A note on zero-sum 5-flows in regular graphs

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\textbf{Abstract}

Let \(G\) be a graph. A zero-sum flow of \(G\) is an assignment of non-zero real numbers to the edges such that the sum of the values of all edges incident with each vertex is zero. Let \(k\) be a natural number. A zero-sum \(k\)-flow is a flow with values from the set \(\{\pm 1, \ldots, \pm (k-1)\}\).

It has been conjectured that every \(r\)-regular graph, \(r \geq 3\), admits a zero-sum 5-flow. In this paper we give an affirmative answer to this conjecture, except for \(r = 5\).

\textbf{1. Introduction}

Nowhere-zero flows on graphs were introduced by Tutte [7] in 1949 and since then have been extensively studied by many authors. A great deal of research in the area has been motivated by Tutte’s 5-Flow Conjecture which states that every 2-edge connected graph can have its edges

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directed and labeled by integers from \{1, 2, 3, 4\} in such a way that Kirchhoff’s current law is satisfied at each vertex. In 1983, Bouchet [4] generalized this concept to bidirected graphs. A bidirected graph \( G \) is a graph with vertex set \( V(G) \) and edge set \( E(G) \) such that each edge is oriented as one of the four possibilities: 

\[ \begin{align*}
\bullet & \rightarrow \bullet \\
\bullet & \leftarrow \bullet \\
\bullet & \rightarrow \bullet \\
\bullet & \leftarrow \bullet
\end{align*} \]

Let \( G \) be a bidirected graph. For every \( v \in V(G) \), the set of all edges with tails (respectively, heads) at \( v \) is denoted by \( E^+(v) \) (respectively, \( E^-(v) \)). The function \( f : E(G) \rightarrow \mathbb{R} \) is a bidirected flow of \( G \) if for every \( v \in V(G) \), we have

\[ \sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e). \]

If \( f \) takes its values from the set \( \{\pm 1, \ldots, \pm (k-1)\} \), then it is called a nowhere-zero bidirected \( k \)-flow.

Consequently, Bouchet proposed the following interesting conjecture.

**Bouchet’s Conjecture.** [4, 8] Every bidirected graph that has a nowhere-zero bidirected flow admits a nowhere-zero bidirected 6-flow.

Bouchet proved that his conjecture is true if 6 is replaced by 216. Then Zyka reduced 216 to 30 [9].

Let \( G \) be a graph. A zero-sum flow for \( G \) is an assignment of non-zero real numbers to the edges such that the sum of the values of all edges incident with each vertex is zero. Let \( k \) be a natural number. A zero-sum \( k \)-flow is a flow with values from the set \( \{\pm 1, \ldots, \pm (k-1)\} \). The following conjecture was posed on the zero-sum flows in graphs.

**Zero-Sum Conjecture (ZSC).** [1] If \( G \) is a graph with a zero-sum flow, then \( G \) admits a zero-sum 6-flow.

The following conjecture is an improved version of ZSC for regular graphs.

**Conjecture A.** [2] Every \( r \)-regular graph (\( r \geq 3 \)) admits a zero-sum 5-flow.

Recently, in connection with this conjecture the following two theorems were proved.

**Theorem 1.** [1] Let \( r \) be an even integer with \( r \geq 4 \). Then every \( r \)-regular graph has a zero-sum 3-flow.

**Theorem 2.** [2] Let \( G \) be an \( r \)-regular graph. If \( r \) is divisible by 3, then \( G \) has a zero-sum 5-flow.
Remark 1. There are some regular graphs with no zero-sum 4-flow. To see this consider the graph given in Figure 1. To the contrary assume this the graph has a zero-sum 4-flow. Since the sum of the values of all edges incident with each vertex is zero, for every \( v \in V(G) \), \(-2\) or \(2\) should appear in the neighborhood of \( v \). On the other hand two numbers with absolute value \(2\) can not appear in the neighborhood of a vertex. So all edges of \( G \) with values \(\pm 2\) form a perfect matching. But by celebrated Tutte’s Theorem [3, p.76], \( G \) has no perfect matching, a contradiction.

![Figure 1. A 3-regular graph with no zero-sum 4-flow](image)

In 2010, the following result was proved.

**Theorem 3.** [2] Bouchet’s Conjecture and ZSC are equivalent.

Motivated by Bouchet’s Conjecture and along with Theorem 3 we focused our attention to establish the Conjecture A. We show that except \( r = 5 \), Conjecture A is true.

2. The Main Result

In this section we prove that every \( r \)-regular graph, \( r \geq 3, r \neq 5 \), admits a zero-sum 5-flow. Before establishing our main result we need some notations and definitions.

Let \( G \) be a finite and undirected graphs with vertex set \( V(G) \) and edge set \( E(G) \), where multiple edges and loops are admissible. A \( k \)-regular graph is a graph where each vertex is of degree \( k \). A subgraph \( F \) of a graph \( G \), is a factor of \( G \) if \( F \) is a spanning subgraph of \( G \). If a factor \( F \) has all of its degrees equal to \( k \), it is called a \( k \)-factor. Thus a 2-factor is a disjoint union of finitely
many cycles that cover all the vertices of $G$. A $k$-factorization of $G$ is a partition of the edges of $G$ into disjoint $k$-factors. For integers $a$ and $b$, $1 \leq a \leq b$, an $[a,b]$-factor of $G$ is defined to be a factor $F$ of $G$ such that $a \leq d_F(v) \leq b$, for every $v \in V(G)$. For any vertex $v \in V(G)$, let $N_G(v) = \{ u \in V(G) \mid uv \in E(G) \}$.

The following two theorems are also needed.

**Theorem 4.** [6] Every $2k$-regular multigraph admits a $2$-factorization.

**Theorem 5.** [5] Let $r \geq 3$ be an odd integer and let $k$ be an integer such that $1 \leq k \leq \frac{2r}{3}$. Then every $r$-regular graph has a $[k-1,k]$-factor each component of which is regular.

**Lemma 1.** Let $G$ be an $r$-regular graph. Then for every even integer $q$, $2r \leq q \leq 4r$, there exists a function $f : E(G) \to \{2, 3, 4\}$ such that for every $u \in V(G)$, $\sum_{v \in N_G(u)} f(uv) = q$.

**Proof.** First assume that $r$ is an odd integer. For every edge $e = uv$, we add a new edge $e' = uv$ to the graph $G$ and call the resultant graph by $G'$. Clearly, $G'$ is a $2r$-regular multigraph. By Theorem 4, $G'$ admits a 2-factorization with 2-factors $F_1, \ldots, F_r$. Now, for every $e \in F_i, 1 \leq i \leq r$, we define a function $g : E(G') \to \{1, 2\}$ as follows:

$$g(e) = \begin{cases} 2, & 1 \leq i \leq \frac{q-2r}{2}; \\ 1, & \frac{q-2r}{2} < i. \end{cases}$$

Therefore for each $v \in V(G')$, $\sum_{e \in N_G'(u)} g(uv) = q$. Now, define a function $f : E(G) \to \{2, 3, 4\}$ such that for every $e = uv \in E(G)$, $f(e) = g(e) + g(e')$, where $e' = uv$ in $G'$. Then for every $u \in V(G)$, $\sum_{v \in N_G(u)} f(uv) = q$, as desired.

Now, let $r$ be an even integer. Since $G$ is an $r$-regular graph, by Theorem 4, $G$ admits a 2-factorization with 2-factors $F_1, \ldots, F_{\frac{r}{2}}$. Now, for every $e \in F_i, 1 \leq i \leq \frac{r}{2}$, we define a function $f : E(G) \to \{2, 3, 4\}$ as follows:

$$f(e) = \begin{cases} 4, & 1 \leq i \leq \lfloor \frac{q-2r}{4} \rfloor; \\ 3, & \lfloor \frac{q-2r}{4} \rfloor < i \leq \lfloor \frac{q-2r}{4} \rfloor; \\ 2, & \lfloor \frac{q-2r}{4} \rfloor < i. \end{cases}$$

It is not hard to verify that for every $u \in V(G)$, $\sum_{v \in N_G(u)} f(uv) = q$, as desired. □
Now, we are in a position to prove our main theorem.

**Theorem 6.** Let \( r \geq 3 \) and \( r \neq 5 \). Then every \( r \)-regular graph has a zero-sum 5-flow.

**Proof.** First we prove the theorem for \( r = 7 \). Let \( G \) be a 7-regular graph. Then by Theorem 5, \( G \) has a \([3, 4]\)-factor, say \( H \), whose components are regular. Let \( H_1 \) be the union of the 3-regular components of \( H \) and let \( H_2 \) be the union of 4-regular components of \( H \). By Theorem 4, \( H_2 \) can be decomposed into two 2-factors \( H_2' \) and \( H_2'' \). Assign 1 and 2 to all edges of \( H_2' \) and \( H_2'' \), respectively. By Lemma 1, there exists a function \( f : E(H_1) \to \{2, 3, 4\} \) such that for every \( u \in V(H_1) \), \( \sum_{v \in N_{H_1}(u)} f(uv) = 8 \). Now, assign \(-2\) to every edge in \( E(G) \setminus E(H) \) and we are done.

Now, let \( r \geq 9 \) be an odd integer. By Theorem 5, for every \( k, k \leq \frac{2r}{3}, G \) has a \([k - 1, k]\)-factor whose components are regular. Let \( k = \lceil \frac{2r}{3} \rceil \), \( k' = r - k \), and \( H \) be a \([k - 1, k]\)-factor of \( G \) such that \( H_1 \) be the union of \((k - 1)\)-regular subgraph of \( H \) and \( H_2 = H \setminus H_1 \). It can be easily checked that \( k \leq 2k' \leq 2k - 4 \). Hence by Lemma 1, there exists a function \( f : E(H_1) \to \{2, 3, 4\} \) such that for every \( u \in V(H_1) \), \( \sum_{v \in N_{H_1}(u)} f(uv) = 4k' + 4 \). Also by Lemma 1, there exists a function \( f : E(H_2) \to \{2, 3, 4\} \) such that for every \( v \in V(H_2) \), \( \sum_{u \in N_{H_2}(v)} f(uv) = 4k' \). Finally assign \(-4\) to every edge of \( E(G) \setminus E(H) \). Now, by Theorem 1 and Theorem 2 the proof is complete. \( \square \)

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