Opportunities as chances: maximising the probability that everybody succeeds*

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This version: January 28, 2014

Abstract

We propose to model opportunities in society as 'chances of success', that is as they are commonly described by practitioners. We show that a classical liberal principle of justice together with a limited principle of social rationality imply that the social objective should be to maximise the chance that everybody in society succeeds. Technically, this means that opportunity profiles should be evaluated by means of a 'Nash' criterion. A particular consequence is that the failure of even only one individual must be considered maximally detrimental. We also study a refinement of this criterion and its extension to problems of intergenerational justice.

JEL: D63; D70.

Keywords: opportunities, chances in life, Harm Principle, Nash ordering.

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*We thank for comments Geir Asheim, Benedetto Gui, Carmen Herrero, Serge Kolm, Paola Manzini, Jorge Nieto, and especially José Carlos Rodriguez Alcantud. We also thank audiences at numerous conferences and seminars. Marco Mariotti gratefully acknowledges financial support through a Leverhulme Fellowship. This paper substantially extends and improves "Opportunities, Chances in Life and Inequality" ('Equality Exchange' working paper http://mora.rente.nhh.no/projects/EqualityExchange/Portals/0/articles/mariotti1.pdf), and it supersedes that paper.

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"If we are in politics for one thing it is to make sure that all children are given the best chance in life."[1]

1 Introduction

'Opportunities' are a central concept both in the public discourse and in economics. To model opportunities, we assume that each individual is regarded as a binary experiment with either 'success' or 'failure' as possible outcomes. Then, opportunities in society are expressed by the profile of 'chances of success' across individuals. By means of this simplification, we are able to offer several insights on the issue of the allocation of opportunities. For example, what is the social cost of one person in society not having any chance of success? Is it conceivable that such a sacrifice be justified by a sufficient increase in opportunities for the rest of society? Our theoretical framework offers a general method to address this type of question.

When in a social policy study it is claimed that some categories of individuals have low opportunities, what is usually meant is that the *probabilities* - measured through empirical frequencies - of those individuals to attain success in a certain dimension are lower than some benchmark. So it is quite common to read statements of this kind: "An adolescent of ethnic origin X and social background Y has half the average chances to be eventually admitted into a top university". Statements such as this link some attribute of individuals in a group to the attainment of a desirable target simply by looking at the objective statistical frequencies of attainment for that group.[2] The meaning of the term 'opportunities' in natural language is close to the one adopted in this literature.[3] People with more opportunities are people who face more favourable circumstances, and, therefore, who will tend to succeed more frequently. On this interpretation there is no mention of 'effort', 'responsibility' or 'talent'. Politicians express themselves openly in terms of chances. For example, beside the opening quote from Tony Blair, in a major independent report [14], Labour MP Frank Field says that "im-

[1] Tony Blair, speech given at the Labour Party conference, 28th of September 1999.

[2] The literature is too vast for a comprehensive set of references. We limit ourselves to refer to Mayer's book [30], whose very title is representative: 'What Money Can't Buy: Family Income and Children's Life Chances'.

[3] Consider the Webster's definition of an opportunity: "a favourable juncture of circumstances". Similarly, in the Oxford Dictionary: "a good chance; a favourable occasion".

proving the life chances of under fives is the key to cutting social inequality". Academic economists too sometimes emphasise the 'favourable juncture of circumstances' aspect of opportunities. For example, Deaton's [11] notion of 'escape' is not distant from our notion of 'success'.

But, more often, economists have adopted more sophisticated and indirect views of the notion of opportunity. Concepts such as talent and responsibility are placed at the forefront.[4] It is also often taken as a given that opportunities, once properly formulated, should be equalised.

In this paper we take seriously the direct interpretation of practitioners. In spite of its limitations, this approach has the advantage of interpreting opportunities in a way that is very amenable to straightforward measurement. A target for social policy to equalise the proportion of students in top schools among the various ethnic groups, or the proportion in high-level jobs of students from different types of schools, is concrete and easy to understand and verify empirically, in a way in which, say, 'equalise capabilities across ethnic groups' is not.[5]

Our simplification is drastic but pays off with some interesting insights in respect of a well-known difficulty with justifying egalitarian principles. Equalisation of any value measure across individuals can always be criticised (just like simple welfare egalitarianism) on the grounds that many individuals might have to face large aggregate losses for the sake of increasing only marginally the value for one individual. Our analysis, however, leads to a preference for some degree of equality that does not stem from the *nature* of the 'equalisandum' (opportunities as opposed to welfare), but rather from outside the stock of egalitarian principles, via a classical liberal 'Harm principle'.

This principle is liberal because it asserts a form of non-interference with individuals in society. The details are explained in section 4, but its core is

[4] The literature here is vast too: an illustrative but far from comprehensive selection of contributions includes: Sen [38]; Fleurbaey [15], [16]; Herrero [20]; Bossert and Fleurbaey [8]; Kranich [22]; Roemer [35], [36]; Laslier et al. [23]; Tungodden [40].

The contribution by Bénabou and Ok [5] does not refer to responsibility and is in this respect closer in spirit to this paper. However, our focus is different since we attempt to derive the desiderability of equality from first principles.

[5] For example, the Deputy Leader of the British Government expresses in this way his worry about the perceived failure of the school system: "Clegg's aides drew attention on Monday to the fact that just over 7% of children in England go to private schools, but go on to make up 75% of judges and 70% of finance directors." From *The Guardian* newspaper, http://www.guardian.co.uk/education/2011/feb/08/nick-clegg-university-access

the requirement that an individual who has suffered damage *without harming others* should not be interfered with.

By means of this and other properties we characterise some 'Nash-like' criteria: society should, broadly speaking, maximise the *product of opportunities*. In the usual setting of social welfare, a drawback of the Nash product is that it raises a difficulty of interpretation: what does a product of utilities mean? In contrast, classical criteria such as the Utilitarian and maximin ones, for example, are clearly interpretable as 'total' utility or the utility of the worst off.[6] However, in the context of this paper the Nash product, too, acquires a transparent meaning: to maximise the Nash product means to maximise the *probability that everybody in society succeeds* under the assumption that the individuals are independent experiments.[7]

An interesting feature of the Nash criterion in a context of opportunity profiles is a strongly egalitarian implication. In fact, *it is sufficient for a profile to include one agent who fails with certainty for this profile to be the worst possible one (no matter how many other individuals succeed).* This answers the question of the opening paragraph.

Furthermore, we adress a refinement issue. The straightforward application of the Nash criterion suffers from some lack of discriminatory power at the boundaries: we cannot distinguish situations where many individuals fail from situations in which only one of them fails (only weak, and not strong, Pareto is satisfied on the set of profiles in which some of the individuals fail). To address this issue, we also formulate a new variant of the Nash criterion, the *Two-Step Nash* criterion. This criterion refines the indifference classes and satisfies strong Pareto. However, this variant mitigates the strong form of inequality aversion at the boundary shown by the standard Nash criterion.

Finally, we study situations where the number of agents is infinite. This case is relevant for the evaluation of intergenerational allocation problems. A concrete and focal example of 'success' for a generation is the ability to enjoy a clean environment. At a more abstract level, in an "Aristotelian" perspective, *self-realisation* - intended as developing human capacities - could be taken as the fundamental objective of mankind. In this interpretation, the probability of success of a generation is the probability that the generation

[6] Provided of course that the appropriate assumptions on the comparability of the units and origin of the utility scale are made.

[7] Note that the independence assumption just concerns the *interpretation* of our model. It is not a formal assumption that underlies the formal results. We discuss independence in the conclusions.

will develop its inherently human capacities. At the formal level, the main novelty in this part of the paper is the introduction of the *Nash catching up* and the *Nash overtaking* criteria. This part of the analysis complements a voluminous stream of recent work (including Alcantud [1], Asheim and Banerjee [3], Basu and Mitra [4], Bossert et al. [9]. For a detailed survey, see Asheim [2]), and is necessarily more technical in nature.

While we have focused on the contribution to the opportunity literature, there is a technical angle from which this paper can be read, in connection with the abstract social choice literature on the 'Nash Social Welfare Ordering'. From this angle, our main result constitutes a new characterisation of this ordering, which dispenses with some of the standard key axioms (such as Scale Invariance and Continuity) and replaces them with ones that are different both formally and conceptually. We expand on these issues in section 9.

2 The framework

There are $\mathcal{T}$ individuals in society. An *opportunity* for individual t is a number between 0 and 1. This number is interpreted as a 'chance of success' either in some given field or in life as a whole,[8] so that opportunities can be manipulated just as probabilities. We are interested in how opportunities should be allocated among the $\mathcal{T}$ individuals. The underlying idea is that some (limited) resources (possibly money) can be allocated so as to influence the distribution of opportunities.[9] An *opportunity profile* (or simply a *profile*) is a point in the '*box of life*' $B^{\mathcal{T}} = [0,1]^{\mathcal{T}}$, where $\mathcal{T}$ is either a natural number T or ∞, interpreted as the cardinalities of a finite set of agents $\mathcal{N}$ or of an infinite set of agents $\mathbb{N}$, respectively. So, in the latter case, B^{∞} denotes the

[8]Leading examples of 'success' that appear in the social policy literature are the following: no teenage childbearing; not dropping out of school; attainment of x years of formal education; attainment of fraction α of the average hourly wage, or yearly income; no male idleness (this is defined in Mayer [30] as the condition of a 24-year old not in school and not having done paid work during the previous year); no single motherhood. In a health context, success may be defined, for instance, by: surviving until age y; surviving a given operation; (for a group) mortality and morbidity below percentage β of a reference group's average. In a social psychology context, success may be related to reported happiness being within a certain quantile of the population. And so on.

[9]See Mayer [30] for an interesting counterpoint to the effect of money on children's life chances.

set of countably infinite streams of probabilities of success for agents in $\mathbb{N}$. Here we develop the notation for the finite case.

A profile $a = (a_1, a_2, ..., a_T) \in B^T$ lists the opportunities, or 'chances of success' of agents in $\mathcal{N}$ if a is chosen.

The points $\mathbf{0} = (0, 0, ..., 0) \in B^T$ and $\mathbf{1} = (1, 1, ..., 1) \in B^T$ can be thought of as *Hell* (no opportunities for anybody) and *Heaven* (full opportunities for everybody), respectively. We will also say that individual t *is in Hell* (resp., Heaven) at a if $a_t = 0$ (resp., $a_t = 1$).

Let $B_+^T = \left\{a \in B^T | a \gg \mathbf{0}\right\}$.[10]

A *permutation* π is a bijection of $\mathcal{N}$ onto itself. For all $a \in B^T$, let $\overline{a}$ be the permutation of a which ranks its elements in ascending order.

3 Opportunities in the box of life: finite societies

We aim to specify desirable properties for a *social opportunity relation* $\succcurlyeq^S$ on the box of life B^T.[11]

Two properties for $\succcurlyeq^S$ are the following, for all $a, b \in B^T$:

Strong Pareto: $a > b \Rightarrow a \succ^S b$.

Anonymity: $a = \pi b$ for some permutation $\pi \Rightarrow a \sim^S b$.

These properties are standard and will not be discussed further. We now define the two relations on the box of life that are the main object of this study.[12]

For all $a, b \in B^T$, the **Nash social opportunity ordering** $\succcurlyeq^N$ aggre-

[10] Vector notation: for all $a, b \in B^T$ we write $a \geq b$ to mean $a_t \geq b_t$, for all $t \in \mathcal{N}$; $a > b$ to mean $a \geq b$ and $a \neq b$; and $a \gg b$ to mean $a_t > b_t$, for all $t \in \mathcal{N}$.

[11] Given a binary relation $\succcurlyeq$ on a set X and $x, y \in X$, we write $x \succ y$ (the asymmetric factor) if and only if $x \succcurlyeq y$ and $y \not\succcurlyeq x$, and we write $x \sim y$ (the symmetric part) if and only if $x \succcurlyeq y$ and $y \succcurlyeq x$.

[12] We recall here some standard terminology. A relation $\succcurlyeq$ on a set X is said to be: *reflexive* if, for any $x \in X$, $x \succcurlyeq x$; *complete* if, for any $x, y \in X$, $x \neq y$ implies $x \succcurlyeq y$ or $y \succcurlyeq x$; *transitive* if, for any $x, y, z \in X$, $x \succcurlyeq y \succcurlyeq z$ implies $x \succcurlyeq z$. $\succcurlyeq$ is a *quasi-ordering* if it is reflexive and transitive, while $\succcurlyeq$ is an *ordering* if it is a complete quasi-ordering. A relation $\succcurlyeq'$ on X is an *extension* of $\succcurlyeq$ if $\sim \subseteq \sim'$ and $\succ \subseteq \succ'$.

gates chances of success by multiplication:

$$a \succcurlyeq^N b \Leftrightarrow \prod_{t=1}^{T} a_t \geq \prod_{t=1}^{T} b_t.$$

Next, we introduce a new refinement of the Nash ordering on the boundary of the box of life, which we call the **Two-Step Nash social opportunity ordering** $\succcurlyeq^{2N}$. For all $a \in B^T$, let $P^a = \{t \in \mathcal{N} : a_t > 0\}$. Then for all $a, b \in B^T$:

$$\begin{aligned} a &\succ^{2N} b \Leftrightarrow \text{either } |P^a| > |P^b|, \\ \text{or } |P^a| &= |P^b| \;\&\; \prod_{t \in P^a} a_t > \prod_{t \in P^b} b_t. \end{aligned}$$

Thus also:

$$a \sim^{2N} b \Leftrightarrow \left(|P^a| = |P^b|\right) \;\&\; \left(\prod_{t \in P^a} a_t = \prod_{t \in P^b} b_t\right),$$

which includes the case $|P^a| = |P^b| = 0$ and $a = b = \mathbf{0}$.[13] So, the Two-Step Nash ordering is equivalent to the standard Nash ordering in the interior of the box of life (that is, in the case $|P^a| = |P^b| = T$), but unlike the standard Nash ordering it does not consider all profiles on the boundary indifferent to each other. If at least one of the two profiles has (at least) a zero component we count the positive entries. If they have the same number of positive entries, we apply Nash to them. If not, then the profile with the higher number of positive entries is preferred.

4 A Non-Interference Principle

Imagine that success is achieved by overcoming a series of 'hurdles'. For example, for success in becoming a doctor, being a dustman's daughter combines hurdles that a doctor's son does not face (less favourable studying environment, lack of a high-level social network, and so on). A different example comes from Deaton's [11] idea of escape we mentioned earlier. He vividly recounts the representative story of his father, a miner in Scotland

[13] We use the convention that $\prod_{t \in P^a} a_t = \prod_{t \in P^b} b_t = 1$ when $P^a = P^b = \varnothing$.

with few prospects of improvement, who saw this hurdle to success in life removed by the draft to the army. It is instructive to cite the list of 'lucks' that Deaton deems crucial for success (escape), because it is a concrete illustration of the "to success through hurdles" view that we propose here:

> "the luck not be among those who died as children, the luck to be rescued from the pit by the war, the luck not to be on wrong commando raid, the luck not to die from tubercolosis, and the luck to get a job in an easy labor market".

Observe that our view of success can encompass two distinct types of situations. In one, hurdles are 'events' that can happen to individuals (e.g. being drafted in the army). In the other, they are more like given individual characteristics (e.g. being a dustman's daughter).[14]

We imagine that hurdles are defined so that the addition or removal of a hurdle has a multiplicative effect on the probability of success.[15] With this interpretation in mind, the next axiom imposes some minimal limits on the interference of society on an individual's opportunities. We assume that an individual has the right to prevent society from acting against her in all circumstances of reduction in her opportunities (due to a change in the hurdles she faces), *whenever* the opportunities of no other individual are affected. By 'acting against her' we mean a switch against the individual in society's strict rankings of the chance profiles, with respect to the ranking of the original profiles (before the change in hurdles for the individual under consideration occurred). Crucially, the principle says nothing on society's possible actions aimed at increasing the individual's opportunities: an individual facing additional hurdles cannot demand (on the basis of our axiom) to be compensated by a switch of society's ranking in her favour. In this sense the principle we propose is libertarian rather than egalitarian.[16]

[14] Obviously whether a given event is a hurdle depends on individual circumstances: for somebody who already has a job with good career prospects rather than a mining job in Scotland, being drafted in the army would be a hurdle, not an advantage.

[15] In other words, all hurdles whose effects are correlated are lumped together. For example, surviving tubercolosis and being drafted in the army can be considered non-correlated and thus separate hurdles, like being female and having a dustman's father.

[16] In Mariotti and Veneziani [26], we explore a more radical formalisation of the principle, applied not to chances but to welfare levels, in which the 'no harm' conclusion follows even when the reduction in welfare is not proportional. This leads to the leximin principle.

From a philosophical viewpoint, we interpret this principle as an incarnation of J.S. Mill's 'Harm Principle'. We dwell on philosophical issues in Veneziani and Mariotti [29].

Probabilistic Harm Principle: Let $a, b, a', b' \in B^T$ be such that $a \succ^S b$ and, for some $t \in \mathcal{N}$ and for some $\rho \in (0, 1)$,

$$\begin{aligned} a'_t &= \rho \cdot a_t, \\ b'_t &= \rho \cdot b_t, \\ a'_j &= a_j, \text{ for all } j \neq t, \\ b'_j &= b_j, \text{ for all } j \neq t. \end{aligned}$$

Then $b' \not\succ^S a'$ whenever $a'_t > b'_t$.

In other words, when comparing two pairs of profiles interpreted as involving losses of opportunities for only individual t from an initial situation a, b to a final situation a', b' as described, there are three possibilities:

- Individual t is *compensated* for her loss (society abandons the strict preference for t's lower-chances profile).
- Individual t is *not harmed* further beyond the given opportunity damage (society prefers always the lower-chances or always the higher-chances profile for t).
- Individual t is *punished* (society switches preference from t's higher chances profile to t's lower chances-profile).

What the Probabilistic Harm Principle does is to exclude the third possibility. Society's choice should not become less favourable to somebody *solely* because her position has worsened, without affecting others' opportunities.

Note how in formulating this principle the *cause* of the reduction in opportunities for individual t (i.e. the specific hurdles that are raised) is completely ignored. It may have happened because of carelessness or because of sheer bad luck. All that matters is that *the other individuals are not affected* by individual t's change.

At the formal level, note that we allow for the possibility that $b_t = b'_t = 0$. Below we also explore another liberal axiom in which we require $b_t > 0$. This is important from both the theoretical and the analytical viewpoint. Theoretically, the question is whether the principle should be restricted to situations where a damage occurs in the strict sense, i.e. where opportunities strictly decrease. This may seem reasonable, but maybe it is not. If $b_t = 0$, so that an agent would be in Hell both before and after the harm should society choose against him, changing social preferences to $b' \succ^S a'$ might be regarded as a very heavy punishment indeed on the logic of the Probabilistic Harm Principle.

Note also the conclusion $b' \not\succ^S a'$ in the statement of the axiom. The veto power of the individual whose opportunities have decreased is limited, in that she cannot impose on society a ranking in complete agreement with her chances. This feature becomes especially relevant if we allow $\succcurlyeq^S$ to be incomplete (as in the impossibility results below), for in this case $b' \not\succ^S a'$ does not imply $a' \succcurlyeq^S b'$ and thus the requirement of the axiom becomes even weaker.

The Probabilistic Harm Principle rules out, for instance, the Utilitarian ordering. The following example demonstrates this and provides an illustration of how the principle works:[17]

Example 1 ***Utilitarianism violates the Probabilistic Harm Principle:*** *Let $\mathcal{N} = \{1, 2\}$. Then $\left(1, \frac{1}{8}\right)$ is Utilitarian-better than $\left(\frac{1}{2}, \frac{1}{2}\right)$ but $\left(\frac{1}{2}, \frac{1}{8}\right)$ is Utilitarian-worse than $\left(\frac{1}{4}, \frac{1}{2}\right)$. Yet in moving from $\left(1, \frac{1}{8}\right)$ to $\left(\frac{1}{2}, \frac{1}{8}\right)$, and from $\left(\frac{1}{2}, \frac{1}{2}\right)$ to $\left(\frac{1}{4}, \frac{1}{2}\right)$, all that has happened is that individual 1's opportunities have been halved, without touching the opportunities of the other individual. The switch in social choice punishes individual 1 for the damage she has suffered!*

5 Impossibilities

When attempting to apply the Probabilistic Harm Principle - together with the other basic requirements of Anonymity and Strong Pareto - we are immediately confronted with a difficulty.

Theorem 2 *There exists no transitive social opportunity relation $\succcurlyeq^S$ on B^T that satisfies* ***Anonymity****,* ***Strong Pareto****, and the* ***Probabilistic Harm Principle****.*

Proof: By example. Consider the profiles

$$a = (a_1, 0, x, x, ..., x)\,,\ b = (0, b_2, x, x, ..., x)\,,$$

[17] The Utilitarian criterion would however satisfy a Non-Interference principle in which the change from one pair of profiles to the other is not 'proportional' but additive (and thus incompatible with the independent hurdle interpretation we have given here). See Mariotti and Veneziani [27], [28].

where $1 \geq a_1 > b_2 > 0$ and $x \in [0, 1]$. By transitivity, together with **Anonymity** and **Strong Pareto**, we have $a \succ^S b$.

Consider next the following profiles obtained from a, b:

$$a' = (a'_1, 0, x, x, ..., x), b' = b = (0, b_2, x, x, ..., x)$$

where $a'_1 = \rho a_1$, $b'_1 = \rho b_1 = 0$, for some $\rho \in (0, 1)$ such that $\rho a_1 < b_2$. Since $\rho a_1 > \rho b_1$, then by the **Probabilistic Harm Principle**, it follows that $b' \not\succ^S a'$. However, by transitivity, together with **Anonymity** and **Strong Pareto**, $b' \succ^S a'$, a contradiction. ■

Observe that this result holds for social opportunity relations which are possibly incomplete. And even transitivity can be dispensed with, provided that Anonymity and Strong Pareto are replaced by the following axiom.

Suppes-Sen Grading Principle: If $a > \pi b$ for some permutation π then $a \succ^S b$.

Corollary 3 *There exists no social opportunity relation $\succcurlyeq^S$ on B^T that satisfies **Suppes-Sen Grading Principle** and the **Probabilistic Harm Principle**.*

Proof: Straightforward modification of the previous proof. ■

It is worth noting that previous impossibility results concerning the application of the Nash criterion in the context of welfare orderings focus on axioms of different nature, emphasising the role of continuity and ratio-scale measurability (see, e.g., Tsui and Weymark's [39] Theorem 1). The Probabilistic Harm Principle is logically strictly weaker than ratio-scale measurability (beside being interpreted very differently), given the restriction $\rho \in (0, 1)$. In addition to that, the consequent in the statement of the axiom only requires that society's strict preference should not be reversed (which in our case allows both for indifference and for noncomparability). A further difference concerns the fact that, as noted, we dispense with both the completeness and the transitivity of $\succcurlyeq^S$.

An equivalent of Theorem 2 holds also for infinite societies using Finite Anonymity (defined in section 10) and the infinite version of the Probabilistic Harm Principle below.

The result originates in the structure of the space of alternatives and the properties of the boundary of the box of life, coupled with the fact that the

Probabilistic Harm Principle applies also to profiles on the boundary, and to boundary values $b_t = 0$. In this sense, while the impossibility is formally robust, in that it holds for several combinations of similar axioms (e.g. Strong Pareto in the statement could be weakened in some ways) we do not deem it as expressing any deep contradiction between normative principles.

We shall explore two possible strategies to avoid the impossibility and thus two alternative ways of weakening the above axioms. The first strategy consists of weakening Strong Pareto. For all $a, b \in B^T$:[18]

Pareto: $a > b \Rightarrow a \succsim^S b$ and $a >> b \Rightarrow a \succ^S b$.

In order to derive our main characterisation, we need to introduce another property, described in the next section.

6 Social Rationality and the Diamond Critique

The new type of property we examine concerns the 'rationality' of the social opportunity relation. Consider first an axiom analogous to the sure-thing type of principle underlying Harsanyi's [19] defense of Utilitarianism (in a welfare context):

Sure Thing: Let $a, b, a', b' \in B^T$. If $a \succsim^S b$ and $a' \succsim^S b'$, then

$$\forall \lambda \in (0,1) : \lambda a + (1-\lambda)\, a' \succsim^S \lambda b + (1-\lambda)\, b',$$

with $\lambda a + (1-\lambda)\, a' \succ^S \lambda b + (1-\lambda)\, b'$ if at least one of the two preferences in the premise is strict.

Sure Thing is a classical independence property, and it can be justified in a standard way as follows. Denote the compound profiles $a'' = \lambda a + (1-\lambda)\, a'$ and $b'' = \lambda b + (1-\lambda)\, b'$. The profile a'' can be thought of as being obtained by means of a two-stage lottery: first, an event E can occur with probability λ. Then, if E occurs the profile is a, and otherwise it is a'. And b'' can be described analogously, as a compound event conditional on the occurrence or not of E. Then, when choosing between a'' and b'', it seems natural to adhere to this decomposition: if E occurs, it would have been better to choose a'' since a is better than b; and if E does not occur it would also have been better

[18]This is property S1 in Diamond's [12] seminal paper.

to choose a'' since a' is better than b'. Therefore, a'' should be regarded as better than b'' before knowing whether E occurs or not.

We think that a property akin to Sure Thing should be imposed but that, as it is formulated, it displays some ethically unattractive features. The following argument parallels the classical 'Diamond critique' of the similar property in Harsanyi's Utilitarianism[19] (note that a utilitarian social opportunity ordering would satisfy Sure Thing). Consider:

$$a = a' = b' = (0,1)\,,\ b = (1,0)\,,\ \lambda = \frac{1}{2}.$$

Then if Anonymity applies we have

$$a \sim^S b' \sim^S a' \sim^S b,$$

and by Sure Thing

$$a'' = (0,1) \sim^S \left(\frac{1}{2}, \frac{1}{2}\right) = b''.$$

But having one individual in Hell and the other in Heaven for sure can hardly be reasonably regarded as socially indifferent to both individuals being half way between Heaven and Hell in the box of life. As Diamond [13] would put it, "b'' seems strictly preferable to me since it gives 1 a fair share while a'' does not".[20]

The reason for this unacceptable situation is, obviously, that 'mixing' opportunities across different individuals may produce ethically relevant effects. The problem of properties like Sure Thing, both in a utility context and in the present one, is precisely the potentially beneficial effect of this sort of 'diagonal mixing' in the box of life.

However, the property is immune from this line of criticism when the allowable mixings are restricted to ones that are parallel to the edges of the box: namely, the compound lotteries only concern *a single individual*. This seems to capture a position *à la* Diamond: "I am willing to accept the sure-thing principle for individual choice but not for social choice" ([13], p. 766).

The following weakening of Sure Thing is then responsive to the Diamond critique:

[19] See also Fleurbaey [17].

[20] [13], p.766. The notation has been adapted to be consistent with the rest of the paper.

Individual Sure Thing: Let $a, b \in B^T$ be such that $a \succsim^S b$ and let $a', b' \in B^T$ be such that there exists $t \in \mathcal{N}$ such that $a'_j = a_j$ and $b'_j = b_j$, for all $j \in \mathcal{N} \backslash \{t\}$, and $a' \succsim^S b'$. Then

$$\forall \lambda \in (0,1) : \lambda a + (1-\lambda) a' \succsim^S \lambda b + (1-\lambda) b',$$

with $\lambda a + (1-\lambda) a' \succ^S \lambda b + (1-\lambda) b'$ if at least one of the two preferences in the premise is strict.

7 Nash Retrouvé

Before proving the main characterisation result of this section, we establish a preliminary result, which is of interest in its own right. The Lemma proves that any two profiles that imply Hell for at least one individual are socially indifferent (we address this feature in the next section).

Lemma 4 *Let the social opportunity ordering $\succsim^S$ on B^T satisfy **Anonymity, Pareto, Probabilistic Harm Principle, and Individual Sure Thing**. Then:*

$$\textit{for all } a, b \in B^T : [a_t = 0,\ b_j = 0, \textit{ some } t, j \in \mathcal{N}] \Rightarrow a \sim^S b.$$

The proofs of all Lemmas are in the appendix.

Given lemma 4, we can now show that an ordering in the box of life can be completely characterised by the four axioms discussed before.[21]

Theorem 5 ***(MAXIMISE THE PROBABILITY OF HEAVEN)**: A social opportunity ordering $\succsim^S$ on B^T satisfies **Anonymity, Pareto, Probabilistic Harm Principle** and **Individual Sure Thing** if and only if $\succsim^S$ is the Nash ordering $\succsim^N$.*

Proof: ($\Rightarrow$) It is immediate to prove that the Nash ordering $\succsim^N$ satisfies all four axioms.

($\Leftarrow$) Suppose that the social opportunity ordering $\succsim^S$ on B^T satisfies **Anonymity**, **Pareto**, **Probabilistic Harm Principle**, and **Individual Sure Thing.** For any $a, b \in B^T$, we shall first prove that $a \sim^N b \Rightarrow a \sim^S b$

[21] The axioms in Theorem 5, and indeed in all characterisation results below, can be shown to be independent. The details are available from the authors upon request.

holds, and then invoke **Pareto** and transitivity to conclude that $a \succ^N b \Rightarrow a \succ^S b$ also holds.

1. Suppose that $a, b \in B^T$ are such that $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t$. If $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t = 1$, then $a \sim^S b$ follows from reflexivity. If $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t = 0$, then it follows from Lemma 4.

2. Therefore suppose that $1 > \prod_{t=1}^T a_t = \prod_{t=1}^T b_t > 0$. If there exists a permutation π such that $a = \pi b$, then the result follows by **Anonymity**. Therefore, suppose that $U = \left\{t \in \mathcal{N} | \ \overline{a}_t > \overline{b}_t\right\} \neq \emptyset$ and $L = \left\{t \in \mathcal{N} | \ \overline{b}_t > \overline{a}_t\right\} \neq \emptyset$. Suppose in contradiction that $a \nsim^S b$. By completeness, and without loss of generality, suppose that $a \succ^S b$. By **Anonymity** and transitivity, consider the ranked profiles $\overline{a}, \overline{b}$.

Let $k = \max_{t \in U} \frac{\overline{a}_t}{\overline{b}_t}$. Then, from $\overline{a}, \overline{b}$ construct a', b' as follows: for any $l \in L$, let $a'_k = \delta \overline{a}_k$, and $b'_l = \delta \overline{b}_l$, where $\delta = \max\left(\frac{\overline{b}_k}{\overline{a}_k}, \frac{\overline{a}_l}{\overline{b}_l}\right)$, and leave all other entries of $\overline{a}$ and $\overline{b}$ unchanged. Note that $0 < \delta < 1$ by construction and so $a', b' \in B^T$. We prove that $a' \succ^S b'$.

3. Consider a profile b^π which is a permutation of $\overline{b}$ such that $b^\pi_k = \overline{b}_l$. By **Anonymity** and transitivity, $\overline{a} \succ^S b^\pi$ Then, from $\overline{a}, b^\pi$ construct a^0, b^0 as follows: $a^0_k = 0$, $b^0_k = 0$, and leave all other entries of $\overline{a}$ and b^π unchanged. By Lemma 4, $a^0 \sim^S b^0$. Therefore by **Individual Sure Thing**, $a^\lambda = \lambda \overline{a} + (1-\lambda) a^0 \succ^S b^\lambda = \lambda b^\pi + (1-\lambda) b^0$, for all $\lambda \in (0,1)$. Then, setting $\lambda = \delta$, and noting that for all $\lambda \in (0,1)$, $a^\lambda_k = \lambda \overline{a}_k$, $b^\lambda_k = \lambda b^\pi_k$, and $a^\lambda_i = \overline{a}_i$, $b^\lambda_i = b^\pi_i$, for all $i \neq k$, by **Anonymity** and transitivity, it follows that $a' \succ^S b'$.

4. Let $a^1 \equiv a'$, and $b^1 \equiv b'$: by construction $\prod_{t=1}^T a^1_t = \prod_{t=1}^T b^1_t > 0$ and $E = \left\{t \in \mathcal{N} | \ \overline{a}_t = \overline{b}_t\right\} \subset E^1 = \left\{t \in \mathcal{N} | \ \overline{a}^1_t = \overline{b}^1_t\right\}$. If $U = \{k\}$ and $L = \{l\}$, it is easy to show that $\frac{\overline{b}_k}{\overline{a}_k} = \frac{\overline{a}_l}{\overline{b}_l}$ and so $E^1 = \mathcal{N}$, yielding the desired contradiction, by **Anonymity**. Otherwise, the previous argument can be iterated $m-1$ times to obtain profiles $a^m, b^m \in B^T$ such that $a^m \succ^S b^m$, but $E^m = \mathcal{N}$, which again yields a contradiction by **Anonymity**.

5. This proves that for all $a, b \in B^T$, $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t$ implies $a \sim^S b$. The proof is completed in a routine way by invoking **Pareto** and transitivity. ■

The interpretation of the Nash social opportunity ordering is of interest. In the present framework, each individual is a binary experiment, with outcome either success or failure. Imagining that such experiments are independent, the requirement to maximise the Nash ordering means that chances in life should be allocated so as to *maximise the probability that everybody succeeds*. As a particular implication, the failure of even only one individual must be considered as maximally detrimental.

Contrast this attempt to maximise the probability of Heaven with a Utilitarian type of ordering, which would maximise the sum of probabilities. In the proposed interpretation, that would amount to *maximising the expected number of successes*. Clearly, such a method would be biased, compared to the one proposed, against a minority of individuals with very low probability of success.

It is also interesting to compare the use of the Nash ordering in the present framework to that in a standard utility framework. In the latter, there are two problems of interpretation.

Firstly, it is not clear what it means to maximise a product of utilities (as noted, e.g., by Rubinstein [37][22]). In a welfare world the utilitarian process of aggregation has a 'natural' meaning, which the Nash product lacks. But in a world of chances, a process of aggregation by product is equally natural.

Secondly, the maximisation of the Nash product on the positive orthant requires the external specification of a 'welfare zero'. In a bargaining context, this is assumed to be the 'disagreement point'; but its determination in a general social choice context is unclear, and it must be based on some external argument. On the contrary, the structure of the box of life, with its internal zero, makes this problem vanish.

8 The Two-Step Nash Ordering

A drawback of the Nash ordering - a consequence of relaxing Strong Pareto - is that it yields some very large indifference classes by considering all points on the boundary of the box of life as equally good (or bad). This may be deemed undesirable from an ethical perspective, and it may be a drawback for practical applications. For, a profile where *all* agents (potentially a very large number of individuals) are in Hell can hardly be seen as indifferent to one in which only one of them suffers.

[22] "The formula of the Nash bargaining solution lacks a clear meaning. What is the interpretation of the product of two von Neumann Morgenstern utility numbers?" (p. 82). The interpretation he goes on to propose is related to non-cooperative bargaining. Here we are rather interested in an interpretation of the Nash ordering as an *ethical* allocation method. A different interpretation in this vein is in Mariotti [25].

In this section, we explore another way out of the impossibility in which Strong Pareto is not abandoned. This requires some adjustments in the axiomatic system. We restrict the application of Probabilistic Harm Principle to strictly positive probabilities (as we discussed after the definition of the principle, this may be a reasonable restriction). Moreover, the same liberal logic that underlies the Probabilistic Harm Principle can be argued to extend to the case of *improvements* in individual opportunities, without being restricted to harms. The new Probabilistic Non Interference principle below incorporates this extension.[23]

Probabilistic Non-Interference : Let $a, b, a', b' \in B^T$ be such that $a \succ^S b$ and, for some $t \in \mathcal{N}$ and for some $\rho > 0$,

$$\begin{aligned} a'_t &= \rho \cdot a_t, \\ b'_t &= \rho \cdot b_t, \\ a'_j &= a_j, \text{ for all } j \neq t, \\ b'_j &= b_j, \text{ for all } j \neq t. \end{aligned}$$

Then $a' \succ^S b'$ whenever $b_t \neq 0$ and $a'_t > b'_t$.

We can now state the main characterisation of this section:

Theorem 6 ***(MAXIMISE THE PROBABILITY OF HEAVEN AND HAVE FEW PEOPLE IN HELL)****: A social opportunity ordering $\succsim^S$ on B^T satisfies* ***Anonymity****, **Strong Pareto**, and* ***Probabilistic Non-Interference*** *if and only if $\succsim^S$ is the Two-Step Nash ordering $\succsim^{2N}$.*

Proof: ($\Rightarrow$) It is immediate to prove that $\succsim^{2N}$ satisfies all three axioms.

($\Leftarrow$) Suppose that the social opportunity ordering $\succsim^S$ on B^T satisfies **Anonymity**, **Strong Pareto**, and **Probabilistic Non-Interference**. For any $a, b \in B^T$, we shall prove that (i) $a \sim^{2N} b \Rightarrow a \sim^S b$; and (ii) $a \succ^{2N} b \Rightarrow a \succ^S b$.

Claim (i). We need to consider two cases.

Case 1. Suppose that $a, b \in B^T_+$ are such that $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t$. If $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t = 1$, then the result follows from reflexivity. Therefore suppose that $1 > \prod_{t=1}^T a_t = \prod_{t=1}^T b_t > 0$.

[23] In [29] we discuss in detail the conceptual underpinnings and implications of this extended liberal principle.

If there exists a permutation π such that $a = \pi b$, then the result follows by **Anonymity**. Therefore, suppose that $U = \left\{t \in \mathcal{N} | \; \overline{a}_t > \overline{b}_t\right\} \neq \emptyset$ and $L = \left\{t \in \mathcal{N} | \; \overline{b}_t > \overline{a}_t\right\} \neq \emptyset$. Suppose in contradiction that $a \nsim^S b$. By completeness, and without loss of generality, suppose that $a \succ^S b$. By **Anonymity** and transitivity, consider the ranked profiles $\overline{a}, \overline{b}$.

Let $k = \min_{t \in U} t$. Take any $l \in L$. Suppose that $l < k$ (An analogous argument applies if $l > k$). By construction $\overline{a}_k > \overline{b}_k \geq \overline{b}_l > \overline{a}_l$. By **Anonymity** and transitivity, consider a profile b^π which is a permutation of $\overline{b}$ such that $b^\pi_k = \overline{b}_l$. Then, from $\overline{a}, b^\pi$ construct a', b' as follows. Let $b'_k = \rho b^\pi_k = \rho \overline{b}_l$, $a'_k = \rho \overline{a}_k$, where $\rho = \frac{\overline{b}_k}{\overline{a}_k} < 1$; and leave all other entries of $\overline{a}$ and b^π unchanged. By construction $a', b' \in B^T$ and $a'_k = \overline{b}_k$. Further, by **Probabilistic Non-Interference**, $a' \succ^S b'$.

Let $a^1 \equiv a'$, and $b^1 \equiv b'$: by construction $\prod_{t=1}^T a^1_t = \prod_{t=1}^T b^1_t > 0$ and $E = \left\{t \in \mathcal{N} | \; \overline{a}_t = \overline{b}_t\right\} \subset E^1 = \left\{t \in \mathcal{N} | \; \overline{a}^1_t = \overline{b}^1_t\right\}$. If $U = \{k\}$ and $L = \{l\}$, it is easy to show that $E^1 = \mathcal{N}$, yielding the desired contradiction, by **Anonymity**. Otherwise, the previous argument can be iterated $m-1$ times to obtain profiles $a^m, b^m \in B^T$ such that $a^m \succ^S b^m$, but $E^m = \mathcal{N}$, which again yields a contradiction by **Anonymity**.

Case 2. Suppose that $|P^a| = |P^b| < T$ and $\prod_{t \in P^a} a_t = \prod_{t \in P^b} b_t$. If $|P^a| = |P^b| > 0$, then by focusing on the strictly positive entries of $a, b \in B^T$, the same reasoning as for case 1 can be applied to obtain the desired result. If $|P^a| = |P^b| = 0$, then $a \sim^S b$ by reflexivity.

Claim (ii). We need to consider three cases.

Case 1. Suppose that $a, b \in B^T_+$ are such that $\prod_{t=1}^T a_t > \prod_{t=1}^T b_t$. Then there exists $a^\varepsilon \in B^T_+$, where $a^\varepsilon_k = a_k - \varepsilon$ some $k \in \mathcal{N}$, $a^\varepsilon_t = a_t$ for all $t \neq k$, and $\prod_{t=1}^T a^\varepsilon_t = \prod_{t=1}^T b_t$. By **Strong Pareto** $a \succ^S a^\varepsilon$, and therefore $a \succ^S b$ follows from claim (i) and transitivity.

Case 2. Suppose that $a, b \in B^T$ are such that $|P^a| = |P^b| < T$ and $\prod_{t \in P^a} a_t > \prod_{t \in P^b} b_t$. Then by focusing on the subset of strictly positive entries of $a, b \in B^T$, the previous reasoning can be used to prove $a \succ^S b$.

Case 3. Suppose that $a, b \in B^T$ are such that $|P^a| > |P^b|$. By **Anonymity** and transitivity, consider the ranked profiles $\overline{a}, \overline{b}$. Let $k = \min \{t \in \mathcal{N} : \overline{a}_t > 0\}$. Note that by assumption, $\overline{a}_k > \overline{b}_k = 0$. Next, let $l = \min \left\{t \in \mathcal{N} : \overline{b}_t > 0\right\}$. If for all $i \geq l$, $\overline{a}_i \geq \overline{b}_i$, then the result follows by **Strong Pareto**. Therefore suppose that $\overline{b}_h > \overline{a}_h$, some $h \geq l$, and, in contradiction to claim (ii), $\overline{b} \succ^S \overline{a}$. Then by **Anonymity** and transitivity, consider profile b^π which is a permutation of $\overline{b}$ such that $b^{\pi}_k = \overline{b}_h$. Then, from $\overline{a}, b^{\pi}$ construct a', b' as follows: let $\rho > 0$ be such that $a'_k = \rho \overline{a}_k$, $b'_k = \rho b^{\pi}_k = \rho \overline{b}_h$, $a'_j = a^{\pi}_j$ all $j \neq k$, $b'_j = b^{\pi}_j$ all $j \neq k$, and such that $b'_k = \rho \overline{b}_h < \overline{a}_h$. Since $h \geq l > k$, then $b'_k > a'_k$, and given that $a'_k \neq 0$, by **Probabilistic Non-Interference**, it follows that $b' \succ^S a'$. Consider the ranked profiles $\overline{a}', \overline{b}'$. Note that $k' = \min \{t \in \mathcal{N} : \overline{a}'_t > 0\} = k$. If $\overline{a}' > \overline{b}'$, then the desired contradiction follows from **Strong Pareto**, **Anonymity**, and transitivity. Otherwise repeat the procedure (always using the k-th entry of the ranked profiles $\overline{a}$, $\overline{a}'$, and so on) until the desired contradiction ensues.

This proves that $|P^a| > |P^b|$ implies $a \succcurlyeq^S b$. Suppose, contrary to claim (ii), that $a \sim^S b$. Then, for a sufficiently small $\varepsilon > 0$, it is possible to construct a profile $a^{\varepsilon} \in B^T$ such that $a^{\varepsilon}_t = a_t - \varepsilon > 0$ for some $t \in \mathcal{N}$, $a^{\varepsilon}_j = a_j$ all $j \neq t$, and $|P^{a^{\varepsilon}}| = |P^a| > |P^b|$. By transitivity and **Strong Pareto**, $b \succ^S a^{\varepsilon}$, and the previous argument can be applied. ■

As the reader will have noticed, another major difference in the conditions of theorems 5 and 6, beside those already discussed, is the absence of Individual Sure Thing in the latter. In fact, the two-step Nash ordering does not satisfy Individual Sure Thing, as the following example demonstrates:

Example 7 *$a = (\frac{3}{10}, \frac{4}{10}) \sim^{2N} b = (\frac{2}{10}, \frac{6}{10})$ and $a' = (\frac{3}{10}, 0) \succ^{2N} b' = (\frac{2}{10}, 0)$. However,*

$$\forall \lambda \in (0,1) : a^{\lambda} = \lambda a + (1-\lambda) a' \sim^{2N} b^{\lambda} = \lambda b + (1-\lambda) b'.$$

In fact, $\forall \lambda \in (0,1)$ $a^{\lambda} = (\frac{3}{10}, \lambda \frac{4}{10})$ and $b^{\lambda} = (\frac{2}{10}, \lambda \frac{6}{10})$, and thus $\prod_{t=1}^{2} a^{\lambda}_t = \prod_{t=1}^{2} b^{\lambda}_t$.

This example implies immediately, together with the characterisation, that it is impossible to impose on a social opportunity ordering $\succcurlyeq^S$ the four properties of Anonymity, Strong Pareto, Probabilistic Non-Interference, and Individual Sure Thing. The next result provides a direct proof of this claim, and it demonstrates that the clash between axioms remains even if one drops transitivity.

Theorem 8 *There exists no complete social opportunity relation $\succcurlyeq^S$ on B^T that satisfies **Anonymity, Strong Pareto, Probabilistic Non-Interference,** and **Individual Sure Thing**.*

Proof: By example. Let $x \in [0,1]$. Consider

$$\begin{aligned} a &= \left(\frac{1}{2}, \frac{1}{2}, x, x, ..., x\right), \ a' = \left(0, \frac{1}{2}, x, x, ..., x\right), \\ b &= \left(\frac{3}{4}, \frac{1}{3}, x, x, ..., x\right), \ b' = \left(0, \frac{1}{3}, x, x, ..., x\right). \end{aligned}$$

By **Strong Pareto** we have $a' \succ^S b'$. Now consider two possibilities. If $a \succcurlyeq^S b$ then by **Individual Sure Thing**

$$\begin{aligned} \frac{2}{3}a + \frac{1}{3}a' &\succ^S \frac{2}{3}b + \frac{1}{3}b' \\ &\Leftrightarrow \left(\frac{1}{3}, \frac{1}{2}, x, x, ..., x\right) \succ^S \left(\frac{1}{2}, \frac{1}{3}, x, x, ..., x\right), \end{aligned}$$

contradicting **Anonymity**.

On the other hand, if $b \succ^S a$ then by **Probabilistic Non-Interference**

$$\left(\frac{2}{3} \cdot \frac{3}{4}, \frac{1}{3}, x, x, ..., x\right) \succ^S \left(\frac{2}{3} \cdot \frac{1}{2}, \frac{1}{2}, x, x, ..., x\right),$$

again contradicting **Anonymity**. ■

The insight in Theorem 8 is robust. Impossibilities obtain even if Probabilistic Non-Interference is restricted only to the Harm part (note that in the proof of Theorem 8 we only use $\rho < 1$) and if Strong Pareto is weakened to following minimal mononicity property: there exists $x > 0$ for which $(x, 0, 0, ...) \succ^S (0, 0, 0, ...)$. Finally, even completeness can be dropped and replaced by transitivity.[24]

9 Relation with the literature

To reiterate, the main goal of this paper is to study an operational version of opportunities and to illustrate a new interpretation of the Nash criterion in this context. Nevertheless, in this section we collect for the interested reader some observations on the formal relation between our work and the literature on the Nash social welfare orderings (SWOs).

[24] The proof of these assertions is available upon request. We thank J.C. Alcantud for pointing out these possible weakenings.

A first observation to make is that the characterisation of Theorem 5 constitutes a rather radical departure from those of the SWO literature. In the latter, a key axiom is typically one of Scale Invariance, while Theorem 5 uses a combination of a liberal and a social rationality principle. These principles are both formally and conceptually distinct from Scale invariance properties.

The older part of the SWO literature focuses on the strictly positive orthant only (Boadway and Bruce [6]; Moulin [31]. See also Bosi, Candeal and Indurain [7]) and as we have seen profiles with zero entries create special technical problems. While still using a different domain (that of the box of life) our setting is closer to two more recent contributions by Tsui and Weymark [39] and Naumova and Yanovskaya [32], who explore larger domains.

Beside the one already mentioned, a further main technical difference from these papers is that we focus on Anonymity and we do not assume any continuity property, whereas continuity axioms are central in both papers. Consequently, the arguments involved are entirely different. Notably, we do not use any results from functional analysis, nor properties of social welfare functions, since we cannot assume that our social welfare ordering is representable.

To be more specific, Tsui and Weymark ([39], Theorem 5, p.252) elegantly characterise, using techniques from functional analysis, 'Cobb-Douglas' SWOs (of which the Nash ordering is a special case) on $\mathbb{R}^n$ by a continuity axiom, Weak Pareto and Ratio Scale measurability. Once transferred to the appropriate domain, our ranking can be seen as the anonymous case within this class (obtained via Anonymity instead of continuity). They do not characterise SWOs similar to our Two-Step Nash ordering. Naumova and Yanovskaya [32] provide a general analysis of SWOs on $\mathbb{R}^n$ that satisfy Ratio-Scale measurability, and they do characterise some *lexicographic* social welfare functions. Essentially, as compared to [39], they weaken the continuity properties. For example, they focus on the requirement that continuity should hold within *orthants*, which are unbounded sets of vectors whose individual components have always the same sign, positive, negative or zero (therefore the vectors $(1, 0, 1)$, $(1, 1, 0)$, and $(0, 1, 1)$, for instance, belong to the box of life B^3 but to three different orthants in the sense of [32]). The lexicographic SWOs characterised there differ markedly from ours in that they require a linear ordering of the orthants and therefore vectors on the boundary of the box of life (e.g., $(1, 0, 1)$, $(1, 1, 0)$, and $(0, 1, 1)$ in B^3) will never be indifferent. Therefore, contrary to our analysis, Anonymity is violated.

10 Intergenerational justice and the Nash criterion

The focus on joint probability of success seems, at the conceptual level, as attractive an opportunity criterion when the agents are infinite in number as when there is only a finite number of them. And yet, a large set of infinite streams of probabilities yield a zero probability of joint success, making the criterion vacuous for practical purposes.

We propose two solutions to this dilemma. They consist of adapting two well-known methods for comparing infinite streams of utilities: namely, the *overtaking* and the *catching-up* criteria. In order to obtain the desired extensions of the social opportunity relations, we add properties that permit a link with the infinite case to (analogs of) the characterising axioms of the finite case. In this way, we obtain an overtaking version of the Nash criterion and a catching-up version of the Two-Step Nash criterion.

Almost without exception all uses of the Nash criterion we are aware of apply to a finite number of agents, and therefore our proposals may be of interest in their own right.[25]

The previous notation is extended in a straightforward way to the infinite context, with the following specific additions. A profile is now denoted ${}_1a = (a_1, a_2, ...) \in B^\infty$, where a_t is the probability of success of generation $t \in \mathbb{N}$. For $T \in \mathbb{N}$, ${}_1a_T = (a_1, ..., a_T)$ denotes the T-*head* of ${}_1a$ and ${}_{T+1}a = (a_{T+1}, a_{T+2}, ...)$ denotes its T-*tail*, so that ${}_1a = ({}_1a_T, {}_{T+1}a)$.

For any $x \in B$, $\mathbf{x} = (x, x, ...) \in B^\infty$ denotes the stream of constant probabilities equal to x. Let $B_+^\infty = \{{}_1a \in B^\infty | {}_1a \gg \mathbf{0}\}$. For all ${}_1a \in B^\infty$ and $T \in \mathbb{N}$, let $P^{{}_1a_T} = \{t \in \{1, ..., T\} : a_t > 0\}$.

A *permutation* π is now a bijective mapping of $\mathbb{N}$ onto itself. A permutation π of $\mathbb{N}$ is finite if there is $T \in \mathbb{N}$ such that $\pi(t) = t$, for all $t > T$, and Π is the set of all finite permutations of $\mathbb{N}$. For any ${}_1a \in B^\infty$ and any $\pi \in \Pi$, let $\pi({}_1a) = \left(a_{\pi(t)}\right)_{t \in \mathbb{N}}$ be a permutation of ${}_1a$. For any ${}_1a \in B^\infty$, let ${}_1\overline{a}_T$ denote the permutation of the T-head of ${}_1a$, which ranks the elements of ${}_1a_T$ in ascending order.

We are now ready to consider the first infinite horizon version of the Nash criterion.

The Nash overtaking criterion: For all ${}_1a, {}_1b \in B^\infty$, ${}_1a \sim^{N^*} {}_1b \Leftrightarrow$

[25] The only partial exception we are aware of is Cato [10], which however only considers the Nash overtaking criterion on the strictly positive orthant.

$\exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T}$: $\prod_{t=1}^{T} a_t = \prod_{t=1}^{T} b_t$; and ${}_1a \succ^{N^*} {}_1b \Leftrightarrow \exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T}$: $\prod_{t=1}^{T} a_t > \prod_{t=1}^{T} b_t$.

The characterisation results below are based on the following axioms which are analogous to those used in the finite context.

Finite Anonymity: For all ${}_1a \in B^\infty$ and for all $\pi \in \Pi$, $\pi({}_1a) \sim^S {}_1a$.

Monotonicity: For all ${}_1a$, ${}_1b \in B^\infty$, ${}_1a > {}_1b \Rightarrow {}_1a \succsim^S {}_1b$.

Restricted Dominance: For all $x, y \in B$, $x < y \Rightarrow \mathbf{y} \succ^S (x,_2 \mathbf{y})$.

Probabilistic Harm Principle*: Let ${}_1a,_1 b \in B^\infty$ be such that ${}_1a = ({}_1a_T,_{T+1} b)$ for some $T \in \mathbb{N}$, and ${}_1a \succ^S {}_1b$; and let ${}_1a',_1 b' \in B^\infty$ be such that for some $t \in \mathbb{N}$, and some $\rho \in (0, 1)$,

$$\begin{aligned} a'_t &= \rho a_t, \\ b'_t &= \rho b_t, \\ a'_j &= a_j \text{ , for all } j \neq t, \\ b'_j &= b_j \text{ , for all } j \neq t. \end{aligned}$$

Then ${}_1b' \not\succ^S {}_1a'$ whenever $a'_t > b'_t$.

Individual Sure Thing*: Let ${}_1a,_1 b \in B^\infty$ be such that ${}_1a = ({}_1a_T,_{T+1} b)$ for some $T \in \mathbb{N}$, and ${}_1a \succsim^S {}_1b$ and let ${}_1a',_1 b' \in B^\infty$ be such that for some $t \leq T$, $a'_j = a_j$ and $b'_j = b_j$, for all $j \neq t$, and ${}_1a' \succsim^S {}_1b'$. Then

$$\forall \lambda \in (0, 1) : \lambda_1 a + (1 - \lambda)\ {}_1a' \succsim^S \lambda_1 b + (1 - \lambda)\ {}_1b',$$

with $\lambda_1 a + (1 - \lambda)\ {}_1a' \succ^S \lambda_1 b + (1 - \lambda)\ {}_1b'$ if at least one of the two preferences in the premise is strict.

Like in the finite case, Strong Pareto must necessarily be weakened to avoid impossibilities: Monotonicity and Restricted Dominance are two such weakenings that have been used in the literature (for a discussion, see Asheim [2]).

In addition to the above axioms, a weak consistency requirement is imposed.

Weak Consistency: For all ${}_1a,_1 b \in B^\infty$: (i) $\exists \tilde{T} \in \mathbb{N} : ({}_1a_T,_{T+1} \mathbf{1}) \succ^S ({}_1b_T,_{T+1} \mathbf{1})$ $\forall T \geq \tilde{T} \Rightarrow {}_1a \succ^S {}_1b$; (ii) $\exists \tilde{T} \in \mathbb{N} : ({}_1a_T,_{T+1} \mathbf{1}) \sim^S ({}_1b_T,_{T+1} \mathbf{1})$ $\forall T \geq \tilde{T} \Rightarrow {}_1a \sim^S {}_1b$.

Weak Consistency provides a link to the finite setting by transforming the comparison of two infinite utility paths into an infinite number of comparisons of utility paths each containing a finite number of generations. Axioms similar to Weak Consistency are common in the literature (see, e.g., Basu and Mitra [4], Asheim [2], Asheim and Banerjee [3]).[26]

Finally, the next axiom requires that $\succcurlyeq^S$ be complete at least when comparing elements of B^∞ with the same tail. This requirement is weak and it seems uncontroversial, for it is obviously desirable to be able to rank as many profiles as possible.[27]

Minimal Completeness: For all ${}_1a$, ${}_1b \in B^\infty$, ${}_1a \neq {}_1b$: ${}_{T+1}a = {}_{T+1}b$ for some $T \in \mathbb{N} \Rightarrow {}_1a \succcurlyeq^S {}_1b$ or ${}_1b \succcurlyeq^S {}_1a$.

Before proving our main characterisation result, we state the following Lemma which extends to B^∞ the equivalent result obtained in the finite context.[28]

Lemma 9 *Let the social opportunity quasi-ordering $\succcurlyeq^S$ on B^∞ satisfy* ***Finite Anonymity, Monotonicity, Probabilistic Harm Principle*****, **Individual Sure Thing***, and **Minimal Completeness**. Then: for all ${}_1a$, ${}_1b \in B^\infty$ such that ${}_{T+1}a = {}_{T+1}b$ for some $T \in \mathbb{N}$, $[a_t = 0, b_j = 0,$ some $t, j \in \{1, ..., T\}] \Rightarrow {}_1a \sim^S {}_1b$.*

The next Lemma derives a useful implication of **Monotonicity**, **Restricted Dominance**, and **Individual Sure Thing*.**

Lemma 10 *Let the social opportunity quasi-ordering $\succcurlyeq^S$ on B^∞ satisfy* ***Monotonicity, Restricted Dominance****, and **Individual Sure Thing***. Then: for all ${}_1a$, ${}_1b \in B^\infty$ such that ${}_{T+1}a = {}_{T+1}b = {}_{T+1}\mathbf{1}$ for some $T \in \mathbb{N}$, ${}_1a_T \gg {}_1b_T \Rightarrow {}_1a \succ^S {}_1b$.*

The next Theorem proves that the above axioms jointly characterise the Nash overtaking quasi-ordering.

[26] Under Strong Pareto Optimality normally one needs only part (i) of the Weak Consistency axiom (or similar axiom), see e.g. Asheim and Banerjee [3], in particular Proposition 2. Here we only assume Monotonicity and Restricted dominance and therefore the results in [3] do not hold. We thank Geir Asheim for alerting us to this issue.

[27] Lombardi and Veneziani [24] use minimal completeness to characterise the infinite leximin and maximin social welfare relations.

[28] The proof of Lemma 9 is a straightforward modification of the proof of Lemma 4 and therefore is omitted. Details are available from the authors upon request.

Theorem 11 *(NASH OVERTAKING): A social opportunity quasi-ordering $\succcurlyeq^S$ on B^∞ is an extension of $\succcurlyeq^{N^*}$ if and only if it satisfies **Finite Anonymity**, **Monotonicity**, **Restricted Dominance**, **Probabilistic Harm Principle***, **Individual Sure Thing***, **Weak Consistency**, and **Minimal Completeness**.*

The proofs of the two theorems of this section are in the Appendix.

Next, we provide an extension of the Two-Step Nash criterion to the infinite context in the framework of Bossert, Sprumont and Suzumura [9]. As announced, the characterisation is based on infinite-versions of the axioms used in Section 8. In addition to Finite Anonymity, we consider

Strong Pareto: For all ${}_1a,\ {}_1b \in B^\infty$, ${}_1a > {}_1b \Rightarrow {}_1a \succ^S {}_1b$.

Probabilistic Non-Interference*: Let ${}_1a, {}_1b \in B^\infty$ be such that ${}_1a = ({}_1a_T, {}_{T+1}b)$ for some $T \in \mathbb{N}$, and ${}_1a \succ^S {}_1b$; and let ${}_1a', {}_1b' \in B^\infty$ be such that for some $t \in \mathbb{N}$ and some $\rho > 0$,

$$
\begin{aligned}
a'_t &= \rho a_t, \\
b'_t &= \rho b_t, \\
a'_j &= a_j \text{ , for all } j \neq t, \\
b'_j &= b_j \text{ , for all } j \neq t.
\end{aligned}
$$

Then ${}_1a' \succ^S {}_1b'$ whenever $b_t \neq 0$ and $a'_t > b'_t$.

Suppose that for each $T \in \mathbb{N}$, the Two-Step Nash ordering on B^T is denoted as $\succcurlyeq_F^{2N}$. In analogy with Bossert, Sprumont and Suzumura [9], the Two-Step Nash social opportunity relation on B^∞ can be formulated as follows. Define $\succcurlyeq_T^{2N} \subseteq B^\infty \times B^\infty$ by letting, for all ${}_1a, {}_1b \in B^\infty$,

$$
{}_1a \succcurlyeq_T^{2N} {}_1b \Leftrightarrow {}_1a_T \succcurlyeq_F^{2N} {}_1b_T \text{ and } {}_{T+1}a \geq {}_{T+1}b. \tag{1}
$$

The relation $\succcurlyeq_T^{2N}$ is reflexive and transitive for all $T \in \mathbb{N}$. Then the Two-Step Nash social opportunity relation is $\succcurlyeq^{2N^*} = \bigcup_{T \in \mathbb{N}} \succcurlyeq_T^{2N}$.

Theorem 12 *(NASH CATCHING-UP) $\succcurlyeq^S$ on B^∞ is an ordering extension of $\succcurlyeq^{2N^*}$ if and only if $\succcurlyeq^S$ on B^∞ satisfies **Finite Anonymity**, **Strong Pareto**, and **Probabilistic Non-Interference***.*

11 Concluding Remarks

We have proposed formulating opportunities as chances of success, an interpretation close to the standard use of the term by practitioners. This interpretation is easily amenable to concrete measurement, suitable to the formulation of social policy targets, and close to common usage in the public debate.

We have highlighted some interesting conflicts between principles and discussed how such conflicts can be overcome. We have shown that strong limits to inequality in the profile of opportunities are implied by a liberal principle of justice and of social rationality. Beside the inequality aversion (concavity) of the social criterion, even only one person failing with certainty brings down the value of *any* profile to the minimum possible.

The use of the Nash social opportunity ordering acquires a natural interpretation in this context as the probability that everybody succeeds. Although not purely egalitarian, this 'maximise the probability of Heaven' criterion is likely in practice to avoid major disparities in opportunities, as profiles involving very low opportunities for one individual will appear very low in the social ordering. And, in the two-step refinement we have proposed, Hell should also be a sparsely populated place: that is, in practice, societies in which opportunities are confined to a tiny elite should be frowned upon. These partially egalitarian conclusions look stronger when one considers that they are obtained without any reference to issues of 'talent' or 'responsibility': the conclusions are partial but unconditional.[29]

One feature of our analysis is that in the 'Maximise the probability of Heaven' interpretation of the Nash criterion we have treated individuals as *independent* experiments. Note first that this relates only to the interpretation and not to the results themselves: the Nash criterion continues to follow from the axioms even without assuming such independence. Secondly, at least to some extent, independence can be guaranteed by defining the notion of success in such a way as to factor out the common variables affecting success across individuals. For example, the chances of attaining a high paying job for the dustman's daughter and for the doctor's son are both affected

[29] One aim of our approach is to simplify the issue of egalitarianism in a context of 'social risk' as much as possible, which is obtained by assuming that success is binary. If social risk were to be considered allowing individual outcomes to be measured along a utility scale, the definition of an appropriate concept of egalitarianism would raise many additional thorny issues. See Fleurbaey [17] for a recent insightful contribution.

by the possibility of an economic recession, and must therefore be partially correlated. To obtain independence, one might define a high-paying job independently for each state of nature or as an average across states. Thirdly, it seems nevertheless of interest to consider a framework in which the *input* of the analysis is the probability distribution over all logically conceivable profiles of success and failure, so as to include explicitly the possible correlations, instead of social preferences over profiles of 'marginal' distributions. This would be appropriate in cases where the correlation device is a relevant variable under the control of the social decision maker - imagine for instance the decision whether two officials on a wartime mission should travel on the same plane or on separate planes (with each plane having a probability p of crashing). Correlations are at the core of Fleurbaey's [17] study of risky social situations, which characterises a (mild) form of ex-post egalitarianism, allowing individual outcomes to be measured along a utility scale, for a fixed and strictly positive vector of probabilities on a given set of states of the world. An interesting development of our proposal would be to study the issue of correlations in our framework, with variable probabilities and a restricted range of outcomes. This is left for future research.

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12 Appendix: Proof of Lemmas and of Theorems on Infinite Societies

Proof of Lemma 4: For any $a, b \in B^T$, suppose without loss of generality that $T > z = |P^b| \geq |P^a|$, and denote $h = |P^b| - |P^a|$. We proceed by induction on h.

1. ($h = 0$) Consider $a, b \in B^T$ and suppose that $T > |P^a| = |P^b| = z$. If $z = 0$, then the result follows by reflexivity. If $z > 0$ and there is a permutation π such that $a = \pi b$, the result follows by **Anonymity**. Therefore suppose that $z > 0$ and there is no permutation π such that $a = \pi b$.

Suppose, by way of contradiction, that $a \nsim^S b$. By completeness, and without loss of generality, suppose that $a \succ^S b$. By **Anonymity** and transitivity, we can focus on the ranked profiles $\overline{a}, \overline{b}$ where by assumption:

$$\overline{a} = (0, 0, ..., 0, \overline{a}_l, ..., \overline{a}_T), \ \overline{b} = \left(0, 0, ..., 0, \overline{b}_l, ..., \overline{b}_T\right),$$

and $z = T - l + 1$. We need to consider two cases.

Case 1. Suppose that for all $l \leq j \leq T$, $\overline{a}_j \geq \overline{b}_j$ and consider any k such that $\overline{a}_k > \overline{b}_k$. Consider a profile a^π which is a permutation of $\overline{a}$ such that $a_1^\pi = \overline{a}_k$, $a_k^\pi = \overline{a}_1 = 0$, and all other entries are the same. By **Anonymity** and transitivity, $a^\pi \succ^S \overline{b}$.

Then, consider the profiles $a', b' \in B^T$ obtained from $a^\pi, \overline{b}$ as follows: $a'_1 = \rho a_1^\pi = \rho \overline{a}_k < \overline{b}_k$, $b'_1 = \rho \overline{b}_1 = 0$, and $a'_j = a_j^\pi$, $b'_j = \overline{b}_j$ all $j \neq 1$. By the **Probabilistic Harm Principle**, $a' \succcurlyeq^S b'$. Therefore by **Individual Sure Thing**, $a'' = \lambda a^\pi + (1 - \lambda) a' \succ^S \lambda \overline{b} + (1 - \lambda) b' = \overline{b}$, for all $\lambda \in (0, 1)$. Since $a_1^\pi = \overline{a}_k > \overline{b}_k$ and $a'_1 < \overline{b}_k$, it follows that there is $\lambda \in (0, 1)$ such that $a''_1 = \overline{b}_k$. If for all $j \neq k, l \leq j \leq T$, $\overline{a}_j = \overline{b}_j$, then we obtain a contradiction by **Anonymity**. If, instead, there is k' such that $\overline{a}_{k'} > \overline{b}_{k'}$, then the same argument can be applied iteratively m times to all entries of $\overline{a}$ and $\overline{b}$ such that $\overline{a}_t > \overline{b}_t$ to obtain profiles $a^m, b^m \in B^T$ such that by the **Probabilistic Harm Principle** and **Individual Sure Thing**, $a^m \succ^S b^m$, but $\overline{a}^m = \overline{b}^m$, yielding a contradiction by **Anonymity**.

Case 2. Suppose that there exists $l \leq j \leq T$, $\overline{a}_j < \overline{b}_j$. Let $J = \left\{l \leq j \leq T | \overline{a}_j < \overline{b}_j\right\}$. Then consider the profile $\overline{b}'$ such that $\overline{b}'_k = \overline{b}_k$ for all $k \notin J$ and $\overline{b}'_j = \overline{a}_j$, for all $j \in J$. By **Pareto** together with transitivity, it follows that $\overline{a} \succ^S \overline{b}'$. Then the same argument as in case 1 can be applied to derive the desired contradiction.

2. (Induction step) Suppose the result holds for $T - 1 > h - 1 \geq 0$. Consider $a, b \in B^T$ such that $|P^b| < T$ and $|P^b| - |P^a| = h > 0$. Suppose, by way of contradiction, that $a \nsim^S b$. By completeness, suppose that $a \succ^S b$. By **Anonymity** and transitivity, we can focus on the ranked profiles $\overline{a}, \overline{b}$ where by construction:

$$\overline{a} = (0, 0, ..., 0, \overline{a}_l, ..., \overline{a}_T), \ \overline{b} = \left(0, 0, ..., 0, \overline{b}_{l-h}, ..., \overline{b}_T\right),$$

with $l > l - h > 1$. Then consider the profile $\overline{a}' \in B^T$ which is obtained from $\overline{a}$ by setting $1 > \overline{a}'_{l-1} > 0$: $\overline{a}' = \left(0, 0, ..., 0, \overline{a}'_{l-1}, \overline{a}_l, ..., \overline{a}_T\right)$. By construction $|P^{\overline{b}}| - |P^{\overline{a}'}| = h - 1$ and thus by the induction hypothesis, $\overline{a}' \sim^S \overline{b}$. Then, by **Individual Sure Thing**, it follows that $\overline{a}'' = \lambda\overline{a} + (1-\lambda)\,\overline{a}' \succ^S \overline{b} = \lambda\overline{b} + (1-\lambda)\,\overline{b}$, for all $\lambda \in (0,1)$. However, since $|P^{\bar{b}}| - |P^{\overline{a}''}| = h-1$ it must be $\overline{a}'' \sim^S \overline{b}$ by the induction hypothesis, a contradiction.

A similar argument rules out the possibility that $b \succ^S a$. ■

Proof of Lemma 10: We proceed by induction on T.

1. ($T=1$) Take any ${}_1a$, ${}_1b \in B^\infty$ such that $a_1 > b_1$ and ${}_2a = {}_2b = {}_2\mathbf{1}$. Note that by **Restricted Dominance**, $\mathbf{1} \succ^S (b_1,{}_2\mathbf{1})$, and so if $a_1 = 1$, the result immediately follows. Suppose $a_1 < 1$. By reflexivity, ${}_1b \sim^S {}_1b$. But then, by **Individual Sure Thing*** it follows that $\forall\lambda \in (0,1) : \lambda\mathbf{1} + (1-\lambda)$ ${}_1b \succ^S \lambda_1 b + (1-\lambda)\ {}_1b = {}_1b$, and noting that $1 > a_1 > b_1$, we obtain ${}_1a \succ^S$ ${}_1b$.

2. (Induction step.) Suppose that the result holds for $T-1 \geq 1$. Consider any ${}_1a$, ${}_1b \in B^\infty$ such that ${}_{T+1}a = {}_{T+1}b = {}_{T+1}\mathbf{1}$ for some $T > 1$, and ${}_1a_T \gg {}_1b_T$. Suppose first that $a_T = 1$. By the induction hypothesis, it follows that $({}_1a_{T-1},{}_T\,\mathbf{1}) \succ^S ({}_1b_{T-1},{}_T\,\mathbf{1})$. By **Monotonicity**, $({}_1b_{T-1},{}_T\,\mathbf{1}) \succcurlyeq^S ({}_1b_{T-1}, b_T,{}_{T+1}\,\mathbf{1})$ and therefore by transitivity, $({}_1a_{T-1},{}_T\,\mathbf{1}) \succ^S ({}_1b_{T-1}, b_T,{}_{T+1}\,\mathbf{1})$, which yields the desired result. Suppose next that $a_T < 1$. By **Monotonicity**, $({}_1a_{T-1}, b_T,{}_{T+1}\,\mathbf{1}) \succcurlyeq^S ({}_1b_{T-1}, b_T,{}_{T+1}\,\mathbf{1})$. But then, by **Individual Sure Thing*** it follows that $\forall\lambda \in (0,1) : \lambda\,({}_1a_{T-1},{}_T\,\mathbf{1}) + (1-\lambda)\,({}_1a_{T-1}, b_T,{}_{T+1}\,\mathbf{1}) \succ^S \lambda\,({}_1b_{T-1}, b_T,{}_{T+1}\,\mathbf{1}) + (1-\lambda)\,({}_1b_{T-1}, b_T,{}_{T+1}\,\mathbf{1}) = ({}_1b_T,{}_{T+1}\,\mathbf{1})$, and noting that $1 > a_T > b_T$, we obtain ${}_1a \succ^S {}_1b$. ■

Proof of Theorem 11: ($\Rightarrow$) Let $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$. It is easy to see that $\succcurlyeq^S$ meets **Finite Anonymity**, **Monotonicity**, and **Restricted Dominance**. By observing that $\succcurlyeq^{N^*}$ is complete for comparisons between profiles with the same tail, it is also easy to see that $\succcurlyeq^S$ satisfies **Weak Consistency** and **Minimal Completeness**. We need to show that $\succcurlyeq^S$ meets **Probabilistic Harm Principle*** and **Individual Sure Thing***.

To prove that $\succcurlyeq^S$ satisfies **Probabilistic Harm Principle***, take any ${}_1a,{}_1b \in B^\infty$ such that ${}_1a = ({}_1a_T,{}_{T+1}\,b)$ for some $T \in \mathbb{N}$, and ${}_1a \succ^S {}_1b$. Since $\succcurlyeq^{N^*}$ is complete for comparisons between profiles with the same tail, it follows that ${}_1a \succ^{N^*} {}_1b$. Then, let ${}_1a',{}_1b' \in B^\infty$ be such that for some $t' \in \mathbb{N}$, and some $\rho \in (0,1)$, $a'_{t'} = \rho a_{t'}$, $b'_{t'} = \rho b_{t'}$, $a'_j = a_j$, all $j \neq t'$, $b'_j = b_j$, all $j \neq t'$. We need to prove that ${}_1b' \not\succ^S {}_1a'$ whenever $a'_{t'} > b'_{t'}$.

By definition, ${}_1a \succ^{N^*} {}_1b$ implies that $\exists\tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T}$: $\prod_{t=1}^T a_t > \prod_{t=1}^T b_t$. Consider any $T' \geq \max\left\{t', \tilde{T}\right\}$. Then note that $\forall T \geq T'$, $\prod_{t=1}^T a_t > \prod_{t=1}^T b_t$ implies $\prod_{t=1}^T a'_t = \rho\prod_{t=1}^T a_t > \prod_{t=1}^T b'_t = \rho\prod_{t=1}^T b_t$, for all

$\rho \in (0,1)$. Therefore ${}_1a' \succ^{N^*} {}_1b'$, and since $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$, it follows that ${}_1b' \not\succcurlyeq^S {}_1a'$.

To prove that $\succcurlyeq^S$ satisfies **Individual Sure Thing***, take any ${}_1a, {}_1b \in B^\infty$ such that ${}_1a = ({}_1a_{\widehat{T}}, {}_{\widehat{T}+1}b)$ for some $\widehat{T} \in \mathbb{N}$, and ${}_1a \succcurlyeq^S {}_1b$, and let ${}_1a', {}_1b' \in B^\infty$ be such that for some $t' \leq \widehat{T}$, $a'_j = a_j$ and $b'_j = b_j$, all $j \neq t'$, and ${}_1a' \succcurlyeq^S {}_1b'$. Since $\succcurlyeq^{N^*}$ is complete for comparisons between profiles with the same tail, it follows that ${}_1a \succcurlyeq^{N^*} {}_1b$ and ${}_1a' \succcurlyeq^{N^*} {}_1b'$. We show that

$$\forall \lambda \in (0,1): \ {}_1a'' = \lambda {}_1a + (1-\lambda) \ {}_1a' \succcurlyeq^S \ {}_1b'' = \lambda {}_1b + (1-\lambda) \ {}_1b',$$

with ${}_1a'' \succ^S {}_1b''$ if at least one of the two preferences in the premise is strict.

Suppose that ${}_{\widehat{T}+1}a = {}_{\widehat{T}+1}b \gg \mathbf{0}$. By definition, and noting that ${}_{\widehat{T}+1}a = {}_{\widehat{T}+1}b$, ${}_1a \succcurlyeq^{N^*} {}_1b$ implies that either $\forall T \geq \widehat{T} : \prod_{t=1}^T a_t > \prod_{t=1}^T b_t$, or $\forall T \geq \widehat{T}$: $\prod_{t=1}^T a_t = \prod_{t=1}^T b_t$. And a similar argument holds for ${}_1a' \succcurlyeq^{N^*} {}_1b'$. By assumption it must be $\prod_{t=1}^{\widehat{T}} a_t \geq \prod_{t=1}^{\widehat{T}} b_t$ and $\prod_{t=1}^{\widehat{T}} a'_t \geq \prod_{t=1}^{\widehat{T}} b'_t$. Furthermore, by construction, $a''_j = a_j = a'_j$ and $b''_j = b'_j = b_j$, all $j \neq t'$, $a''_{t'} = \lambda a_{t'} + (1-\lambda) a'_{t'}$, and $b''_{t'} = \lambda b_{t'} + (1-\lambda) b'_{t'}$. Therefore for all $T \geq \widehat{T}$, the following holds: $\prod_{t=1}^T a''_t = (\lambda a_{t'} + (1-\lambda) a'_{t'}) \prod_{t \neq t'} a_t$, and noting that $\prod_{t \neq t'} a_t = \prod_{t \neq t'} a'_t$, $\prod_{t=1}^T a''_t = \lambda \prod_{t=1}^T a_t + (1-\lambda) \prod_{t=1}^T a'_t$. A similar argument shows that $\prod_{t=1}^T b''_t = \lambda \prod_{t=1}^T b_t + (1-\lambda) \prod_{t=1}^T b'_t$.

Therefore if $\forall T \geq \widehat{T} : \prod_{t=1}^T a_t = \prod_{t=1}^T b_t$ and $\prod_{t=1}^T a'_t = \prod_{t=1}^T b'_t$, it follows that $\forall T \geq \widehat{T} : \prod_{t=1}^T a''_t = \prod_{t=1}^T b''_t$. Instead, if either $\forall T \geq \widehat{T} : \prod_{t=1}^T a_t > \prod_{t=1}^T b_t$, or $\forall T \geq \widehat{T} : \prod_{t=1}^T a'_t > \prod_{t=1}^T b'_t$, holds, it follows that $\forall T \geq \widehat{T} : \prod_{t=1}^T a''_t > \prod_{t=1}^T b''_t$. In the former case, ${}_1a'' \sim^{N^*} {}_1b''$, whereas in the latter case ${}_1a'' \succ^{N^*} {}_1b''$. Since $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$, the desired result follows.

If $a_{\widetilde{T}} = b_{\widetilde{T}} = 0$ for some $\widetilde{T} > \widehat{T}$, then $\forall T \geq \widetilde{T} : \prod_{t=1}^T a_t = \prod_{t=1}^T b_t = \prod_{t=1}^T a'_t = \prod_{t=1}^T b'_t = 0$, and so $\forall T \geq \widetilde{T} : \prod_{t=1}^T a''_t = \prod_{t=1}^T b''_t = 0$. This implies ${}_1a'' \sim^{N^*} {}_1b''$ and the desired result again follows from $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$.

($\Leftarrow$) Suppose that $\succcurlyeq^S$ on B^∞ satisfies **Finite Anonymity**, **Monotonicity**, **Restricted Dominance**, **Probabilistic Harm Principle***, **Individual Sure Thing***, **Weak Consistency**, and **Minimal Completeness**. We show that $\succcurlyeq^{N^*} \subseteq \succcurlyeq^S$, that is, for all ${}_1a$, ${}_1b \in B^\infty$,

$${}_1a \sim^{N^*} {}_1b \Rightarrow {}_1a \sim^S {}_1b, \tag{2}$$

and

$${}_1a \succ^{N^*} {}_1b \Rightarrow {}_1a \succ^S {}_1b. \tag{3}$$

Consider (2). Take any ${}_1a,\ {}_1b \in B^\infty$ such that $\exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T}$: $\prod_{t=1}^{T} a_t = \prod_{t=1}^{T} b_t$.

Case 1. ${}_1a \gg \mathbf{0}$ and ${}_1b \gg \mathbf{0}$. If $\exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T}$: $\prod_{t=1}^{T} a_t = \prod_{t=1}^{T} b_t$, then ${}_{\tilde{T}+1}a = {}_{\tilde{T}+1}b$. Suppose, in contradiction, that ${}_1a \nsim^S {}_1b$. By **Minimal Completeness**, and without loss of generality, suppose that ${}_1a \succ^S {}_1b$. Fix $T \geq \tilde{T}$. With an argument analogous to the finite case, we can use **Monotonicity**, **Probabilistic Harm Principle***, **Individual Sure Thing***, **Finite Anonymity**, **Minimal Completeness**, and transitivity iteratively to derive profiles ${}_1a^m, {}_1b^m \in B^\infty$ such that ${}_1a^m = ({}_1a_T^m, {}_{T+1}a) \succ^S {}_1b^m = ({}_1b_T^m, {}_{T+1}b)$, but there is a permutation $\pi \in \Pi$ such that ${}_1a^m = \pi({}_1b^m)$, which contradicts **Finite Anonymity**.

Case 2. $a_{T'} = 0$ for some $T' \in \mathbb{N}$ and $b_{T''} = 0$ for some $T'' \in \mathbb{N}$. Take any $T \geq \max\{T', T''\}$ and consider the profiles $({}_1a_T, {}_{T+1}\mathbf{1})$ and $({}_1b_T, {}_{T+1}\mathbf{1})$. Clearly, $({}_1a_T, {}_{T+1}\mathbf{1})$ and $({}_1b_T, {}_{T+1}\mathbf{1})$ are in B^∞ and by Lemma 9 $({}_1a_T, {}_{T+1}\mathbf{1}) \sim^S ({}_1b_T, {}_{T+1}\mathbf{1})$. Hence, by **Weak Consistency** we conclude that ${}_1a \sim^S {}_1b$.

Consider (3). Take any ${}_1a,\ {}_1b \in B^\infty$ such that $\exists \tilde{T} \in \mathbb{N}$ such that $\forall T \geq \tilde{T}$: $\prod_{t=1}^{T} a_t > \prod_{t=1}^{T} b_t$. Take any $T \geq \tilde{T}$ and consider the profiles $({}_1a_T, {}_{T+1}\mathbf{1})$ and $({}_1b_T, {}_{T+1}\mathbf{1})$. Clearly, $({}_1a_T, {}_{T+1}\mathbf{1})$ and $({}_1b_T, {}_{T+1}\mathbf{1})$ are in B^∞ and $({}_1a_T, {}_{T+1}\mathbf{1}) \succ^{N^*} ({}_1b_T, {}_{T+1}\mathbf{1})$. Let ${}_1x \equiv ({}_1b_T, {}_{T+1}\mathbf{1})$ and ${}_1y \equiv ({}_1a_T, {}_{T+1}\mathbf{1})$.

Note that $\prod_{t=1}^{T} a_t > \prod_{t=1}^{T} b_t$ implies that ${}_1a_T \gg \mathbf{0}$. Hence there is a sufficiently small profile $\varepsilon \in B_+^T$ such that ${}_1a_T^\varepsilon = (a_1 - \varepsilon_1, a_2 - \varepsilon_2, ..., a_T - \varepsilon_T) \gg \mathbf{0}$ and $\prod_{t=1}^{T} (a_t - \varepsilon_t) = \prod_{t=1}^{T} b_t$. By (2), it follows that ${}_1y^\epsilon \equiv ({}_1a_T^\varepsilon, {}_{T+1}\mathbf{1}) \sim^S {}_1x$. By Lemma 10, ${}_1y \succ^S {}_1y^\epsilon$ and therefore ${}_1y \succ^S {}_1x$ by transitivity.

Therefore $({}_1a_T, {}_{T+1}\mathbf{1}) \succ^S ({}_1b_T, {}_{T+1}\mathbf{1})$ and since the argument holds for any $T \geq \tilde{T}$, it follows from **Weak Consistency** that ${}_1a \succ^S {}_1b$. ■

Proof of Theorem 12: ($\Rightarrow$) We first prove that the relations $\succsim_T^{2N}$ and $\succ_T^{2N}$ are nested. That is, for all $T \in \mathbb{N}$

$$\succsim_T^{2N} \subseteq \succsim_{T+1}^{2N},$$

and

$$\succ_T^{2N} \subseteq \succ_{T+1}^{2N} .$$

To prove the former set inclusion, suppose that ${}_1a \succsim_T^{2N} {}_1b$. By definition, ${}_1a \succsim_T^{2N} {}_1b \Leftrightarrow {}_1a_T \succsim_F^{2N} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$. Then, either ${}_1a_T \succ_F^{2N} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$ or ${}_1a_T \sim_F^{2N} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$. In either case, it is immediate to prove that ${}_1a \succsim_{T+1}^{2N} {}_1b$ and ${}_{T+2}a \geq {}_{T+2}b$, and so ${}_1a \succsim_{T+1}^{2N} {}_1b$.

To prove the latter set inclusion, suppose that ${}_1a \succ^{2N}_{T} {}_1b$. By definition at least one of the following statements is true:

(i) ${}_1a_T \succ^{2N}_{F} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$

(ii) ${}_1a_T \succcurlyeq^{2N}_{F} {}_1b_T$ and ${}_{T+1}a > {}_{T+1}b$.

If (i) holds, then it is immediate to prove that ${}_1a_{T+1} \succ^{2N}_{F} {}_1b_{T+1}$ and ${}_{T+2}a \geq {}_{T+2}b$ and so ${}_1a \succ^{2N}_{T+1} {}_1b$.

So, suppose (ii) holds but (i) does not. If $a_{T+1} = b_{T+1}$, then ${}_1a_T \succcurlyeq^{2N}_{F} {}_1b_T$ and ${}_{T+1}a > {}_{T+1}b$ implies ${}_1a_{T+1} \succcurlyeq^{2N}_{F} {}_1b_{T+1}$ and ${}_{T+2}a > {}_{T+2}b$. If $a_{T+1} > b_{T+1}$, then ${}_1a_T \succcurlyeq^{2N}_{F} {}_1b_T$ and ${}_{T+1}a > {}_{T+1}b$ implies ${}_1a_{T+1} \succ^{2N}_{F} {}_1b_{T+1}$ and ${}_{T+2}a \geq {}_{T+2}b$. In either case ${}_1a \succ^{2N}_{T+1} {}_1b$.

In sum, we have proved that $\succcurlyeq^{2N}_{T} \subseteq \succcurlyeq^{2N}_{T+1}$ and $\succ^{2N}_{T} \subseteq \succ^{2N}_{T+1}$.

Then, using the same arguments as in Bossert et al. ([9], Theorem 1, p.584) it can be shown that $\succcurlyeq^{2N^*}$ is reflexive and transitive, and that it satisfies the following property ([9], p. 586, equation (14)):

$$\forall\ {}_1a, {}_1b \in B^{\infty} : \exists T \in \mathbb{N} \text{ such that } {}_1a \succ^{2N}_{T} {}_1b \Leftrightarrow {}_1a \succ^{2N^*} {}_1b. \tag{4}$$

In order to complete the proof of necessity, we need to prove that any ordering extension $\succcurlyeq^{S}$ of $\succcurlyeq^{2N^*}$ satisfies the properties in the statement.

To prove that **Strong Pareto** is satisfied, take any ${}_1a, {}_1b \in B^{\infty}$ such that ${}_1a > {}_1b$. Let $T = \min\{t \in \mathbb{N} : a_t > b_t\}$. By definition, ${}_1a \succ^{2N}_{T} {}_1b$ and therefore, by property (4), ${}_1a \succ^{2N^*} {}_1b$ and the result follows from $\succcurlyeq^{2N^*} \subseteq \succcurlyeq^{S}$.

To prove that **Finite Anonymity** is satisfied, take any ${}_1a \in B^{\infty}$ and any $\pi \in \Pi$. By definition of $\pi \in \Pi$, there is $T \in \mathbb{N}$, such that $\pi(t) = t$, for all $t > T$. Take such $T \in \mathbb{N}$. By definition of $\succcurlyeq^{2N}_{T}$, it follows that ${}_1a \sim^{2N}_{T} \pi({}_1a)$, which in turn implies ${}_1a \sim^{2N^*} \pi({}_1a)$, and the result follows from $\succcurlyeq^{2N^*} \subseteq \succcurlyeq^{S}$.

To prove that **Probabilistic Non-Interference*** is satisfied, let ${}_1a, {}_1b \in B^{\infty}$ be such that ${}_1a = ({}_1a_T, {}_{T+1}b)$ for some $T \in \mathbb{N}$ and ${}_1a \succ^{S} {}_1b$. Suppose that ${}_1a', {}_1b' \in B^{\infty}$ are such that for some $t' \in \mathbb{N}$, and some $\rho > 0$, $a'_{t'} = \rho a_{t'}$, $b'_{t'} = \rho b_{t'}$, and $a'_j = a_j$, $b'_j = b_j$, all $j \neq t'$. We want to prove that if $\succcurlyeq^{2N^*} \subseteq \succcurlyeq^{S}$ then ${}_1a' \succ^{S} {}_1b'$ whenever $b_{t'} \neq 0$ and $a'_{t'} > b'_{t'}$.

Since $\succcurlyeq^{2N^*}$ is complete for comparisons between profiles with the same tail, it follows that ${}_1a \succ^{2N^*} {}_1b$. Therefore by property (4), there exists $T' \in \mathbb{N}$ such that ${}_1a \succ^{2N}_{T'} {}_1b$. Without loss of generality, let $T' = T$. Then ${}_1a \succ^{2N}_{T} {}_1b$ implies ${}_1a_T \succ^{2N}_{F} {}_1b_T$ and ${}_{T+1}a = {}_{T+1}b$. If ${}_1a_T, {}_1b_T \in B^T_+$ and $\prod_{t=1}^{T} a_t > \prod_{t=1}^{T} b_t$, then ${}_1a'_T, {}_1b'_T \in B^T_+$ and $\prod_{t=1}^{T} a'_t = \rho \prod_{t=1}^{T} a_t > \prod_{t=1}^{T} b'_t = \rho \prod_{t=1}^{T} b_t$. If $|P^{{}_1a_T}| > |P^{{}_1b_T}|$, then $|P^{{}_1a'_T}| = |P^{{}_1a_T}| > |P^{{}_1b'_T}| = |P^{{}_1b_T}|$. Finally, if $|P^{{}_1a_T}| = |P^{{}_1b_T}| < T$ and $\prod_{t \in P^{{}_1a_T}} a_t > \prod_{t \in P^{{}_1b_T}} b_t$, then $|P^{{}_1a'_T}| = |P^{{}_1a_T}| =$

$|P^{{}_1b'_T}| = |P^{{}_1b_T}|$, $\prod_{t \in P^{{}_1a'_T}} a'_t = \rho \prod_{t \in P^{{}_1a_T}} a_t > \prod_{t \in P^{{}_1b'_T}} b'_t = \rho \prod_{t \in P^{{}_1b_T}} b_t$. In all three cases, ${}_1a'_T \succ_F^{2N} {}_1b'_T$ and ${}_{T+1}a' = {}_{T+1}b'$, so that ${}_1a' \succ_T^{2N} {}_1b'$ and therefore by property (4), ${}_1a' \succ^{2N^*} {}_1b'$. The result follows noting that $\succcurlyeq^{2N^*} \subseteq \succcurlyeq^S$.

($\Leftarrow$) (The proof of sufficiency simply adapts the one given for the leximin catching up by Bossert, Sprumont and Suzumura [9]. We report it in its entirety for clarity.) Suppose that $\succcurlyeq^S$ is an ordering on B^∞ that satisfies **Finite Anonymity**, **Strong Pareto**, and **Probabilistic Non-Interference***. Fix $T \in \mathbb{N}$ and ${}_1c \in B^\infty$, and for any ${}_1a$, ${}_1b \in B^\infty$ define the relation $\succcurlyeq_{{}_1c}^T \subseteq B^T \times B^T$ as follows:

$$ {}_1a_T \succcurlyeq_{{}_1c}^T {}_1b_T \Leftrightarrow ({}_1a_T, {}_{T+1}c) \succcurlyeq^S ({}_1b_T, {}_{T+1}c) \, . $$

$\succcurlyeq_{{}_1c}^T$ is an ordering because $\succcurlyeq^S$ is. Moreover, for any ${}_1a$, ${}_1b \in B^\infty$,

$$ {}_1a_T \succ_{{}_1c}^T {}_1b_T \Leftrightarrow ({}_1a_T, {}_{T+1}c) \succ^S ({}_1b_T, {}_{T+1}c) \, . $$

The three axioms imply that $\succcurlyeq_{{}_1c}^T$ must satisfy the T-person versions of the axioms. Hence, using the characterisation of the T-person Two-Step Nash social opportunity ordering in theorem 6, it follows that

$$ \succcurlyeq_{{}_1c}^T = \succcurlyeq_F^{2N} \, . $$

Because T and ${}_1c$ were chosen arbitrarily, the latter statement is true for all $T \in \mathbb{N}$ and for any ${}_1c \in B^\infty$.

To prove that $\succcurlyeq^S$ is an ordering extension of $\succcurlyeq^{2N^*}$, we first establish that $\succcurlyeq^{2N^*} \subseteq \succcurlyeq^S$. Suppose that ${}_1a$, ${}_1b \in B^\infty$ are such that ${}_1a \succcurlyeq^{2N^*} {}_1b$. By the definition of $\succcurlyeq^{2N^*}$, there exists a T such that ${}_1a \succcurlyeq_T^{2N} {}_1b$, that is, ${}_1a_T \succcurlyeq_F^{2N} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$. Then, since $\succcurlyeq_{{}_1c}^T = \succcurlyeq_F^{2N}$, it follows that ${}_1a_T \succcurlyeq_{{}_1c}^T {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$, for all ${}_1c \in B^\infty$. Choosing ${}_1c = {}_1b$ and using the definition of $\succcurlyeq_{{}_1c}^T$, it follows that $({}_1a_T, {}_{T+1}b) \succcurlyeq^S ({}_1b_T, {}_{T+1}b)$. Because ${}_{T+1}a \geq {}_{T+1}b$, either reflexivity or **Strong Pareto**, together with transitivity imply $({}_1a_T, {}_{T+1}a) \succcurlyeq^S ({}_1b_T, {}_{T+1}b)$.

The proof is completed by showing that $\succ^{2N^*} \subseteq \succ^S$. Suppose that ${}_1a$, ${}_1b \in B^\infty$ are such that ${}_1a \succ^{2N^*} {}_1b$. By (4), there exists $T \in \mathbb{N}$ such that ${}_1a \succ_T^{2N} {}_1b$. By definition, at least one of the following statements is true:

${}_1a_T \succ_F^{2N} {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$,

${}_1a_T \succcurlyeq_F^{2N} {}_1b_T$ and ${}_{T+1}a > {}_{T+1}b$.

In the former case, since $\succcurlyeq_{{}_1c}^T = \succcurlyeq_F^{2N}$, it follows that ${}_1a_T \succ_{{}_1c}^T {}_1b_T$ and ${}_{T+1}a \geq {}_{T+1}b$, for all ${}_1c \in B^\infty$. Choosing ${}_1c = {}_1b$ and using the definition of $\succ_{{}_1c}^T$, it follows that $({}_1a_T, {}_{T+1}b) \succ^S ({}_1b_T, {}_{T+1}b)$. Then using either reflexivity or **Strong Pareto**, together with transitivity as in the proof of $\succsim^{2N^*} \subseteq \succsim^S$, we obtain $({}_1a_T, {}_{T+1}a) \succ^S ({}_1b_T, {}_{T+1}b)$.

In the latter case, since $\succsim^T_{{}_1c} = \succsim^{2N}_F$, it follows that ${}_1a_T \succsim^T_{{}_1c} {}_1b_T$ and ${}_{T+1}a > {}_{T+1}b$, for all ${}_1c \in B^\infty$. Choosing ${}_1c = {}_1b$ and using the definition of $\succsim^T_{{}_1c}$, it follows that $({}_1a_T, {}_{T+1}b) \succsim^S ({}_1b_T, {}_{T+1}b)$. Then by **Strong Pareto** and transitivity, it follows that $({}_1a_T, {}_{T+1}a) \succ^S ({}_1b_T, {}_{T+1}b)$.

Therefore $\succ^{2N^*} \subseteq \succ^S$, which concludes the proof. ■