Abstract. In a cable-stayed bridge the bridge deck is supported by a large number of inclined cables of varying lengths and inclinations, which are suspended from a bridge tower. Vertical excitation of the bridge deck, for example, due to traffic loading, is influencing the cables. This may result in quite complex dynamics of those cables, whose basic vibration modes are near resonances with the excitation. These dynamics are nonlinear due to gravitational sag and tension variations in the cable.

We consider here a single inclined cable of length $L$, whose lower support is periodically excited in the vertical direction with a frequency that is close to the second natural frequency $\omega_2$ of the cable. This system is described mathematically by a system of six ordinary differential equations for the contributions of three basic modes of vibration: the second in-plane mode (of vertical motion) and the first and second out-of-plane modes (of horizontal motion). The participating basic modes and the parameters are chosen to represent a scaled-down cable experiment that was designed to represent a full-scale cable-stayed bridge.

More specifically, we present a comprehensive bifurcation study of the different possible coupled-mode responses of the cable, which are distinguished by which of the two out-of-plane modes participate in the dynamics apart from the directly excited in-plane mode. These coupled modes are equilibria of the ODE model, and they are continued numerically with the package AUTO in the dimensionless amplitude $\Delta/L$ and the frequency ratio $\Omega/\omega_2$ of the excitation. Each type of coupled-mode response, of which there are four, is computed and represented as a surface of equilibria in $(\Delta/L, \Omega/\omega_2, ||N||)$-space, where $||N||$ is a suitable norm. We also compute curves of bifurcations, along which the stability of the equilibria changes and different surfaces meet. Overall, we obtain a geometric picture of the two-parameter bifurcation diagram of the coupled-mode response of the cable. Chosen cross sections, for fixed $\Delta/L$ and for fixed $\Omega/\omega_2$, illucidate its intricate properties.

Key words. Cable dynamics, parametric excitation, modal equations, bifurcation analysis, equilibrium surfaces.

AMS subject classifications. 70K30, 70K50, 34C15
1. INTRODUCTION. In a cable-stayed bridge, the towers support the deck by many inclined cables of different length [11]. Therefore, deck excitation due to traffic or other external forces can trigger harmonic vibrations of some of the cables. This motivates the need to explore the dynamic behaviour of inclined cables due to harmonic excitation, such as those present in cable-stayed bridges. Cable dynamics is nonlinear due to gravitational sag and tension variations during oscillations, and these nonlinearities result in the coupling between different basic modes of the cable. Interesting dynamic responses are the result of this coupling, in light of other important properties of the system:

1. cables are very lightly damped;
2. both in-plane (vertical) and out-of-plane (horizontal) motion is possible;
3. the natural frequencies of the basic modes and/or the excitation may be close to resonance.

We consider here a single inclined cable subject to vertical excitation of its lower support; the corresponding physical setup is sketched in figure 1. It has been shown that small amplitude motion of the cable anchorage can result in large amplitude cable vibrations [16]. Parametric excitation was studied by using Mathieu-Hill equations [27, 30]; the most significant effect was shown to occur when the ratio between the excitation frequency and the first natural frequency of the cable is 2:1 [13]. Out-of-plane motion was considered in [8], where auto-parametric resonance and nonlinearities for the multiple cable modes are included. The coupling between the in-plane and out-of-plane modes of vibration of the cable was studied by using nonlinear Mathieu-type equations to model parametric resonance; see [5, 6, 7]. The triggering of an out-of-plane motion is also widely formulated as an instability of the semi-trivial solution (in which only the directly excited mode is present in the response) [2, 9, 14, 18, 19, 22, 24, 26]. Rich experimental analysis of the cable response in out-of-plane modes of vibration, periodic oscillations and even chaotic behaviour has been reported in [17, 20, 21, 23].

Warnitchai et al. [33] derived model equations of motion for the nonlinearly coupled basic modes of an inclined cable subject to support excitation; see [32] for an extended discussion of the derivation. A three-mode model based on Warnitchai’s equations for the second in-plane mode and the first and second out-of-plane modes was considered in [9] These three second-order differential equations are scaled and averaged and led to a system of six autonomous ODEs. Their solutions
represent the coefficients that modulate the sine and cosine members in the mode shape functions of the cable response. In [9] these solutions were used to identify the boundary of the directly excited pure second in-plane response in a parameter plane via examination of the localised stability of each out-of-plane mode about its zero response; these results were validated with experimental data of a scaled model of an inclined cable. Generalised equations of the same type as those in [9] for excitation in every natural frequency and response in all possible modes are provided in [14]; there a theoretical and an experimental study are compared not only for excitation in the second natural frequency of the cable, but also for excitation its first, third and fourth natural frequency. A four-mode model based on Warnitchai’s equations was investigated and verified experimentally in [15], where the response amplitudes of the various modes are studied and compared with experimental measurements for the cable discussed in [14].

In this paper we present a detailed bifurcation study of the three-mode ODE model from [9] to investigate when and how the two basic out-of-plane modes contribute to the dynamics upon the direct excitation of the second in-plane mode. We consider an inclined cable of length $L$ with second natural frequency $\omega_2$ (for both in-plane and out-of-plane vibrations) and excited vertically at its lower end by the periodic excitation $\delta = \Delta \cos(\Omega t)$. Steady-state solutions (equilibria) of the system of autonomous ODE correspond to periodic dynamics of the cable. The equilibria can be distinguished by determining which of the three basic in-plane and out-of-plane modes contribute to the cable response (i.e., have non-zero amplitudes); we refer to these equilibria as (coupled) modes of the cable. The different types of equilibria, and their bifurcations, are continued with the numerical continuation software AUTO [3, 4] in two parameters: the frequency ratio $\Omega/\omega_2$ and the dimensionless amplitude $\Delta/L$. The equilibria, and hence the different types of modes, are represented by surfaces in $(\Omega/\omega_2, \Delta/L, ||N||)$-space by plotting the Euclidean norm $||N||$ of their amplitude contributions. Modes that are stable correspond to the observable dynamics of the cable, and they are distinguished by darker shading. Different surfaces meet or encounter changes of stability along curves of bifurcations that are also computed. In this way, we obtain a comprehensive geometric picture of surfaces of the different coupled modes of vibration. Its knowledge allows one to identify the contributions of the three basic in-plane and out-of-plane modes to the observed cable response for given (or changing) amplitude and frequency of the periodic excitation $\delta(t)$. This is illuminated with the discussion of one-parameter bifurcation diagrams at cross sections of interest.

The paper is organized as follows. The differential equation model of the cable, which describes the evolution of its three basic modes, are introduced in section 2 together with the specific settings of the parameters. Section 3 is devoted to the bifurcation analysis of the system as a function of frequency and amplitude of the excitation. The overall picture of connected surfaces of equilibria is built up gradually in sections 3.1–3.5 by adding surfaces corresponding to the different coupled modes of vibration one by one. One-parameter bifurcation diagrams as a function of the excitation frequency, which are intersections of the surfaces of equilibria for fixed amplitude of excitation, are considered in section 3.6. Finally, some conclusions are drawn in section 4.

2. Three-mode model of inclined cable dynamics. In this study we concentrate on sinusoidal support excitations $\delta(t)$ close to the second in-plane natural frequency of an inclined cable, as sketched in figure 1. We chose parameters to represent the cable that was experimentally tested in [9], which was designed as a scaled-down model for larger, realistic cable. The experiment used a 1.98m long, 0.8mm diameter steel cable, inclined at $20^\circ$ to the horizontal, with a mass of 0.67 kg/m (which is achieved by using a series of lead weights).

We consider here the dynamics of the directly excited second in-plane mode with natural frequency $\omega_2 = 2\omega_1$, and the first and second out-of-plane modes. These two out-of-plane modes are included as they have 2:1 and 1:1 internal resonances with the second in-plane mode (i.e., natural frequencies $\omega_1 = \omega_2/2$ and $\omega_2$, respectively). The first in-plane mode, on the other hand,
is not included as its frequency is 1.07ω₁ (see [9]) and, therefore, exhibits a significant response only at excitation frequencies beyond those considered here; the inclusion of this basic mode in the model is discussed in [28] and [29].

The modal equations of motion that form the basis of this study were derived in their general form by Warnitchai et al. [33]; see also the discussion in [32]. When scaled and averaged (see [9] for the details) these equations give the following expressions for the amplitude dynamics of the modes:

\[
y'_{1c} = -\xi \omega_1 y_{1c} - \left( \mu \omega_1 + \frac{N_1 \Delta}{4 \omega_1} - \frac{3W_{12}}{32 \omega_1} Y_1^2 - \frac{W_{12}}{4 \omega_1} (Y_2^2 + Z_2^2) \right) y_{1s}, \\
y'_{1s} = \left( \mu \omega_1 - \frac{N_1 \Delta}{4 \omega_1} - \frac{3W_{12}}{32 \omega_1} Y_1^2 - \frac{W_{12}}{4 \omega_1} (Y_2^2 + Z_2^2) \right) y_{1c} - \xi \omega_1 y_{1s}, \\
y'_{2c} = \left( \frac{W_{12} z_{2c}}{2 \omega_1} - 2 \xi \omega_1 \right) y_{2c} - \left( 2 \mu \omega_1 - \frac{W_{12}}{8 \omega_1} \left( Y_1^2 + 6(Y_2^2 + Z_2^2) - 4z_{2s}^2 \right) \right) y_{2s}, \\
y'_{2s} = \left( 2 \mu \omega_1 - \frac{W_{12}}{8 \omega_1} \left( Y_1^2 + 6(Y_2^2 + Z_2^2) - 4z_{2s}^2 \right) \right) y_{2c} - \left( 2 \xi \omega_1 + \frac{W_{12} z_{2c}}{2 \omega_1} \right) y_{2s}, \\
z'_{2c} = \left( \frac{W_{12} y_{2c} z_{2s}}{2 \omega_1} - 2 \xi \omega_1 \right) z_{2c} - \left( 2 \mu \omega_1 - \frac{W_{12}}{8 \omega_1} \left( Y_1^2 + 6(Y_2^2 + Z_2^2) - 4y_{2s}^2 \right) \right) z_{2s}, \\
z'_{2s} = \left( 2 \mu \omega_1 - \frac{W_{12}}{8 \omega_1} \left( Y_1^2 + 6(Y_2^2 + Z_2^2) - 4y_{2s}^2 \right) \right) z_{2c} - \left( 2 \xi \omega_1 + \frac{W_{12} y_{2c} z_{2s}}{2 \omega_1} \right) z_{2s} - B \Delta \omega_1. \tag{2.1}
\]

Here \( z \) and \( y \) represent in-plane and out-of-plane modes, respectively, where subscripts 1 and 2 indicate the mode number and subscripts \( c \) and \( s \) indicate sine and cosine components of the response. The averaged amplitudes of the normalised second in-plane and the first and second out-of-plane modes are written as \( Z_2 = \sqrt{z_{2c}^2 + z_{2s}^2}, Y_1 = \sqrt{y_{1c}^2 + y_{1s}^2} \) and \( Y_2 = \sqrt{y_{2c}^2 + y_{2s}^2} \), respectively. Throughout we consider the norm

\[
||N|| = \sqrt{Z_2^2 + Y_1^2 + Y_2^2} / L, \tag{2.2}
\]

which represents the combined amplitude of the cable response associated with the (steady-state) solutions of (2.1).

The external lower support displacement excitation is \( \delta = \Delta \cos(\Omega t) \), where \( \Omega \) is chosen to be close to \( \omega_2 \); therefore, a small detuning parameter \( \mu \) defined as \( \Omega = 2\omega_1(1 + \mu) \) has been introduced. Moreover, the scaled time \( \tau = (1 + \mu)t \) has been used in (2.1) and \( (\cdot)' \) denotes the derivative of \( (\cdot) \) with respect to this scaled time.

We vary the normalized amplitude \( \Delta / L \) and frequency \( \Omega / \omega_2 \) of the excitation as the continuation parameters. The other parameters are fixed to the values given in Table 1. Here, the damping ratio \( \xi \) is assumed to be the same for all three basic modes considered, which is in line with the experimental study in [9]. The parameters in Table 1 have been calculated for the specific cable under consideration from the relationships

\[
W_{nk} = v_{nk} / m, \quad N_n = 2n \sin \theta / m, \quad B = \zeta_2 \cos \theta / m, \\
E_q = \frac{1}{1 + \lambda^2 / 12}, \quad \lambda^2 = \frac{E}{\sigma_s} \left( \frac{\gamma \ell}{\sigma_s} \right)^2, \quad \gamma = \rho g \cos \theta, \\
\nu_{nk} = \frac{E A \pi^4 n^2 k^2}{8 \ell^3}, \quad \eta_n = \frac{E q A \pi^2 n^2}{4 \ell^2}, \quad \zeta_n = \frac{2m}{n \pi}, \quad \omega_n = \frac{n \pi}{L} \sqrt{\frac{\sigma_s}{\rho}}. \tag{2.3}
\]
frequency boundaries this pure the

While results for $\Omega/\omega_2$ close to unity; continuation in $\Delta/L$ then allows us to follow the branch of the $Z_2$-mode. This branch is initially stable but loses stability in different types of bifurcations, of which we find fold (or saddle-node) bifurcations, branch point (or pitchfork) bifurcations and Hopf bifurcations [10, 12, 31].

3. Bifurcation analysis of the mode structure. Our starting point is the bifurcation diagram in the $(\Omega/\omega_2, \Delta/L)$-plane of the (pure) $Z_2$-mode (with zero contributions from $Y_1$ and $Y_2$), which was already considered in [9]. It is shown in figure 2 and features a curve $F_{Z_2}$ of fold bifurcations and curves $B_1$ and $B_2$ of branch point bifurcations. (Mathematically, the latter bifurcations are pitchfork bifurcations; the two bifurcating branches of equilibria differ only in the
Fig. 2. Bifurcation diagram in the $(\Omega/\omega, \Delta/L)$-plane of the $Z_2$-mode of (2.1), showing a curve $F_{Z_2}$, of fold bifurcations and curves $B_1$ and $B_2$ of branch point bifurcations. Along solid parts of the curves the bifurcation concerns a stable solution and along dashed parts it concerns an unstable one; the region where the $Z_2$-mode is stable is shaded dark blue. The inset shows an enlargement near $\Omega/\omega = 1$.

phase relationship between the contributing basic modes, which physically corresponds to clockwise or counter-clockwise whirling motion of the cable. Since it is common in the field to represent the coupled modes in terms of their norm $||N||$, these two branches of equilibria are not distinguished; hence, one refers to this bifurcation as a branch point bifurcation.) For any value of $\Omega/\omega$ the $Z_2$-mode is stable for sufficiently small $\Delta/L$ (dark blue region). Stability of the $Z_2$-mode is lost when the solid parts of these curves are crossed; such a loss of stability means that $Y_1$ and/or $Y_2$ develop a non-zero amplitude. The inset of figure 2, which is an enlargement near $\Omega/\omega = 1$, shows that there is a cusp bifurcation point $C$ on the fold curve $F_{Z_2}$; moreover, the two curves $B_1$ and $B_2$ of branch point bifurcations interact at point $N$ where they exchange their roles as stability boundary for the $Z_2$-mode.

Figure 2 shows that, when $\Delta/L$ is increased the $Z_2$-mode loses stability in different ways depending on the value of $\Omega/\omega$. For fixed $\Omega/\omega < 1.003$, the frequency at which point $C$ is located, the branch point curve $B_1$ is crossed and $Y_1$ gradually becomes nonzero. For $\Omega/\omega > 1.003$, the upper part of the fold point curve $F_{Z_2}$ (with $\Delta/L$ above the cusp point $C$) is crossed, which leads to a transition to quite different dynamics, where $Z_2$, $Y_1$ and $Y_2$ all contribute. As the inset of figure 2 shows, the stability region of the $Z_2$-mode extends (from between the points $C$ and $N$) as a very thin tongue that is bounded by the curve $B_2$ and the lower part of the fold curve $F_{Z_2}$ (with $\Delta/L$ above the cusp point $C$). When $B_2$ is crossed $Y_2$ gradually becomes nonzero; when the lower part of $F_{Z_2}$ is crossed there is a transition to another stable $Z_2$-mode (of lower amplitude). Hence, this thin tongue constitutes a region of bistability of the $Z_2$-mode.

This discussion makes clear that the bifurcation diagram in figure 2 is already quite complicated, even though it only concerns the (pure) $Z_2$-mode. Moreover, it would be quite impractical to try to add to the $(\Omega/\omega, \Delta/L)$-plane all the bifurcation curves that bound stability regions of the other coupled mode responses that the system may support. We therefore adopt the more intuitive and
insightful way of representing the stability properties of all modes of the system by representing them as surfaces in \((\Omega/\omega_2, \Delta/L, ||N||)\)-space. Starting from the knowledge of the \(Z_2\)-mode, branch switching at bifurcation points allows us to follow branches of equilibria with nonzero amplitudes of \(Y_1\) and/or \(Y_2\). Surfaces of the different types of equilibria are then constructed in \((\Omega/\omega_2, \Delta/L, ||N||)\)-space from continuation runs in \(\Delta/L\) for suitably many fixed choices of \(\Omega/\omega_2\). The rendering of the surface images in this paper is performed with Matlab. Curves of bifurcations are computed by two-parameter continuations and then also plotted in \((\Omega/\omega_2, \Delta/L, ||N||)\)-space, where they lie on corresponding surfaces. We find and consider one by one the following types of equilibria of (2.1), which we refer to as the (coupled) modes of (2.1):

1. the \(Z_2\)-mode of pure second in-plane mode response (where \(Y_1 = Y_2 = 0\)); the surface of the \(Z_2\)-mode is denoted \(L\).
2. the \((Z_2,Y_1)\)-mode of coupled response between the second in-plane mode and the first out-of plane mode (where \(Y_2 = 0\)); the surface of the \((Z_2,Y_1)\)-mode is denoted \(L_1\).
3. the \((Z_2,Y_2)\)-mode of coupled response of the second in-plane mode and the second out-of plane mode (where \(Y_1 = 0\)); the surface of the \((Z_2,Y_2)\)-mode is denoted \(L_2\).
4. the \((Z_2,Y_1,Y_2)\)-mode of coupled response to which all three basic modes contribute; the surface of the \((Z_2,Y_1,Y_2)\)-mode is denoted \(L_{12}\).

### 3.1. The Surface \(L\) of the \(Z_2\)-Mode

The surface \(L\) of the \(Z_2\)-mode in \((\Omega/\omega_2, \Delta/L, ||N||)\)-space is shown in figures 3(a) and 4(a) in two different views; also shown are two sets of cross sections in panels (c) and (d) that show the corresponding bifurcation diagram for different fixed values of \(\Delta/L\) and \(\Omega/\omega_2\), respectively. Notice how the surface \(L\) is folded along the curve \(F_{Z_2}\). The dark blue part of \(L\) corresponds to the \(Z_2\)-mode being stable; this stability region is a single region that is bounded by \(F_{Z_2}\) and the respective parts of the branch bifurcation curves \(B_1\) and \(B_2\). Notice how these bifurcation curves from figure 2 are lifted to the surface \(L\), which now also represents the norm \(||N||\) of the \(Z_2\)-mode; indeed, the bifurcation diagram in the \((\Omega/\omega_2, \Delta/L)\)-plane is obtained by projection along \(||N||\).

Figure 3(a) shows a view of the surface \(L\) with two intersection curves for fixed \(\Delta/L\) on it, for \(\Delta/L = 0.3 \times 10^{-5}\) and \(\Delta/L = 7 \times 10^{-5}\), respectively. These curves represent the corresponding cross sections through \(L\), which are one-parameter bifurcation diagrams of the \(Z_2\)-mode as a function of the excitation frequency \(\Omega/\omega_2\); they are plotted in the \((\Omega/\omega_2, ||N||)\)-plane in panels (b) and (c). Figure 3(b) shows that for sufficiently small \(\Delta/L\) the resonant peak of the pure \(Z_2\)-response as what one would expect for a linear system: there is a clear peak near 1, the \(Z_2\)-mode is a function of \(\Omega/\omega_2\), and it is stable throughout. However, as the excitation amplitude \(\Delta/L\) is increased the \(Z_2\)-response becomes nonlinear. Figure 3(c) is for \(\Delta/L = 7 \times 10^{-5}\), that is, for a cross section in between the points \(C\) and \(N\); see the inset of figure 2. The resonance peak in figure 3(c) is considerably bent to the right, resulting in an S-shape with two fold points. However, there is practically no bi-stability, because the \(Z_2\)-mode loses its stability at a branch point \(B_2\) when \(\Omega/\omega_2\) is increased. The \(Z_2\)-mode actually regains stability at a the second branch point labelled \(B_2\), only to lose its stability immediately at the fold point \(F_{Z_2}\) (which is virtually indistinguishable from \(B_2\) in figure 3(c); compare with figure 2).

Figure 4(a) shows the surface \(L\) from a different view point, this time with two intersection curves for fixed excitation frequency \(\Omega/\omega_2\). Panel (b) shows the intersection of \(L\) for \(\Omega/\omega_2 = 0.97\) in the \((\Delta/L, ||N||)\)-plane, where the \(Z_2\)-mode is a monotonically increasing curve; it is stable for sufficiently small \(\Delta/L\) and then loses stability at the branch bifurcation point \(B_1\). Figure 4(c) shows the \(Z_2\)-mode in the \((\Delta/L, ||N||)\)-plane for \(\Omega/\omega_2 = 1.03\), which is to the right of the points \(C\) and \(N\) in figure 2. Because the curve \(F_{Z_2}\) is now intersected, the branch of the \(Z_2\)-mode now has a characteristic S-shape. Stability is lost at the first (right) fold point along the branch. The branch turns around and effectively remains unstable throughout, except for a tiny interval of \(\Delta/L\) values just past the second (left) fold point. When \(\Delta/L\) is increased from zero excitation in an
Fig. 3. Panel (a) shows the surface $\mathcal{L}$ of the $Z_2$-mode of (2.1) in $(\Omega/\omega_2, \Delta/L, ||N||)$-space. Also shown on $\mathcal{L}$ are the bifurcation curves $B_1$, $B_2$ and $F_{Z_2}$ from figure 2; the $Z_2$-mode is stable in dark blue part and unstable in light blue part. Panels (b) and (c) show the one-parameter bifurcation diagrams in the $(\Omega/\omega_2, ||N||)$-plane for $\Delta/L = 0.3 \times 10^{-5}$ and for $\Delta/L = 7 \times 10^{-5}$, respectively; the corresponding curves are highlighted on the surface $\mathcal{L}$. Branch points are denoted by squares and fold points by dots; the $Z_2$-mode is stable along blue parts of curves and unstable along red parts of curves.

experiment, one first observes a pure $Z_2$-response until the fold point is reached and the system
transitions to a different, as yet unknown, solution.

Clearly, to understand the response of the cable for values of $\Omega/\omega_2$ where the $Z_2$-mode is unstable one needs to consider the (surface of) bifurcating equilibria with non-trivial contributions from $Y_1$ and/or $Y_2$; they are discussed in the subsequent sections.
3.2. The surface $\mathcal{L}_1$ of the $(Z_2, Y_1)$-mode. The $(Z_2, Y_1)$-mode bifurcates from the $Z_2$-mode along the branch bifurcation curves $B_2$. Figure 5(a) shows the associated surface $\mathcal{L}_1$ (green) of the $(Z_2, Y_1)$-mode in $(\Omega/\omega_2, \Delta/L, ||N||)$-space, which connects to the surface $\mathcal{L}$ (blue) along the curve
The stable part of $L_1$ (dark green) is bounded by the curve $B_1$ on $L$ and by the branch bifurcation curve $B_{12}$ on $L_1$; as we will see in section 3.4, the surface $L_{12}$ of the $(Z_2, Y_1, Y_2)$-mode connects along $B_{12}$. Notice also the fold along $F_{Y_1}$. Highlighted in figure 5(a) are the cross sections of $L$ and $L_1$ for $\Omega/\omega_2 = 0.97$ and $\Omega/\omega_2 = 1.03$. Figure 5(b) shows the corresponding bifurcation diagram for $\Omega/\omega_2 = 0.97$ in the $(\Delta/L, ||N||)$-plane, which shows that a stable branch of the $(Z_2, Y_1)$-mode bifurcates from the $Z_2$-mode at the point $B_1$; compare with figure 4(b). The corresponding bifurcation diagram for $\Omega/\omega_2 = 1.03$ in the $(\Delta/L, ||N||)$-plane is shown in figure 5(c). An unstable branch of $(Z_2, Y_1)$-mode bifurcates from the point $B_1$, where the $Z_2$-mode itself is also unstable. The branch of the $(Z_2, Y_1)$-mode remains unstable after the fold point $F_{Y_1}$ but becomes stable after the branch point $B_{12}$. Overall, figure 5 shows that, for any value of $\Omega/\omega_2$ the $(Z_2, Y_1)$-mode may be observed stably provided the excitation amplitude $\Delta/L$ is sufficiently large.

As it turns out, there is an additional part of the $(Z_2, Y_1)$-mode surface $L_1$ that can be found when one considers it over a larger range of $\Omega/\omega_2$-values. Figure 6(a) shows the two surfaces $L_1$ and $L$ for $\Omega/\omega_2 \in [1.0, 1.07]$. Because of its complicated shape, the new part of the surface $L_1$ is not rendered as a surface but represented by equally spaced (green) intersection curves for fixed $\Omega/\omega_2$. These intersection curves are connected for fixed $\Omega/\omega_2 > 1.05$ to the main part of the surface $L_1$ at two fold curves, which connect to the fold curve $F_{Y_1}$ at $\Omega/\omega_2 \approx 1.05$. However, for fixed $\Omega/\omega_2 < 1.05$ we find figure-of-eight shaped isolas. Panels (b) and (c) of figure 6 show the two slices for $\Omega/\omega_2 = 1.03$ and for $\Omega/\omega_2 = 1.06$, respectively, which are also highlighted in panel (a). Notice the new isola of the $(Z_2, Y_1)$-mode for $\Omega/\omega_2 = 1.03$ in figure 6(b) compared with figure 5(c). In the section for $\Omega/\omega_2 = 1.06$, on the other hand, we find a single connected curve of the $(Z_2, Y_1)$-mode. In both cases, there is a small region of stable $(Z_2, Y_1)$-response on the new curve.

Since these features of the surface $L_1$ are hard to see on the scale of figure 6, we now focus on how the isola in the $(\Delta/L, ||N||)$-plane connects to the remainder of $L_1$ as $\Omega/\omega_2$ is increased through 1.05. Figure 7 shows the intersection curves of $L_1$ with cross sections for $\Omega/\omega_2 = 1.04$, for $\Omega/\omega_2 = 1.03$ and for $\Omega/\omega_2 = 1.06$ in the relevant part of the $(\Delta/L, ||N||)$-plane. Notice the figure-eight shape of the isola for $\Omega/\omega_2 = 1.04$ in figure 7(a); it is not actually connected at the two intersection point with the other intersection curve of $L_1$. There are four fold points on the isola, which are labelled $F_{Y_1}$ and $F'_{Y_1}$. At the moment of transition at $\Omega/\omega_2 \approx 1.05$ the lower fold point $F_{Y_1}$ on the isola coincides with the fold point $F_{Y_1}$ on the other branch; see figure 7(b). For $\Omega/\omega_2 = 1.06$ as in figure 7(c), there is now a single connected intersection curve of $L_1$; the degenerate fold point has split up into two fold points labelled $F'_{Y_1}$. Notice that close to these fold points there are two branch points that are labelled $B_{12}$; as we will see in section 3.4, they lead to the bifurcating $(Z_2, Y_1, Y_2)$-mode.

Overall, figure 7 shows that the connectivity of the local branches near the point of transition changes. This corresponds to a transition of the cross section through a critical point on the surface $L_1$ in $(\Omega/\omega_2, \Delta/L, ||N||)$-space. The two fold curves $F_{Y_1}$ and $F'_{Y_1}$ on $L_1$ have a maximum and a minimum with respect to $\Omega/\omega$, respectively, and intersect transversally at this special point. The curve $F'_{Y_1}$ with its minimum can be seen in figure 6(a). Of the curve $F_{Y_1}$ only the part up to the maximum where it meets $F'_{Y_1}$ is shown; its other part lies on the isola (green loops) and it is not shown in figure 6(a). Note that the curve of branch point bifurcations $B_{12}$ also passes through the critical point on $L_1$ and has a minimum there with respect to $\Omega/\omega$. The isola can be found in the cross section for fixed $\Omega/\omega_2$ in the range $\Omega/\omega_2 \in (1.02, 1.05)$; see figure 6(a). It disappears at $\Omega/\omega_2 \approx 1.02$ by shrinking down to a point, which corresponds to a minimum with respect to $\Omega/\omega_2$ of the curve $F_{Y_1}$ in $(\Omega/\omega_2, \Delta/L, ||N||)$-space.

In all three panels of figure 7 we also find a pair of Hopf bifurcations (denoted by asterisks). Moreover, the $(Z_2, Y_1)$-mode on the new part of $L_1$ is stable between the (upper) fold point $F_{Y_1}$ and the left-most point of Hopf bifurcation. Hence, we have found a new observable $(Z_2, Y_1)$-response of the cable in the region of interest $\Omega/\omega_2 \in [0.95, 1.04]$, which is not connected to the main part
Fig. 6. Panel (a) shows the surfaces $\mathcal{L}_1$ (green) and $\mathcal{L}$ (blue) in $(\Omega/\omega_2, \Delta/L, ||N||)$-space over the range $\Omega/\omega_2 \in [1.0, 1.07]$. The new feature is an additional part of the surface $\mathcal{L}_1$ that is connected to its main part for $\Omega/\omega_2 > 1.05$ along two fold curves, but also extends to smaller values of $\Omega/\omega_2$; due to its complicated shape, this new part of $\mathcal{L}_1$ is represented by intersections curves for fixed $\Omega/\omega_2$. The respective intersections for $\Omega/\omega_2 = 1.03$ and for $\Omega/\omega_2 = 1.06$ are highlighted. The corresponding bifurcation diagrams in the $(\Delta/L, ||N||)$-plane are shown in panels (b) and (c); there are two points of Hopf bifurcation (indicated by asterisks) on the additional branches of the $(Z_2, Y_1)$-mode.

of the surface $\mathcal{L}_1$ in this range of $\Omega/\omega_2$. This discovery justifies the consideration of the extended
Fig. 7. One-parameter bifurcation diagrams of the \((Z_2; Y_1)\)-mode in the \((\Delta/L, ||N||)\)-plane for \(\Omega/\omega_2 = 1.04\) in panel (a), for \(\Omega/\omega_2 = 1.05\) in panel (b), and for \(\Omega/\omega_2 = 1.06\) in panel (c). The ranges of \(\Delta/L\) and \(||N||\) are chosen to show how the isola of the \((Z_2; Y_1)\)-mode for \(\Omega/\omega_2 < 1.05\) connects to the remainder of the surface \(\mathcal{L}_1\) when \(\Omega/\omega_2\) is increased through \(\Omega/\omega_2 \approx 1.05\); compare with figure 6.
range of $\Omega/\omega_2$.

3.3. The surface $\mathcal{L}_2$ of the $(Z_2, Y_2)$-mode. Figure 8(a) shows the surface $\mathcal{L}_2$ of the $(Z_2, Y_2)$-mode, which is attached to the surface $\mathcal{L}$ along the branch bifurcation curve $B_2$. Almost all of $\mathcal{L}_2$
is unstable (light purple), but there is a small stable region (dark purple) near $\Omega/\omega_2 = 1$. On $L_2$ there is a curve $F_{Z_2}$ of fold bifurcation, whose left-most point is a cusp point. Moreover, we find a curve $B_{12}^2$ of branch point bifurcation that emerges from the point $N$ where $B_1$ and $B_2$ intersect on $L$; see figure 2. As with previous plots, two intersection sets are highlighted, namely those for $\Omega/\omega_2 = 1.007$ and $\Omega/\omega_2 = 1.03$, respectively; they are shown in the $(\Delta/L, ||N||)$-plane panels (b) and (c) of figure 8.

The section for $\Omega/\omega_2 = 1.007$ in figure 8(b) passes through the stable part of $L_2$. The branch $L_2$ of the $(Z_2, Y_2)$-mode connects to the branch $L$ of the $Z_2$-mode at the two branch bifurcation points labelled $B_2$. The $(Z_2, Y_2)$-mode is stable right after the left-most branch bifurcation point $B_2$ on $L$, and it loses its stability at the branch point $B_{12}^2$; as we will see in 3.4, the $(Z_2, Y_1, Y_2)$-mode can be observed stably past this point. Notice from figure 8(b) that, as the excitation amplitude $\Delta/L$ is increased from zero, one first observes the $Z_2$-mode until the fold point $F_{Z_2}$ is reached, where the system transitions to a stable $(Z_2, Y_2)$-response.

The section for $\Omega/\omega_2 = 1.03$ in figure 8(c), on the other hand, misses the stable part on $L_2$. Consequently, the branch $L_2$ bifurcating from $B_2$ is unstable throughout, until it also reconnects with the branch of the $Z_2$-mode at a second branch bifurcation point $B_2$ (which is outside the $\Delta/L$-range shown in figure 8(c)). Notice the S-shaped nature of the branch $L_2$ with two fold curves labelled $F_{Z_2}$.

3.4. The surface $L_{12}$ of the $(Z_2, Y_1, Y_2)$-mode. The surface $L_{12}$ of the $(Z_2, Y_1, Y_2)$-mode is quite complicated because it connects the two surfaces $L_1$ of the $(Z_2, Y_1)$-mode and $L_2$ of the $(Z_2, Y_2)$-mode. In figure 9(a), it is shown together with only the surface $L_2$ from figure 8(a) to which it is attached along the curve $B_{12}^2$. The (dark brown) stable part of $L_{12}$ in figure 9(a) is bounded at the top by the curve $B_{12}^1$, along which this surface is connected to the surface $L_1$ of the $(Z_2, Y_1)$-mode (which is not shown); compare with figure 6(a). The lower boundary of the stable part of $L_{12}$ is more complicated: at low values of $\Omega/\omega_2$ it is the same curve $B_{12}^2$, and for higher values of $\Omega/\omega_2$ it is formed by fold curve $F_{Y_{12}}$ on $L_{12}$. Specifically, between $F_{Y_{12}}$ and $B_{12}^2$ there are two other boundaries of $L_{12}$, namely the fold curve $F_{Y_1}$ and the Hopf bifurcation curve $H_2$. In fact, $F_{Y_{12}}$ emerges from a point on $B_{12}^2$ and $H_2$ emerges from a point on $B_{12}^2$. This structure of the boundary of the $(Z_2, Y_1)$-mode was also found in [15] for a four-mode model that includes the first in-plane mode.

Two of the three highlighted intersection sets in figure 9(a) are shown separately in the $(\Delta/L, ||N||)$-plane as panels (b) and (c); for clarity, the stable and unstable regions of $L$ are also shown as black and grey curves, respectively. In figure 9(b) for $\Omega/\omega_2 = 1.007$, which builds on figure 8(b), the branch $L_{12}$ bifurcates from the point $B_{12}^2$ on the branch $L_2$; along this branch the $(Z_2, Y_1, Y_2)$-mode is stable. Figure 9(c) for $\Omega/\omega_2 = 1.03$ builds on figure 8(c). Now the branch $L_{12}$ has three fold points; it is stable only after the last fold point labelled $F_{Y_{12}}$. Notice that in between the upper fold point and a point $H_{12}$ of Hopf bifurcation (denoted by an asterisk) there is an additional tiny part of the branch where the $(Z_2, Y_1, Y_2)$-mode is stable; this Hopf bifurcation curve is shown in figure 9(a). Finally, we remark that figure 9(c) shows that, as $\Delta/L$ is increased from zero, the transition to different dynamics when the $Z_2$-mode loses stability at $F_{Z_2}$ now results in a $(Z_2, Y_1, Y_2)$-response of the cable.

3.5. The bifurcation diagram consiting of all mode surfaces. Figure 10 shows the surfaces $L$ (blue), $L_1$ (green), $L_2$ (purple) and $L_{12}$ (brown) together in $(\Omega/\omega_2, \Delta/L, ||N||)$-space. It constitutes a bifurcation diagram of the $Z_2$-mode, the $(Z_2, Y_1)$-mode, the $(Z_2, Y_2)$-mode and the $(Z_2, Y_1, Y_2)$-mode, which represent all the steady-state solutions of (2.1) and, hence, the responses of the cable with fixed amplitudes of the contributing three basic in-plane and out-of-plane modes. These different surfaces meet at curves of branch point bifurcations; specifically, $L_1$ meets $L$ along $B_1$ (see section 3.2 and figure 6), $L_2$ meets $L$ along $B_2$ (see section 3.3 and figure 8), and $L_{12}$ meets $L_1$ along $B_{12}^1$ and $L_2$ along $B_{12}^2$ (see section 3.4, and figure 9). Stable parts of the surfaces are
Fig. 9. Panel (a) shows the surface \( \mathcal{L}_{12} \) (brown) of the \((Z_2, Y_1, Y_2)\)-mode of (2.1) in \((\Omega/\omega_2, \Delta/L, ||N||)\)-space, and how it connects to the surface \( \mathcal{L}_2 \) (purple) from figure 8. Also shown on \( \mathcal{L}_2 \) is the fold curve \( F_{Y_{12}} \); the \((Z_2, Y_1, Y_2)\)-mode is stable in the dark brown part of \( \mathcal{L}_{12} \) and unstable in the light brown part. The intersection sets for \( \Omega/\omega_2 = 1.007 \), for \( \Omega/\omega_2 = 1.03 \), and for \( \Omega/\omega_2 = 1.06 \) are highlighted; those for \( \Omega/\omega_2 = 1.007 \) and for \( \Omega/\omega_2 = 1.03 \) are shown in the \((\Delta/L, ||N||)\)-plane, and that for \( \Omega/\omega_2 = 1.007 \) appears as part of figure 11(c).

represented by darker colours and unstable parts by lighter colours; the respective curves of branch point bifurcations are shown as solid curves where they connect two stable surfaces and as dashed
Fig. 10. The surfaces \( \mathcal{L} \) (blue), \( \mathcal{L}_1 \) (green), \( \mathcal{L}_2 \) (purple) and \( \mathcal{L}_{12} \) (brown) shown together in \( (\Omega/\omega_2, \Delta/L, ||N||) \)-space together with the various curves of bifurcations on them. The intersection sets for \( \Omega/\omega_2 = 1.007, \Omega/\omega_2 = 1.03 \) and \( \Omega/\omega_2 = 1.06 \) are highlighted; darker colours represent stable modes.

curves where they connect two unstable surfaces. Also shown in figure 10 are the different curves of fold bifurcation.

The bifurcation diagram in figure 10 is shown for the range \( \Omega/\omega_2 \in [1.0, 1.07] \) where the surfaces and corresponding modes interact strongly. Note that for \( \Omega/\omega_2 \leq 1 \), only the surfaces \( \mathcal{L} \) and \( \mathcal{L}_1 \) meet along the curve \( B_1 \); see figures 5 and 6. This is visible already on the left-hand side of figure 10, so that this bifurcation diagram represents the topological nature of the equilibria of (2.1) and their stability over the entire range of \( \Omega/\omega_2 \in [0.97, 1.0] \) that we consider here. As before,
even though it may be at the limit of validity of the three-mode model (2.1) considered here, the range $\Omega/\omega_2 \in [1.04, 1.07]$ is specifically included due to the nature of the surface $\mathcal{L}_1$.

The three intersection sets for $\Omega/\omega_2 = 1.007$, $\Omega/\omega_2 = 1.03$ and $\Omega/\omega_2 = 1.06$ are highlighted on the surfaces in figure 10. They are shown as one-parameter bifurcation diagrams for all of the modes in the $\Delta/L, ||N||$-plane in 11(a)–(c).

For fixed $\Omega/\omega_2 = 1.007$ in figure 11(a), as $\Delta/L$ is increased from zero, the cable first experiences vibrations in the $Z_2$-mode. At $F_{Z_2}$, a transition occurs to the stable part of the branch $\mathcal{L}_2$ of the $(Z_2, Y_2)$-mode, meaning that now the second out-of-plane mode also contributes to the cable response. When $\Delta/L$ is increased further, past $B_{12}^2$, one observes also the gradual onset of the first out-of-plane mode to the cable response when $B_{12}^2$ is crossed to the stable part of the branch $\mathcal{L}_{12}$. For even higher $\Delta/L$ the contribution from $Y_2$ disappears gradually at $B_{12}^1$, and one is left with the $(Z_2, Y_1)$-mode along the stable part of $\mathcal{L}_1$. When $\Delta/L$ is decreased again, one observes a gradual transition from the $(Z_2, Y_1)$-mode via the $(Z_2, Y_1, Y_2)$-mode and the $(Z_2, Y_2)$-mode to the $Z_2$-mode. However, there is then a transition at $F_{Z_2}$ to the lower branch of $\mathcal{L}$, that is, to a $Z_2$-mode with lower amplitude, which is part of a small hysteresis loop.

For fixed $\Omega/\omega_2 = 1.03$ in figure 11(b) the bifurcation diagram in the $(\Delta/L, ||N||)$-plane is considerably more complicated. The transition from the stable $Z_2$-mode at $F_{Z_2}$, now results in a $(Z_2, Y_1, Y_2)$-mode response of the cable along the stable part of $\mathcal{L}_{12}$. When $\Delta/L$ is increased the $Y_2$-contribution gradually disappears at $B_{12}^1$, and one is again left with the $(Z_2, Y_1)$-mode along the stable part of $\mathcal{L}_1$. When $\Delta/L$ is decreased, starting from the stable part of $\mathcal{L}_{12}$, the $(Z_2, Y_1, Y_2)$-mode response is observed until the fold point $F_{Z_1}$ is reached. Then the system transitions to different dynamics. It is actually not entirely clear from the bifurcation diagram in figure 11(b) to which response the system will move at this point. After all, there is now a considerable amount of multi-stability between different modes. Moreover, there are points of Hopf bifurcation (denoted by asterisks) that give rise to periodic solutions, some of which are stable, for example, past the left-most Hopf bifurcation on the isola of $\mathcal{L}_1$. There is also the possibility that such periodic orbits bifurcate further to more complicated stable dynamics of the cable. The bifurcation study of periodic solutions and their bifurcations is beyond the study presented here and will be reported elsewhere.

Finally, for $\Omega/\omega_2 = 1.06$ in figure 11(c) the one-parameter bifurcation diagram is very much as that for $\Omega/\omega_2 = 1.03$ in figure 11(b). As was discussed in section 3.2, the difference is that $\mathcal{L}_1$ is a single branch that now incorporates the former isola. However, in terms of observed stable behaviour of the cable, the two cases are effectively the same.

From a practical point of view, it is useful to be able to identify for which combination of excitation frequency and amplitude the different types of coupled mode response of the cable can be found. Therefore, we present in figure 12(a) a projection of the stable parts of the surfaces $\mathcal{L}_1$ (green), $\mathcal{L}_2$ (purple) and $\mathcal{L}_{12}$ (brown) onto the relevant part of the $(\Omega/\omega_2, \Delta/L)$-plane. For comparison, the stability region of the $Z_2$-mode, corresponding to the stable part of the surface $\mathcal{L}$, is shown in blue in figure 12(b). Note that the $Z_2$-mode is stable throughout the white region in figure 12(a); moreover, to the right of the point $N$ the stability regions in panels (a) and (b) of figure 12 overlap, yielding a considerable region of multistability between different cable responses. Overall, figure 12 represents a practical bifurcation chart for the observable mode dynamics of the cable.

In figure 12(a) we show only the bifurcation curves that bound the respective stability regions. They include the curves $H_2 \in \mathcal{L}_2$ and $H_{12} \in \mathcal{L}_{12}$ of Hopf bifurcations, which bound the stability regions of $\mathcal{L}_2$ and $\mathcal{L}_{12}$, respectively. Notice, in particular, the very thin strip of the (purple) stability region of $\mathcal{L}_2$ that is bounded by the curves $B_2$ and $H_2$. Similarly, there is an extremely thin strip of the stability region of $\mathcal{L}_{12}$ between the curves $F_{Y_{12}}$ and $H_{12}$, which correspond to a region of bi-stability of the $(Z_2, Y_1, Y_2)$-mode. We remark that these two thin strips of stability would be
Fig. 11. One-parameter bifurcation diagrams in the $(\Delta/L, ||N||)$-plane for $\Omega/\omega_2 = 1.007$ in panel (a), for $\Omega/\omega_2 = 1.03$ in panel (b), and for $\Omega/\omega_2 = 1.06$ in panel (c). Blue curves denote stable and red curves denote unstable solutions, branch points are denoted by squares, fold points by dots, and Hopf bifurcation points by asterisks.
Panel (a) shows the projection of the stable parts of the surfaces $L_1$ (green), $L_2$ (purple) and $L_{12}$ (brown) onto the $(\Omega/\omega_2, \Delta/L)$-plane, where $\Omega/\omega_2 \in [1, 1.04]$ and $\Delta/L \in [0, 0.0075]$. Panel (b) shows the stable part of the surface $L$ in the same region of the $(\Omega/\omega_2, \Delta/L)$-plane.

virtually impossible to find in an experiment. However, if another parameter, such as the damping ratio, is changed then these regions of bistability may be considerably bigger [29].

3.6. Frequency response of the cable. Most one-parameter bifurcation diagrams we presented are in terms of changing excitation amplitude $\Delta/L$ for a specific fixed value of the excitation frequency $\Omega/\omega_2$. In fact, the entire bifurcation diagram in figure 10 has been built up by the corresponding sections for a set of fixed values of $\Omega/\omega_2$, because this choice facilitates the rendering of the different surfaces involved. It is quite common, on the other hand, to characterize a periodically driven system experimentally by its frequency response, that is, by performing a sweep in the excitation frequency for a given fixed value of the excitation amplitude. Indeed, such one-parameter continuations in $\Omega/\omega_2$ can be obtained from the bifurcation diagram in figure 10 as cross sections for fixed $\Delta/L$. We now discuss their properties in more detail.

The starting point of our investigation is the nonlinear response of the $Z_2$-mode in figure 3(c), where the $Z_2$-mode is actually mostly unstable around the resonance peak of the frequency response for $\Delta/L = 7 \times 10^{-5}$. We are now in a position to consider what other modes are observed, and
Fig. 13. One-parameter bifurcation diagram in the $(\Omega/\omega_2, ||N||)$-plane for $\Delta/L = 7 \times 10^{-5}$. Panel (a) shows only the branches $\mathcal{L}$ and $\mathcal{L}_1$, and in panel (b) the branches $\mathcal{L}_2$ and $\mathcal{L}_{12}$ (coloured) are added to $\mathcal{L}$ and $\mathcal{L}_1$ (black and grey).

The result is shown in figure 13 over an extended range of $\Omega/\omega_2$. Panel (a) repeats the branch $\mathcal{L}$ of the $Z_2$-mode and also shows the bifurcating branch $\mathcal{L}_1$ of the $(Z_2, Y_1)$-mode. The branch $\mathcal{L}_1$ is quite a complicated curve with a large number of fold points. Figure 13(b) repeats the branches $\mathcal{L}$ and $\mathcal{L}_1$ (in black and grey) and also shows the branches $\mathcal{L}_2$ and $\mathcal{L}_{12}$. The overall picture of frequency response is quite complicated; this is a manifestation of the considerable nonlinearity in the coupling between the basic in-plane and the two out-of-plane modes. At low frequencies the cable response is purely in the $Z_2$-mode. As $\Omega/\omega_2$ increases past the branch point $B_2$ a change to the coupled $(Z_2, Y_2)$-mode is observed. At a slightly higher frequency of $\Omega/\omega_2 \approx 1.005$ the point $B_{12}^2$ is passed and there is a gradual transition to a coupled $(Z_2, Y_1, Y_2)$-mode response. The stable part of the branch $\mathcal{L}_{12}$ ends at $F_{Y12}$, which suggest that the cable response transitions down to the pure $Z_2$-mode. Notice the hysteresis loop due to the transition up to the $(Z_2, Y_1, Y_2)$-mode at the point $F_{Y2}$ when $\Omega/\omega_2$ is decreased again.

Figure 14 shows the frequency response for $\Delta/L = 6.3 \times 10^{-4}$, where again panel (a) only shows $\mathcal{L}$ and $\mathcal{L}_1$, and panel (c) adds $\mathcal{L}_2$ and $\mathcal{L}_{12}$. Notice that there is now a larger range of $\Omega/\omega_2$ where the $(Z_2, Y_1)$-mode is stable before the $(Z_2, Y_1, Y_2)$-mode is observed for larger values of $\Omega/\omega_2$. Moreover, the $(Z_2, Y_1, Y_2)$-mode is then stable throughout the entire large range of $\Omega/\omega_2$ in figure 14.
Fig. 14. One-parameter bifurcation diagram in the \((\Omega/\omega_2, ||N||)\)-plane for \(\Delta/L = 6.3 \times 10^{-4}\). Panel (a) shows only the branches \(\mathcal{L}\) and \(\mathcal{L}_1\), and in panel (b) the branches \(\mathcal{L}_2\) and \(\mathcal{L}_{12}\) (coloured) are added to \(\mathcal{L}\) and \(\mathcal{L}_1\) (black and grey).

A direct comparison of the frequency responses for \(\Delta/L = 7 \times 10^{-5}\) and for \(\Delta/L = 6.3 \times 10^{-4}\) is shown in figure 15 in the smaller parameter range \(\Omega/\omega_2 \in [0.98, 1.06]\), similar to the one that we considered before, for example, in figure 10. In figure 15(a) for \(\Delta/L = 7 \times 10^{-5}\) we find, for increasing \(\Omega/\omega_2\) the response sequence \(\mathcal{L}\) to \(\mathcal{L}_2\) to \(\mathcal{L}_{12}\) and back to \(\mathcal{L}\), and vice versa with a hysteresis loop for decreasing \(\Omega/\omega_2\). For \(\Delta/L = 6.3 \times 10^{-4}\) in figure 15(b), on the other hand, we find for increasing \(\Omega/\omega_2\) the response sequence \(\mathcal{L}\) to \(\mathcal{L}_1\) to \(\mathcal{L}_{12}\), with the \((Z_2, Y_1, Y_2)\)-mode being always observed for large \(\Omega/\omega_2\). Starting from the \(Z_2\)-mode on the stable part of \(\mathcal{L}\) for large and then decreasing \(\Omega/\omega_2\), we find a transition at \(F_{Z_2}\) to the response sequence \(\mathcal{L}_{12}\) to \(\mathcal{L}_1\) to \(\mathcal{L}\).

Overall, figures 3(b) and 15 show that the frequency response of the cable is very sensitive to even small changes of the excitation amplitude \(\Delta/L\). This a strong indication of the nonlinear nature of the coupling between the basic in-plane mode (represented by \(Z_2\)) and the two basic out-of-plane modes (represented by \(Y_1\) and \(Y_2\)).

4. Conclusions. We presented a bifurcation study of an inclined cable with support excitation near a 2 : 1 resonance with the second in-plane mode of the cable. This system was described by an ODE model for the contributions of this directly excited in-plane mode and of the first and second
out-of-plane modes of the cable. The focus was on the different types of coupled-mode response of the cable, which are equilibria of the ODE that are distinguished by which of the out-of-plane modes is active: none, the first, the second, or both. Each of the four types of equilibria was computed by means of numerical continuation as a surface in the three-dimensional space of their norm versus the amplitude and frequency of the excitation. The different surfaces connect at curves of bifurcations and constitute a geometric picture of the overall two-parameter bifurcation diagram of the coupled-mode response of the cable. The quite intricate structure of this bifurcation diagram confirms the nonlinear nature of the problem. In particular, we found that the frequency response of the cable depends very sensitively on the excitation amplitude.

An interesting subject for future research is the study of periodic orbits that bifurcate from Hopf bifurcations of the coupled modes. Physically, they correspond to a cable response with periodically varying amplitudes of the contributing basic in-plane and out-of-plane modes, which manifests itself in locked or quasiperiodic two-frequency dynamics. Initial investigations show that these periodic orbits of the model equations may bifurcate further, for example, in period-doubling and homoclinic bifurcations, which lead to a chaotic cable response. A systematic study of these cable dynamics with non-constant amplitudes of the basic modes is under way and will be reported.
elsewhere.

The results we presented are for a range of the excitation frequency (close to the 2 : 1 resonance) where the first in-plane mode is not active and, therefore, the three-mode model considered here provides a suitable description of the cable response. Previously this model was studied in [9] mainly in terms of the stability properties of the pure in-plane response, and good agreement was found with the corresponding experiment. Similar experiments are reported in [15] for a longer cable, where again there was good agreement between measurements and the stability boundaries of in-plane motion. These results were obtained for a four-mode model that includes the first in-plane mode; however, the stability boundaries agree with those of the three-mode model for sufficiently small frequency detunings. It appears feasible for these reasons that the other cable responses predicted here, to which the out-of-plane modes contribute, may also be confirmed experimentally.

Note, however, that such experiments are not straightforward, because the cable is only very lightly damped. Hence, experimental runs require a long time for transients to die down sufficiently before the dynamics can be recorded reliably (e.g., with video-capture at the half- and quarter-points).

Finally, we mention that in a cable-stayed bridge the cable dynamics may also have an influence on the bridge deck under certain conditions. The study presented here shows that the motion of the cable itself, under uni-directional excitation from the deck, may develop considerable amplitudes. Hence, there is the potential for positive feedback loops between vibrations of the cable and the deck. One approach to study such effects is to extend the model for the cable by adding one or more damped oscillators that model the basic modes of the bridge deck. This type of system could be studied in a similar spirit via a bifurcation analysis of the coupled-mode responses of the overall system, which now also include the basic deck modes.

Acknowledgements. The research of V.T. was supported by grant EP/F030711/1 from the Engineering and Physical Sciences Research Council (EPSRC).

REFERENCES