Codes and Designs Related to Lifted MRD Codes

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Abstract—The lifted maximum rank distance codes are considered. It is shown that these codes form structures which are transversal designs and also have some similarity to $q$-analog of transversal designs. Upper bounds on the sizes of codes which contain the lifted maximum rank distance codes are derived. A new construction for codes which attain one of these upper bounds is given. Low density parity check codes are derived from the lifted maximum rank distance codes and some of these codes are quasi-cyclic codes attaining the Griesmer bound.

I. INTRODUCTION

Let $F_q$ be the finite field of size $q$. The set of all $k$-dimensional subspaces of the vector space $F_q^n$, for any two given integers $k$ and $n$, 0 $\leq$ $k$ $\leq$ $n$, forms the Grassmannian space (Grassmannian in short) over $F_q$, which is denoted by $G_q(n, k)$. It is well known that $|G_q(n, k)| = \frac{n!}{k!(n-k)!}$, where $\frac{n!}{k!(n-k)!}$ is the $q$-ary Gaussian coefficient. A subset $C$ of the Grassmannian is called an $(n, M, d_S, k)_q$ constant dimension code if it has size $M$ and minimum distance $d_S$, where the distance function in $G_q(n, k)$ is defined by
\begin{equation}
    d_S(X,Y) \overset{\text{def}}{=} \dim X + \dim Y - 2 \dim (X \cap Y),
\end{equation}
for any two subspaces $X$ and $Y$ in $G_q(n, k)$.

These codes gained renewed interest due to the work by Koetter and Kschischang [1], where they presented an application of such codes for error-correction in random network coding.

There is a close connection between error-correcting codes in the Hamming space and combinatorial designs. For example, the codewords of weight 3 in the Hamming code form a Steiner structure (the $q$-analog of Steiner system), if exist, yield perfect constant weight codes in the Grassmannian; constructions of constant dimension codes from spreads (the only known nontrivial Steiner structures) are given in [2] and [3].

In this paper, we consider constant dimension codes and designs based on lifted maximum rank distance codes. These lifted codes were constructed and analyzed in [1], [7]. We first prove that each such code forms a transversal design in sets and it also forms a structure akin (but not $q$-analog) to transversal design in subspaces. In Section II we will introduce the necessary definitions for this discussion and in Section III we will present the connections to combinatorial designs. In Section IV we will derive two upper bounds on the size of a constant dimension code which contains the lifted maximum rank distance code. Some codes which attain one of the upper bounds are known. In Section V we will present a new construction for codes which contain the lifted maximum rank distance code and attain the second upper bound of Section IV. In Section VI we present LDPC codes based on the lifted maximum rank distance code. We prove that some of these codes are quasi-cyclic and attain the Griesmer bound. Conclusion is given in Section VII.

II. BASIC DEFINITIONS

The projective space of order $n$ over $F_q$, denoted by $P_q(n)$, is a collection of all subspaces of $F_q^n$. A projective point in $P_q(n)$ is an one-dimensional subspace of $F_q^n$. A $k$-spread in $G_q(n, k)$ is a set of $k$-dimensional subspaces which partition $F_q^n$. Note that throughout the paper we are making some abuse of notation. Mainly, by saying sometimes that the elements of a $k$-dimensional subspace are the projective points of $P_q(n)$ and hence a $k$-spread consists of $k$-dimensional subspaces which partition the set of projective points of $P_q(n)$ and not elements of $F_q^n$. In both cases we refer to the same object. A $k$-spread in $P_q(n)$ (which is the same as in $G_q(n, k)$) exists if and only if $k$ divides $n$. A partition of all $k$-dimensional subspaces of $P_q(n)$ (also $G_q(n, k)$) into disjoint $k$-spreads is called a $k$-parallelism.

For two $m \times \eta$ matrices $A$ and $B$ over $F_q$ the rank distance is defined by
\begin{equation}
    d_R(A,B) \overset{\text{def}}{=} \text{rank}(A-B).
\end{equation}
A code $C$ whose codewords are $m \times \eta$ matrices over $F_q$, with the rank distance as the distance measure will be called a rank-metric code. It is a maximum rank distance (MRD) code if its minimum distance is $\delta$ and its size is equal $q^{\theta}$, where $\theta = \min\{m(\eta - \delta + 1), \eta(m - \delta + 1)\}$. If $C$ is a linear code then we say that it is an $[m \times \eta, \phi, \delta]$ rank-metric code. Its codewords form a linear subspace of dimension $\phi$ of $F_q^{m \times \eta}$. Rank-metric codes were well studied [8], [9], [10]. It was proved (see [9]) that for an $[m \times \eta, \phi, \delta]$ rank-metric code $C$ we have $\phi \leq \min\{m(\eta - \delta + 1), \eta(m - \delta + 1)\}$. This bound is attained for all possible parameters with MRD codes.

If an $m \times \eta$ codeword $A$ of a rank-metric code is concatenated to the identity matrix $I$, then the subspace $X_A$ defined by the generator matrix, i.e. a matrix whose rows form a basis for the subspace, $(I \ A)$ is called a lifting of $A$. A constant dimension code $C$ such that all its codewords are lifted codewords of an MRD code is called a lifted MRD code [7]. We denote such a code by $C^{MRD}$. From a $[k \times (n-k), (n-k)(k-\delta+1), \delta]$
MRD code, \( k \leq n - k \), we obtain the code \( C^{\text{MRD}} \in \mathcal{G}_q(n,k) \), which is an \( (n, q^{(n-k)(k-\delta+1)}, 2^k, k)_q \) code.

We turn now to define some concepts of combinatorial designs. All these designs are connected to codes and some are essential in our discussion. A **transversal design** of groupsize \( \eta \), blocksize \( \kappa \), strength \( t \) and index \( \lambda \), denoted by \( TD_\lambda(t,\kappa,\eta) \) is a triple \((V,\mathcal{G},\mathcal{B})\), where

1. \( V \) is a set of \( \kappa \eta \) elements (the points);
2. \( \mathcal{G} \) is a partition of \( V \) into \( \kappa \) classes (the groups), each of size \( \eta \);
3. \( \mathcal{B} \) is a collection of \( \kappa \)-subsets of \( V \) (the blocks);
4. each block meets each group in exactly one point;
5. every \( t \)-subset of points that meets each group in at most one point is contained in exactly \( \lambda \) blocks.

When \( t = 2 \), the strength is usually not mentioned, and the design is denoted by \( TD_\lambda(\kappa,\eta) \). A \( TD_\lambda(t,\kappa,\eta) \) is resolvable if its blocks can be partitioned into sets \( \mathcal{B}_1, \ldots, \mathcal{B}_s \), where every element of \( V \) occurs exactly once in each \( \mathcal{B}_i \). The classes \( \mathcal{B}_1, \ldots, \mathcal{B}_s \) are called parallel classes.

An \( N \times k \) array \( A \) with entries from a set of \( s \) elements is an orthogonal array with \( s \) levels, strength \( t \) and index \( \lambda \), denoted by \( OA_\lambda(N, k, s, t) \), if every \( N \times t \) subarray of \( A \) contains each \( t \)-tuple exactly \( \lambda \) times as a row. It is known \([11]\) that a \( TD_\lambda(k,\eta) \) is equivalent to an orthogonal array \( OA_\lambda(\lambda \cdot \eta^2, k, \kappa, \eta, 2) \).

The well-known concept of \( q \)-analogs replaces subsets by subspace of a vector space over a finite field and their orders by the dimensions of the subspaces. In particular, the \( q \)-analog of a constant weight code is a constant dimension code.

A **Steiner system** \( S(t, k, n) \) is a set \( S \) of \( k \)-subsets of an \( n \)-set \( \mathcal{N} \), called blocks, such that each \( t \)-subsets of \( \mathcal{N} \) is contained in exactly one block of \( S \). The number of blocks in an \( S(t, k, n) \) is \( \binom{n}{k} / \binom{t}{k} \) \([12]\) p. 63]. The \( q \)-analog of a Steiner system is a Steiner structure. A Steiner structure \( S_q(t, k, n) \) is a set \( S \) of \( k \)-dimensional subspaces in \( \mathbb{F}_q^n \), called blocks, such that each \( t \)-dimensional subspace of \( \mathbb{F}_q^n \) is contained in exactly one block of \( S \). The number of blocks in an \( S_q(t, k, n) \) is \( \binom{n}{t} \binom{t}{k} \) \([3]\). Steiner systems and Steiner structures are connected to perfect codes in the Hamming scheme and the Grassmann scheme, respectively. The only known nontrivial Steiner structures are the \( k \)-spreads which are \( S_q[1,k,n] \).

### III. LIFTED MRD CODES AND TRANSVERSAL DESIGNS

Let \( C^{\text{MRD}} \) be an \( (n, q^{(n-k)(k-\delta+1)}, 2^k, k)_q \), lifted MRD code. Let \( \mathcal{L} \) be the set of \( q \)-ary vectors of length \( n \) in which not all the first \( k \) entries are \( z \). This set is clearly of size \( q^n - q^{n-k} \). The first lemma is an immediate result from the structure of lifted words.

**Lemma 1:** All the nonzero vectors which are contained in codewords of the code \( C^{\text{MRD}} \) belong to \( \mathcal{L} \).

Let \( \mathcal{V} \) be the set of projective points formed from \( \mathcal{L} \). In other words, \( \mathcal{V} \) is a set of all the one-dimensional subspaces of \( \mathbb{F}_q^n \) whose nonzero vectors are contained in \( \mathcal{L} \); therefore, \( |\mathcal{V}| = q^n - q^{n-k} \). Let \( \mathbb{P} \) be the set of projective points of \( \mathcal{P}(k) \); clearly, the size of \( \mathbb{P} \) is \( \frac{q^k - 1}{q - 1} \). For a set \( S \subseteq \mathbb{P} \), let \( (S) \) denote the subspace of \( \mathbb{F}_q^n \) spanned by the elements of \( S \). Each \( x \in \mathbb{P} \) is identified with the vector \( v_x \in \mathcal{V} \) in which the first nonzero entry is an \( \text{one} \). For each \( x \in \mathbb{P} \) we define \( \mathcal{V}_x = \{ v \in \mathcal{V} | v = x, y, y \in \mathbb{F}_q^{n-k} \} \). There are \( \frac{q^k - 1}{q - 1} \) disjoint such sets, one for each projective point of \( \mathbb{P} \). For each such point \( x \), \( \mathcal{V}_x \) contains \( q^{n-k} \) projective points. Therefore, each codeword of \( C^{\text{MRD}} \) contains \( q^{n-k} - 1 \) elements of \( \mathcal{V} \). Hence, as a consequence from Lemma 2 we have

**Corollary 1:** A codeword of \( C^{\text{MRD}} \) contains exactly one element from \( \mathcal{V}_x \) for each \( x \in \mathbb{P} \).

**Lemma 2:** Each \( (k - \delta + 1) \)-dimensional subspace \( Y \) of \( \mathbb{F}_q^n \) whose nonzero vectors are contained in \( \mathcal{L} \), is contained in exactly one codeword of \( C^{\text{MRD}} \).

**Proof:** Let \( S = \{ Y \in \mathcal{G}_q(n,k-\delta+1) \}: |Y \cap \mathcal{L}| = q^{k-\delta+1} - 1 \}. Since the minimum distance of \( C^{\text{MRD}} \) is \( 2 \delta \), it follows that the intersection of any two codewords of \( C^{\text{MRD}} \) is at most of dimension \( k - \delta \). In addition, the codewords of \( C^{\text{MRD}} \) contains only nonzero vectors from \( \mathcal{L} \). Hence, each subspace \( Y \in S \) is contained in at most one codeword of \( C^{\text{MRD}} \). The size of \( \mathcal{S}_q(k-\delta+1) \) and each codeword of \( C^{\text{MRD}} \) contains \( q^{(n-k)(k-\delta+1)} \) distinct \( (k - \delta + 1) \)-dimensional subspaces.

We compute now the size of \( S \). There are \( \binom{n}{k} \binom{k-\delta+1}{k} \) ways to choose a \( (k - \delta + 1) \)-dimensional subspaces of \( \mathbb{F}_q^n \). For each such subspace \( Y \) we choose an arbitrary basis \( x_1, x_2, \ldots, x_{\delta+1} \); there are \( q^{n-k} \) ways to choose a projective point from \( \mathcal{V}_x \), \( 1 \leq i \leq k - \delta + 1 \), for a total of \( \binom{n-\delta+1}{n-k} q^{(n-k)(k-\delta+1)} \) ways to choose a basis for a \( (k - \delta + 1) \)-dimensional subspaces of \( \mathbb{F}_q^n \) meeting each \( \mathcal{V}_x \), \( 1 \leq i \leq k - \delta + 1 \), in exactly one projective point (see Corollary 1). Hence, \( |S| = \binom{n}{k} \binom{k-\delta+1}{k} q^{n-k} \).

Thus, the lemma follows.

**Corollary 2:** Each \( (k - \delta - i) \)-dimensional subspace \( Y \) of \( \mathbb{F}_q^n \) whose nonzero vectors are contained in \( \mathcal{L} \), is contained in exactly \( q^{(n-k)(k-\delta-i)} \) codewords of \( C^{\text{MRD}} \).

**Proof:** This is a direct consequence from Lemma 2 and the following enumeration steps. The size of \( C^{\text{MRD}} \) is \( q^{(n-k)(k-\delta+1)} \). Each codeword of \( C^{\text{MRD}} \) contains \( \binom{n}{k} \binom{k-\delta-i}{k} \) \( (k - \delta - i) \)-dimensional subspaces. S contains \( q^{(n-k)(k-\delta-i)} \) distinct \( (k - \delta - i) \)-dimensional subspaces.

**Corollary 3:** Any two projective points \( u_1, u_2 \in \mathcal{V} \), such that
Corollary 4: Any projective point \( u \in \mathbb{V} \) is contained in exactly \( q^{(n-k)(k-\delta)} \) codewords of \( C_{\text{MRD}} \).

For simplicity, in the sequel we will consider only linear MRD codes constructed by Gabidulin \( [8] \). It does not restrict our discussion as such codes exist for all parameters.

Lemma 4: The set of codewords of \( C_{\text{MRD}} \) can be partitioned into \( q^{(n-k)(k-\delta)} \) classes each of size \( q^{n-k} \), such that each element of \( \mathbb{V} \) is contained in exactly one codeword of each class.

Proof: First we prove that a lifted MRD code contains a lifted MRD subcode without disjoint codewords. Let \( G \) be the generator matrix of an \([k \times (n-k), (n-k)(k-\delta+1), \delta]\) MRD code \( C_{\text{MRD}} \), \( n-k \geq k \). Then \( G \) has the following form

\[
G = \begin{pmatrix}
g_1 & g_2 & \cdots & g_k \\
g_1^q & g_2^q & \cdots & g_k^q \\
\vdots & \vdots & \ddots & \vdots \\
g_1^{q^{k-\delta}} & g_2^{q^{k-\delta}} & \cdots & g_k^{q^{k-\delta}}
\end{pmatrix},
\]

where \( g_i \in \mathbb{F}_{q^n} \) are linearly independent over \( \mathbb{F}_q \). If \( k-\delta \) rows (the first or the last ones) are removed from \( \mathbb{G} \), then it results in an MRD subcode of \( C \) with the minimum distance \( k \). In other words, an \([k \times (n-k), n-k, k]\) MRD code \( \hat{C} \) of \( C \) is obtained. The corresponding lifted code is an \((n, q^{n-k}, 2k, k)_q \) lifted MRD subcode of \( C_{\text{MRD}} \).

Let \( \hat{C}_1 = \hat{C}, \hat{C}_2, \ldots, \hat{C}_{q^{(n-k)(k-\delta)}} \) be the \( q^{(n-k)(k-\delta)} \) cosets of \( \hat{C} \) in \( C \). All these \( q^{(n-k)(k-\delta)} \) cosets are nonlinear rank-metric codes with the same parameters as the \([k \times (n-k), n-k, k]\) code. Therefore, their lifted codes form a partition of \( C_{\text{MRD}} \) into \( q^{(n-k)(k-\delta)} \) classes each one of size \( q^{n-k} \), such that each point of \( \mathbb{V} \) is contained in exactly one codeword of each class.

Corollary 5: The codewords of a lifted MRD \((n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q \) code form the blocks of resolvable transversal design \( T D_{\lambda}((q^2-1)/(q-1), q^{n-k}) \), with \( \lambda = q^{(n-k)(k-\delta+1)} \) and with \( q^{(n-k)(k-\delta)} \) parallel classes, each one of size \( q^{n-k} \).

Proof: Let \( \mathbb{V}_x \) be the set of \( q^{\frac{n-k}{q-1}} \) points for the design. Each set \( \mathbb{V}_x \), \( x \in \mathbb{F}_{q^n} \) is defined to be a group and there are \( q^{k-1} \) groups, each one of size \( q^{n-k} \). The \( k \)-dimensional subspaces (codewords) of \( C_{\text{MRD}} \) are the blocks of the design. By Corollary \( 1 \) each block meets each point in exactly one point. By Corollary \( 3 \) each 2-subset which meets each group in at most one point is contained in exactly \( q^{(n-k)(k-\delta+1)} \) blocks. Finally, by Lemma \( 2 \) the design is resolvable with \( q^{(n-k)(k-\delta)} \) parallel classes, each one of size \( q^{n-k} \).

Remark 1: By the equivalence of transversal designs and orthogonal arrays, an \((n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q \) code \( C_{\text{MRD}} \) induces an \( OA(q^{(n-k)(k-\delta+1)}, q^{k-1}, q^{n-k}, 2) \).

Remark 2: A \([k \times (n-k), (n-k)(k-\delta+1), \delta]\) MRD code \( C \) is an MDS code if it is viewed as a code of length \( k \) over \( GF(q^{n-k}) \). Thus its codewords form an orthogonal array \( OA(q^{(n-k)(k-\delta+1)}, q^{k-1}, q^{n-k}, k-\delta+1) \).

The definition of \( q \)-analog for transversal design is somehow ambiguous. It might be better to generalize the definition of the transversal design to obtain a definition in terms of subspaces as follows. Let \( \mathbb{U} \) be the set of projective points of \( \mathbb{P}_q(n) \) that contains only vectors starting with \( \kappa \) zeroes. A subspace transversal design of groupsize \( \eta \), blocksize \( k \), strength \( t \) and index \( \lambda \), denoted by \( STD_{\lambda}(t, \kappa, \eta) \) is a triple \((\mathbb{V}, \mathbb{G}, \mathbb{B})\), where:

1. \( \mathbb{V} \) is the subset of all projective points of \( \mathbb{P}_q^{n+\kappa} \) which does not contain the points of \( \mathbb{U} \), \( |\mathbb{V}| = \frac{(q^{n+\kappa})^t}{q^\kappa} \); 2. \( \mathbb{G} \) is a partition of \( \mathbb{V} \) into \( q^{n-\kappa} \) classes (the groups), each of size \( q^\kappa \); 3. \( \mathbb{B} \) is a collection of \( k \)-dimensional subspaces which contain only points from \( \mathbb{V} \) (the blocks); 4) each block meets each group in exactly one point; 5) every \( t \)-dimensional subspace (with points from \( \mathbb{V} \)) which meets each group in at most one point is contained in exactly \( \lambda \) blocks.

As a direct consequence form Lemma \( 3 \) and Corollary \( 5 \) we have the following theorem.

Theorem 5: An \((n, q^{(n-k)(k-\delta+1)}), 2\delta, k)_q \) code \( C_{\text{MRD}} \) forms a resolvable \( STD_{1}(k-\delta+1, k, n-k) \).

Remark 3: By Corollary \( 2 \) it follows that a subspace transversal design of strength \( t \) is also a subspace transversal design of strength \( t' \), \( 1 \leq t' \leq t \).

IV. UPPER BOUNDS ON THE SIZE OF A CODE

Upper bounds on the sizes of constant dimension codes were obtained in several papers, e.g. [1], [4]. In this section we will be interested in upper bound on the size of a code which contains the lifted MRD code.

Let \( T \) be a subspace transversal design derived from \( C_{\text{MRD}} \) by Theorem 5. Let \( \mathbb{V}_0 \) be the set of projective points of \( \mathbb{P}_q(n) \) which contains only vectors that start with \( k \) zeroes. Each constant dimension code \( \mathbb{C} \) such that \( C_{\text{MRD}} \subseteq \mathbb{C} \) have to contain points of \( \mathbb{V}_0 \) since each codeword in \( \mathbb{C} \setminus C_{\text{MRD}} \) have to contain either at least two points in the same group \( T \) or only the points of \( \mathbb{V}_0 \).

Let \( A_q(n, d, k) \) be the maximum size of an \((n, M, d, k)_q \) code.

Theorem 6: If an \((n, M, 2(k-1), k)_q \) code \( \mathbb{C} \) contains the \((n, q^{2(n-k)}, 2(k-1), k)_q \) lifted MRD code then \( M \leq q^{2(n-k)} - A_q(n-k, 2(k-2), k-1) \).

Proof: Let \( T \) be an \( STD_1(k, n-k) \) obtained from an \((n, q^{2(n-k)}, 2(k-1), k)_q \) code \( C_{\text{MRD}} \). Since the minimum distance of \( C \) is \( 2(k-1) \), it follows that the codewords of \( \mathbb{C} \) intersect in at most one-dimensional subspace. Hence, each two-dimensional subspace of \( \mathbb{P}^n_q \) is contained in at most one codeword of \( \mathbb{C} \). Each two-dimensional subspace \( X \) of \( \mathbb{P}^n_q \), such that \( X = \{(u, v)\} \), where \( v \in \mathbb{F}_{q^2} \) and \( u \in \mathbb{F}_q \), for \( x \neq y, x, y \in \mathbb{F}_q \), is covered by the codewords of \( C_{\text{MRD}} \), since they form the blocks of \( T \). Hence, each codeword \( X \in \mathbb{C} \setminus C_{\text{MRD}} \) is either contained in \( \mathbb{V}_0 \) or contained in \( \mathbb{V}_0 \cup \mathbb{V}_x \), for some \( x \in \mathbb{F}_q^2 \). Therefore, \( \dim(X \cap \mathbb{V}_0) = k \) in the first case and \( \dim(X \cap \mathbb{V}_0) = k-1 \) in the second case. Each \((k-1)\)-dimensional subspace of \( \mathbb{V}_0 \) can occur only in one codeword. Moreover, since the minimum distance of the code is \( 2(k-1) \), it follows that if \( X_1, X_2 \in \mathbb{C} \setminus C_{\text{MRD}} \) and \( \dim(X_1 \cap \mathbb{V}_0) = \dim(X_2 \cap \mathbb{V}_0) = k-1 \) then \( d_S(X_1 \cap \mathbb{V}_0, X_2 \cap \mathbb{V}_0) \geq 2(k-2) \).
Therefore, \( \{ X \cap V_0 : X \in \mathbb{C} \setminus \mathbb{C}_{MRD}, \dim(X \cap V_0) = k-1 \} \) is an \((n-k, M, 2(k-2), k-1)_q\) code. If each codeword \( X \in \mathbb{C} \setminus \mathbb{C}_{MRD} \) such that \( \dim(X \cap V_0) = k \) is replaced by one of its \((k-1)\)-dimensional subspaces \( X' \) then the set which includes these \((k-1)\)-dimensional subspaces and the subspaces of \( \{ X \cap V_0 : X \in \mathbb{C} \setminus \mathbb{C}_{MRD}, \dim(X \cap V_0) = k-1 \} \) is an \((n-k, M, 2(k-2), k-1)_q\) code. This implies the result of the theorem.

Similarly to Theorem 6 we can prove the following theorem (some extra arguments are added in the proof).

Theorem 7: If an \((n, M, 2k, 2k)\) code \( \mathbb{C} \) contains the \((n, q^{n-2k}(k-1), 2k, 2k)_q\) lifted MRD code then \( M \leq q^{n-2k}(k-1) + \left( \frac{n-2k}{k} \right) + \frac{q^{n-2k}}{q^2 - q^k} + A_q(n-2k, 2k, 2k, 2k)\).

V. CONSTRUCTION FOR CONSTANT DIMENSION CODES

In this section we will discuss and present constructions of codes which attain the bounds of Section IV. The bound of Theorem 6 seems to be attainable, at least for \( k = 3 \).

In particular Trautmann and Rosenthal [13] presented codes which attain the bound for \( k = 3, 7 \leq n \leq 9 \), and any power of a prime \( q \). A construction based on partitions of subspaces might be good to obtain a general result. We omit the details as we believe that the search for \( k = 3 \) should be in the direction of cyclic codes which were shown to be better for \( q = 2 \) and \( 8 \leq n \leq 12 [4], [14]\).

Now present a construction for an \((8, 4797, 4, 4)_2\) code \( \mathbb{C} \), which attains the bound of Theorem 7. Note that this is the largest currently known \((8, M, 4, 4)_2\) code since a Ferrers diagrams construction [13] a code of size 4573 was obtained, and a lexicode of size 4605 was found in [19].

**Construction A:** Let \( \mathbb{C}_{MRD} \) be an \((8, 2^{12}, 4, 4)_2\) lifted MRD code, and let \( T \) be the corresponding ST\(D_1(3, 4, 4) \). Since each \( 3 \)-dimensional subspace of \( T \) in at most one point is covered by the codewords of \( \mathbb{C}_{MRD} \), it follows that each codeword \( X \in \mathbb{C} \setminus \mathbb{C}_{MRD} \) meets \( T \) in at most three different groups.

We generate the new codewords (blocks) of \( \mathbb{C} \setminus \mathbb{C}_{MRD} \) as follows. There is a partition of all the subspaces of \( G_2(4, 2) \) into seven 2-spreads, each one of size five, i.e., \( 2 \)-parallelism in \( G_2(4, 2) \) [15]. First, we apply this partition on \( V_0 \). Each \( 2 \)-dimensional subspace of \( V_0 \) will be a subspace of 20 codewords in \( \mathbb{C} \setminus \mathbb{C}_{MRD} \). The 15 groups in \( T \) can be considered as the points of a \( 4 \)-dimensional space and hence we can form from their \( 2 \)-dimensional subspaces the same partition into 2-spreads. Let \( B \) be a block (a \( 2 \)-dimensional subspace) of the \( i \)th spread in this partition. For each block \( B^j \) in the same spread, \( 1 \leq j \leq 5 \), we generate four codewords in \( \mathbb{C} \setminus \mathbb{C}_{MRD} \) as follows. Each codeword consists of the points of \( B \) in \( V_0 \) and four points from each group \( V_{x_i} \), where \( x \in B^j \). We denote these three groups by \( V_{x_i}, V_{x_j}, V_{x_k} \). The \( 2 \)-dimensional subspace \( B \) has four cosets \( B \cup \{ 0 \}, B_0, B_1, B_2 \) in the \( 4 \)-dimensional space. The four points from each group \( V_{x_i} \), \( 0 \leq i \leq 2 \), correspond to these cosets. In the first such codeword of \( \mathbb{C} \setminus \mathbb{C}_{MRD} \) all four points in every \( V_{x_i} \), \( 0 \leq i \leq 2 \), correspond to the same coset \( B \cup \{ 0 \} \). In the other three codewords, in which the points of \( B \) are taken in \( V_0 \), in three groups \( V_{x_0}, V_{x_1}, V_{x_2} \), we take the points of \( B_0, B_1, B_2 \) (for each group we take a new coset) in the following way. First we take the points of \( B_i \) in \( V_{x_i} \), second we take the points of \( B_{i+1} \) in \( V_{x_{i+1}} \), and finally we take the points of \( B_{i+2} \) in \( V_{x_{i+2}} \), where \( 0 \leq i \leq 2 \), and the index of a coset \( B_i \) is taken modulo 3.

Now we calculate the number of the new codewords. For each \( 2 \)-dimensional subspace of \( V_0 \) we obtain \( 5 \cdot 4 \) codewords of dimension four, where the first factor is equal to the size of a spread, and the second factor is the number of cosets. In addition to these codewords we add a codeword which includes all the points of \( V_0 \). Thus in the constructed code \( \mathbb{C} \) there are \( 2^{12} + \left( \frac{4}{2} \right) \cdot 20 + 1 \) codewords, and the minimum distance follows from the construction.

**Remark 4:** Construction A can be easily generalized for all \( q \geq 2 \), since there is a \( 2 \)-parallelism in \( G_q(3, n) \) for all \( q \), where \( n \) is power of 2 [15]. Thus from this construction we can obtain a \((8, M, 4, 4)_q\) code with \( M = q^{12} + \left( \frac{4}{2} \right) q (q^2 - 1)q^2 + 1 \).

**Remark 5:** In general, the existence of \( k \)-parallelism in \( G_q(n, k) \) is an open problem. However, there are some cases, where it was proved that there exists such \( k \)-parallelism. One example which is considered in Construction A and Remark 4 is for \( k = 2 \) and all \( q \), where \( n \) is power of 2 [15]. Also it is known that \( 2 \)-parallelism exists for \( q = 2 \) and all \( n \) [16]. Recently it has been proved that there is a \( 3 \)-parallelism for \( q = 2 \) and \( n = 6 [17] \). Thus we believe that Construction A can be generalized to a larger family of parameters assuming that there exists a corresponding parallelism.

VI. LDPC CODES FROM CONSTANT DIMENSION CODES

For each codeword \( X \) of a constant dimension code \( \mathbb{C}_{MRD} \) we define its binary incident vector \( x \) of length \( |\mathbb{V}| = q^k - q^{n-k} \) such that \( x_z = 1 \) if and only if the point \( z \in \mathbb{V} \) is contained in \( X \).

Let \( H \) be the \( |\mathbb{C}_{MRD}| \times |\mathbb{V}| \) binary matrix where the rows of \( H \) are the incidence vectors of codewords of \( \mathbb{C}_{MRD} \). This matrix \( H \) can be considered as the blocks-points incidence matrix of \( TD_\lambda(\frac{q^k - 1}{q-1}, q^{n-k}) \), with \( \lambda = q^{(n-k)(k-d-1)} \). Then by Corollary 4 each column has \( q^{(n-k)(k-d)} \) ones and also each row has \( (q^k - 1)/(q-1) \) ones.

**Lemma 8:** The matrix \( H \) obtained from an \((n, q^{(n-k)(k-b+1)}, 2b, k)\) \( \mathbb{C}_{MRD} \) code can be decomposed into blocks, where each block is a \( q^{n-k} \times q^{n-k} \) permutation matrix.

**Proof:** Follows from Lemma 4 that the related transversal design is resolvable. A suitable rows permutation of \( H \) should be taken, where the rows of each one of the \( q^{(n-k)(k-d)} \) parallel classes of size \( q^{n-k} \) are made consecutive.

Let \( C \) be a linear code with parity check matrix \( H \), and let \( C^T \) be a linear code with parity check matrix \( H^T \). Note, that \( H^T \) is an incidence matrix of a net, the dual structure to the transversal design. Codes obtained from nets and transversal designs were considered for example in [20], [21].

The code \( C \) has length \( q^{(n-k)(q^k - 1)/(q-1)} \) and the code \( C^T \) has length \( q^{(n-k)(k-d+1)} \). The following lemma provides the lower bound on minimum distance of these codes. Note that we do not consider the trivial case where \( d = k \).

**Lemma 9:** The minimum distance \( d \) of \( C \) and the minimum distance \( d^T \) of \( C^T \) satisfy

\[
d \geq \frac{q^{n-k}(q^k - 1)}{q^k - q}; \quad d^T \geq \frac{q^k - 1}{q^{k-d} - 1} + 1.
\]
Proof: The bound on $d$ follows from Tanner’s bound and calculation of the second largest eigenvalue $\mu_2 = q^{(n-k)(k-\delta)}H^T H$; and the bound on $d^T$ follows from the bound of Theorem 1.

Since the number of ones in the incidence matrix of a transversal design is small (there are $q^{(n-k)(k-\delta+1)}$ entries of $H$), it is tempting to consider such code as an LDPC code. To avoid short cycles in the Tanner graph of the resulting code we consider now only lifted MRD codes with minimum distance $2d = 2(k-1)$. Thus the girth of the corresponding Tanner graph will be at least 6. Then the size of such codes is $M = q^{2(n-k)}$ and $H$ is an incidence matrix of $TD_{1}(\frac{q^{n-1}-1}{q-1}, q^{n-k})$.

The stopping distance of a code $C$ with a parity-check matrix $H$ is defined as the largest integer $s(H)$ such that every set of at most $s(H) - 1$ columns of $H$ contains at least one row of weight one. The stopping distance plays a role in iterative decoding similar to the role of the minimum distance in maximum likelihood decoding. It is well known that $s(H)$ is less or equal to the minimum distance of the code $C$. By Theorem 9, the lower bound on minimum distance of Lemma 9 is also the bound on stopping distance, Therefore, from Lemma 9 we have:

**Theorem 10**: Let $C$ and $C^T$ be the LDPC codes obtained from the $(n, q^{2(n-k)}, d_S = 2(k-1), k)_q$ lifted MRD code with parity-check matrices $H$, respectively. Then the length of $C$ and $C^T$ is $q^{n-k}(q^k-1)$ and $q^{2(n-k)}$, respectively, the minimum distances $d$, $d^T$, and the stopping distances $s(H)$, $s(H^T)$, of $C, C^T$, respectively, satisfy

$$d \geq s(H) \geq \frac{q^{n-k}(q^k-1)}{q^k-q}; \quad d^T \geq s(H^T) \geq \frac{q^k-1}{q-1} + 1.$$  

Consider now the family of binary codes which obtained from lifted MRD codes with $k = 2$. A code is called quasi-cyclic if there exists an integer $p$ such that each cyclic shift of a codeword by $p$ positions is a codeword.

**Theorem 11**: The LDPC code $C$ obtained from the $(n, 2^{2(n-k)}, 2, 2)_q$ quasi-cyclic code with $p = 2^{n-2}$.

**Remark 6**: The codes of Theorem 11 are equivalent to the punctured Hadamard codes. They also attain the Griesmer bound (p. 546).

**VII. Conclusion**

Constant dimension codes derived from lifted MRD code were considered. Connections between these codes and designs were derived. A new construction for constant dimension codes which contain these codes as subcodes was given. Finally, LDPC codes were derived from these codes. Many questions in this context remained unsolved and they will be subject for future research. We will briefly mention one of them. The upper bound on $A_2(q, 3)$ is $381$. If a code which attains the bound exists, it contains only 128 codewords of a rank-metric code which should be lifted (compared to 256 codewords of an MRD code). Assume a maximal rank-metric code of size 128 is taken and lifted. To which size we can extend the obtained constant dimension code? Similarly, the upper bound on $A_2(8, 4, 4)$ is $17 \cdot 381 = 6477$. If a code which attains the bound exists, it contains $17 \cdot 128$ codewords of a rank-metric code which should be lifted. Clearly, such a rank-metric code cannot be linear.

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