappa x)}{\sigma^2 x}\varphi_1(x) = \frac{2\rho}{\sigma^2 x^2}$$

for all $x \geq \zeta$. A direct application of the method of integrating factors yields

$$\varphi_1(x) = \left(\int_\zeta^x \frac{2\rho}{\sigma^2 y^2}\exp\left(\frac{2\mu}{\sigma^2}\ln y - \frac{2\kappa}{\sigma^2}y\right)dy + c\right)\exp\left(-\frac{2\mu}{\sigma^2}\ln x + \frac{2\kappa}{\sigma^2}x\right)$$

If

$$\int_\zeta^\infty \frac{2\rho}{\sigma^2 y^2}\exp\left(\frac{2\mu}{\sigma^2}\ln y - \frac{2\kappa}{\sigma^2}y\right)dy + c > 0$$

we get

$$\lim_{x\to\infty}\frac{V_x^*}{x} = +\infty$$

which contradicts the growth condition (A.7). Therefore we need

$$\int_\zeta^\infty \frac{2\rho}{\sigma^2 y^2}\exp\left(\frac{2\mu}{\sigma^2}\ln y - \frac{2\kappa}{\sigma^2}y\right)dy + c \leq 0.$$

Note that in this case

$$\int_\zeta^x \frac{2\rho}{\sigma^2 y^2} \exp\left(\frac{2\mu}{\sigma^2}\ln y - \frac{2\kappa}{\sigma^2}y\right) dy + c \le -\int_x^\infty \frac{2\rho}{\sigma^2 y^2} \exp\left(\frac{2\mu}{\sigma^2}\ln y - \frac{2\kappa}{\sigma^2}y\right) dy < 0.$$

This implies that for $x > \zeta$

$$V_x^*(x) = \left(\int_\zeta^x \frac{2\rho}{\sigma^2 y^2} \exp\left(\frac{2\mu}{\sigma^2}\ln y - \frac{2\kappa}{\sigma^2}y\right) dy + c\right) \exp\left(-\frac{2\mu}{\sigma^2}\ln x + \frac{2\kappa}{\sigma^2}x\right) < 0,$$

which contradicts the assumption that $V_x^*(x) \ge 1$ for all $x \ge \zeta$. □

The above shows that the scenario from Figure 5 is not possible.

Proposition A.4. *The function V_x^* crosses the line $y = 1$ at most three times. Moreover, we have the following possibilities:*

(I) If $\mu^2 - 4\rho\kappa > 0$ then

(i) For $0 \le x < \frac{\mu - \sqrt{\mu^2 - 4\rho\kappa}}{2\kappa}$ the function V_x^* can only pass the line $y = 1$ at most once and the crossing has to be from below.

(ii) For $x > \frac{\mu + \sqrt{\mu^2 - 4\rho\kappa}}{2\kappa}$ the function V_x^* can pass the line $y = 1$ at most once and the crossing has to be from below.

(iii) For $\frac{\mu - \sqrt{\mu^2 - 4\rho\kappa}}{2\kappa} < x < \frac{\mu + \sqrt{\mu^2 - 4\rho\kappa}}{2\kappa}$ the function V_x^* can pass the line $y = 1$ at most once and the crossing has to be from above.

(II) If $\mu^2 - 4\rho\kappa < 0$ then the function V_x^* can pass the line $y = 1$ at most once and the crossing has to be from below.

(III) If $\mu^2 - 4\rho\kappa = 0$ then V_x^* can cross the line $y = 1$ at most three times. In particular, the possible crossing(s) in $(0, \frac{\mu}{2\kappa}) \cup (\frac{\mu}{2\kappa}, \infty)$ must be from below.

Proof. It follows from the HJB equation (A.6) with $\varphi := V_x$ that if $\varphi(x_0) = 1$, then we have

$$g(x_0) := \rho - \mu x_0 + \kappa x_0^2 = \frac{1}{2}\sigma^2 x_0^2 \varphi'(x_0) \begin{cases} < 0 & \text{if } \varphi'(x_0) < 0 \\ = 0 & \text{if } \varphi'(x_0) = 0 \\ > 0 & \text{if } \varphi'(x_0) > 0. \end{cases}$$

Therefore, when φ crosses the line $y = 1$, we obtain some information from g. More precisely, we can infer the following:

(I) When $\mu^2 - 4\rho\kappa > 0$ the quadratic polynomial $g(x) := \rho - \mu x + \kappa x^2$ has two real zeros at

$$\alpha_1 := \frac{\mu - \sqrt{\mu^2 - 4\rho\kappa}}{2\kappa}$$

and

$$\alpha_2 := \frac{\mu + \sqrt{\mu^2 - 4\rho\kappa}}{2\kappa}.$$

(i) for $0 \le x < \alpha_1$ we have $g(x) > 0$, hence φ is only allowed to cross the line $y = 1$ from below in this region;

(ii) for $x > \alpha_2$ we have $g(x) > 0$, hence φ is only allowed to cross the line $y = 1$ from below in this region;

(iii) for $\alpha_1 < x < \alpha_2$, $g(x) < 0$ and φ is only allowed to cross the line $y = 1$ from above in this region.

(II) If $\mu^2 - 4\rho\kappa < 0$ then $g(x) > 0$ for all $x \in \mathbb{R}_+$. The function V_x^* can pass the line $y = 1$ at most once and the crossing has to be from below.

(III) If $\mu^2 - 4\rho\kappa = 0$ then $g(x) := \rho - \mu x + \kappa x^2$ has a double real zero at

$$\alpha_1 = \alpha_2 = \alpha := \frac{\mu}{2\kappa}.$$

As a consequence $g(x) \geq 0$ and the function V_x^* can pass the line $y = 1$ at most thrice: at most once from below in the region $x < \frac{\mu}{2\kappa}$, at most once from below in the region $x > \frac{\mu}{2\kappa}$ and at most once from above or from below at the point $x = \frac{\mu}{2\kappa}$.

□

Remark A.3. *By the analysis above one can note that at the roots $\alpha_{1,2}$ of the quadratic polynomial $g(x)$ the derivative of φ is 0. This makes it more complicated to say, in case there is a crossing at a root, if the crossing is from above or from below. However, this does not require us to change our arguments. For example, if there is a crossing from below on $0 \leq x < \alpha_1$ and there is a crossing at $x = \alpha_1$ then the crossing at α_1 is necessarily from above. This then implies that there can be no crossing for $x \in (\alpha_1, \alpha_2)$ because in this region the crossing has to be from above and there cannot be two crossings from above in a row.*

Proof of Theorem 2.1. A direct consequence of Proposition A.4 is that V_x^* can cross the line $y = 1$ at most three times. We also know, given the roots $\alpha_{1,2}$ of the quadratic polynomial $g(x)$ how these crossings have to happen. Next, we eliminate all but one possibility.

i) If we get a crossing from below in $(0, \alpha_1)$ this means that there exists $\eta > 0$ such that for all $x \in (0, \eta)$ we have $V_x^*(x) = \varphi_2(x) \leq 1$. This is not possible by Proposition A.2. As such there can be no crossings in $(0, \alpha_1)$.

ii) If we have a crossing from below in (α_2, ∞) then there is $\zeta > 0$ such that for all $x \geq \zeta$

$$V_x^*(x) = \varphi_1(x) \geq 1.$$

This is not possible by Proposition A.3. Therefore, there are no crossings in (α_2, ∞).

iii) We cannot have that $V_x^*(x) \geq 1$ for all $x \in (0, \infty)$ because then we get a contradiction by Proposition A.3. Similarly, we cannot have $V_x^*(x) \leq 1$ for all $x \in (0, \infty)$ since we get a contradiction by Proposition A.2.

iv) If $\mu^2 - 4\rho\kappa < 0$ then, in principle, there could be at most one crossing and this would have to be from below. But this creates a contradiction by either using Proposition A.2 or Proposition A.3. If there is no crossing then we get a contradiction by (iii) above.

v) If $\mu^2 - 4\rho\kappa = 0$ then
   (a) If there is no crossing, then we get a contradiction by part iii) above.
   (b) If there are two crossings then we get contradictions from either Proposition A.2 or Proposition A.3.
   (c) If there are three crossings then we must have a crossing from below in $(0, \alpha)$, one from above at $x = \alpha$ and one from below in (α, ∞). This yields a contradiction because of Proposition A.2.
   (d) If there is just one crossing and the crossing is from below then we get a contradiction by Proposition A.3.

vi) By parts i)-iv) we get that there is exactly one crossing of the line $y = 1$, that this crossing is from above and that the crossing happens at a point in the interval $[\alpha_1, \alpha_2]$.

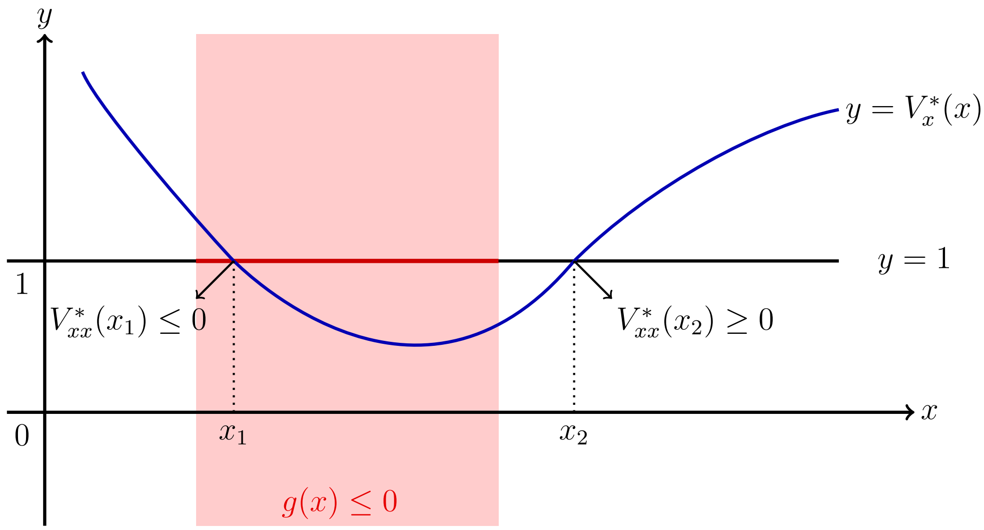

FIGURE 5. An impossible scenario, by Proposition A.3.

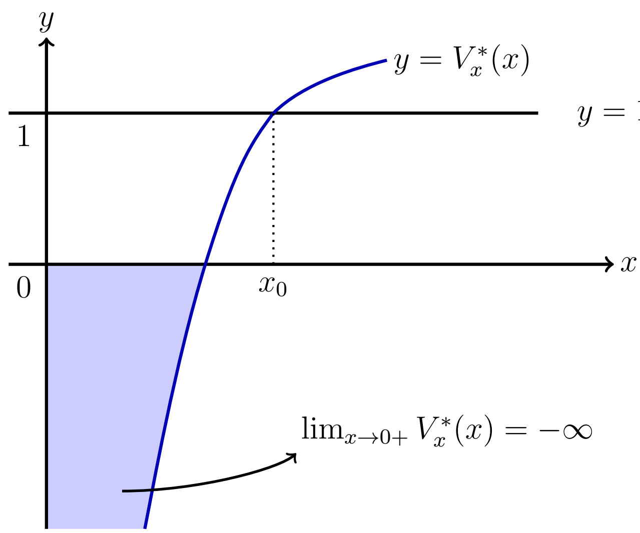

FIGURE 6. If V_x^* crosses $y = 1$ from below at x_0, and it has not crossed from above before then we get a contradiction by Proposition A.2.

This, together with (A.18), implies that the optimal strategy is of bang-bang type

$$v(x) = \begin{cases} 0 & \text{if } 0 < x \le x^* \\ M & \text{if } x > x^*. \end{cases}$$

Moreover, one can see that

$$\mu^2 - 4\rho\kappa \ge 0$$

which in turn forces

$$\rho \le \frac{\mu^2}{4\kappa}.$$

□

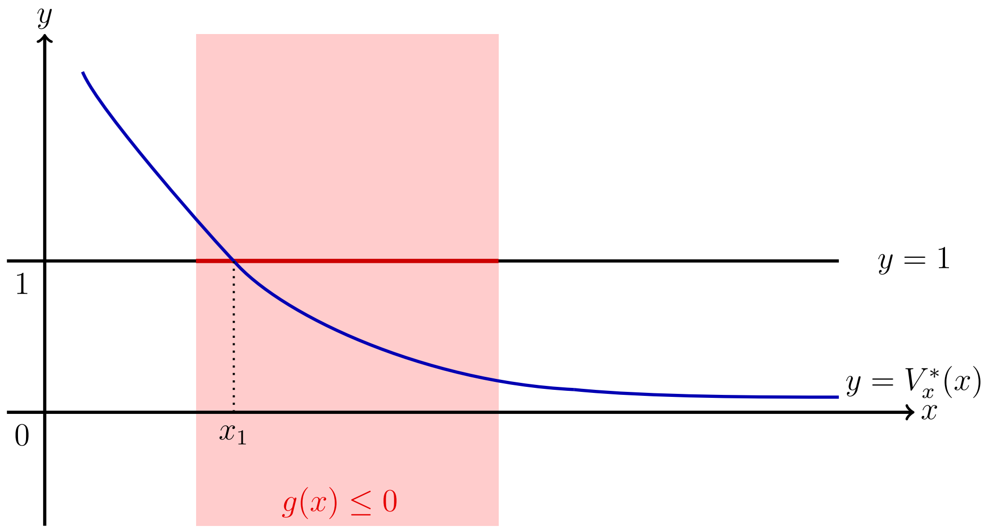

FIGURE 7. The only case which doesn't lead to a contradiction is when V_x^* crosses $y = 1$ only once and the crossing is from above.

APPENDIX B. CONTINUOUS OPTIMAL HARVESTING STRATEGIES

This appendix shows that for a class of yield functions Φ one can get *continuous* optimal harvesting strategies.

For any given Φ, we define

$$F(\omega) := -A\omega + \Phi(\omega),$$

where A is a shorthand of V_x^*, that is,

$$A := V_x^*(x).$$

For any fixed x, we can see A as a constant. Using these shorthands, we can rewrite the maximum principle (A.9) as

$$F(xv) = \max_{\omega \in [0,L]} F(\omega), \tag{B.1}$$

where $L := xM$. A direct computation yields

$$\begin{cases} F(0) = -A \cdot 0 + \Phi(0) = 0 \\ F(L) = -AL + \Phi(L) \\ F'(\omega) = -A + \Phi'(\omega) \end{cases}$$

because $\Phi(0) = 0$. Therefore, the critical point(s) will be given by $\omega_c = [\Phi']^{-1}(A)$, and

$$F(\omega_c) = -A\omega_c + \Phi(\omega_c) = -A[\Phi']^{-1}(A) + \Phi([\Phi']^{-1}(A)).$$

If Φ is assumed to be strictly concave, the maximum on the right hand side of (B.1) can be found easily because $F'' = \Phi''$.

To illustrate the idea, let us assume that Φ is C^2 and strictly concave .

Theorem B.1. *Suppose the yield function* Φ *is* C^2*, increasing, has polynomial growth and is strictly concave. Then optimal harvesting strategy is* ***continuous*** *and given by*

$$v = \begin{cases} 0 & \text{if } [\Phi']^{-1}(V_x^*(x)) \le 0, \\ \dfrac{[\Phi']^{-1}(V_x^*(x))}{x} & \text{if } 0 < [\Phi']^{-1}(V_x^*(x)) < xM, \\ M & \text{if } [\Phi']^{-1}(V_x^*(x)) \ge xM. \end{cases}$$

Furthermore, the HJB equation for the system becomes

(B.2)
$$\rho = \begin{cases} x(\mu - \kappa x) f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} & \text{if } [\Phi']^{-1}(f_x(x)) \le 0, \\ x(\mu - \kappa x) f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} - f_x [\Phi']^{-1}(f_x) + \Phi([\Phi']^{-1}(f_x)) & \text{if } 0 < [\Phi']^{-1}(f_x(x)) < xM, \\ x(\mu - \kappa x - M) f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} + \Phi(xM) & \text{if } [\Phi']^{-1}(f_x(x)) \ge xM. \end{cases}$$

Proof. Since Φ is C^2 and strictly concave we have that $\Phi'' < 0$. In this case, Φ' is strictly decreasing, so its inverse is well-defined. As a result, we have a unique critical point (which is a maximum) $\omega_c = [\Phi']^{-1}(A)$. A standard calculus result yields

$$\begin{aligned} \max_{\omega \in [0,L]} F(\omega) &= \begin{cases} \max\{F(0), F(L)\} & \text{if } \omega_c \notin (0, L) \\ \max\{F(0), F(\omega_c), F(L)\} & \text{if } 0 < \omega_c < L \end{cases} \\ &= \begin{cases} 0 & \text{if } \omega_c \le 0 \\ F(\omega_c) & \text{if } 0 < \omega_c < L \\ F(L) & \text{if } \omega_c \ge L, \end{cases} \end{aligned}$$

where we used the fact that $F(0) = 0$ and the concavity of Φ in the last equality.

Depending on the maximum point, we have the corresponding optimal Markov control:

$$\begin{aligned} v &= \begin{cases} 0 & \text{if } \max_{\omega \in [0,L]} F(\omega) = 0 \\ \dfrac{[\Phi']^{-1}(A)}{x} & \text{if } \max_{\omega \in [0,L]} F(\omega) = F(\omega_c) \\ M & \text{if } \max_{\omega \in [0,L]} F(\omega) = F(L) \end{cases} \\ &= \begin{cases} 0 & \text{if } [\Phi']^{-1}(A) \le 0 \\ \dfrac{[\Phi']^{-1}(A)}{x} & \text{if } 0 < [\Phi']^{-1}(A) < xM \\ M & \text{if } [\Phi']^{-1}(A) \ge xM \end{cases} \end{aligned}$$

because v is the solution to

$$-Axv + \Phi(xv) = \max_{\omega \in [0,L]} F(\omega).$$

In conclusion, in this case, v depends on $A := \dfrac{dV^*}{dx}(x)$ continuously. Hence, since $V^* \in C^2(\mathbb{R}_+)$ we conclude that v is continuous.

The HJB equation (A.6) becomes

$$
\begin{aligned}
\rho &= \max_{u\in U}\left[x(\mu-\kappa x-u)f_x+\frac{1}{2}\sigma^2x^2f_{xx}+\Phi(xu)\right]\\
&= x(\mu-\kappa x)f_x+\frac{1}{2}\sigma^2x^2f_{xx}+\max_{u\in U}[(\Phi(xu)-xuf_x)]\\
&= \begin{cases}
x(\mu-\kappa x)f_x+\frac{1}{2}\sigma^2x^2f_{xx} & \text{if } [\Phi']^{-1}(f_x(x))\le 0,\\
x(\mu-\kappa x)f_x+\frac{1}{2}\sigma^2x^2f_{xx}-f_x[\Phi']^{-1}(f_x)+\Phi([\Phi']^{-1}(f_x)) & \text{if } 0<[\Phi']^{-1}(f_x(x))<xM,\\
x(\mu-\kappa x-M)f_x+\frac{1}{2}\sigma^2x^2f_{xx}+\Phi(xM) & \text{if } [\Phi']^{-1}(f_x(x))\ge xM.
\end{cases}
\end{aligned}
$$

□

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