On $k$-independence in graphs with emphasis on trees

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Abstract

In a graph $G = (V, E)$ of order $n$ and maximum degree $\Delta$, a subset $S$ of vertices is a $k$-independent set if the subgraph induced by $S$ has maximum degree less or equal to $k - 1$. The lower $k$-independence number $i_k(G)$ is the minimum cardinality of a maximal $k$-independent set in $G$ and the $k$-independence number $\beta_k(G)$ is the maximum cardinality of a $k$-independent set. We show that $i_k \leq n - \Delta + k - 1$ for any graph and any $k \leq \Delta$, and $i_k \leq n - \Delta$ if $G$ is connected, that $\beta_k(T) \geq kn/(k + 1)$ for any tree, and that $i_k \leq (n + s)/2 \leq \beta_k$ for any connected bipartite graph with $s$ support vertices. Moreover, we characterize the trees satisfying $i_k = n - \Delta$, $\beta_k = kn/(k + 1)$, $i_k = (n + s)/2$ or $\beta_k = (n + s)/2$.

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1. Introduction

For notation and graph theory terminology, we in general follow [1,4]. In a graph $G = (V, E)$ of order $n(G)$, the neighborhood of a vertex $v \in V$ is $N_G(v) = \{u \in V | uv \in E\}$. If $S$ is a subset of vertices, its neighborhood is $N_G(S) = \bigcup_{v \in S} N_G(v)$. The closed neighborhoods of $v$ and $S$ are $N[v] = N(v) \cup \{v\}$ and $N[S] = N(S) \cup S$. The degree of a vertex $v$ of $G$, denoted by $d_G(v)$, is the order of its neighborhood. A vertex of degree one is called a pendant vertex or a leaf and its neighbor is called a support vertex. If $v$ is a support vertex of a tree $T$ then $L_v$ will denote the set of the leaves attached at $v$. A support vertex $v$ is called strong if $|L_v| > 1$. We also denote the set of leaves of a graph $G$ by $L(G)$, the set of support vertices by $S(G)$, and let $|L(G)| = \ell(G)$, $|S(G)| = s(G)$. If $T = P_2$ then $\ell(P_2) = s(P_2) = 2$. If $u$ is a vertex of a rooted tree $T$, we denote by $T_u$ the subtree of root $u$.

We call $k$-corona of a graph $G$ the graph of order $k|V(G)|$ obtained from $G$ by adding a path of length $k - 1$ to each vertex of $G$ so that the resulting paths are vertex disjoint. A double star $S_{p,q}$ is obtained by attaching $p$ leaves at an endvertex of a path $P_2$ and $q$ leaves at the second one. The tree obtained from a double star $S_{p,q}$ by subdividing once the edge joining the two support vertices is denoted $S^*_{p,q}$. A subdivided star $SS_q$ is obtained from a star $K_1,q$ by subdividing each edge by exactly one vertex. The structure of stars and subdivided stars can be generalized as follows.

Definition 1. For $r + 1$ integers $p_i \geq 1$ such that $\sum_{i=0}^{r} p_i = 1$, the generalized star $GS_{p_0,p_1,...,p_r}$ is the tree of order $n = 1 + \sum_{i=0}^{r} (i + 1) p_i$ and maximum degree $\Delta = \sum_{i=0}^{r} p_i$ obtained from the star $K_{1,p_0+p_1+...+p_r}$ by subdividing $p_1$ rays once, $p_2$ rays twice, ..., $p_r$ rays $r$ times.

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An independent set is a set of vertices whose induced subgraph has no edge. The domination independence number $i(G)$ and the independence number $\beta(G)$ are, respectively, the minimum and the maximum cardinality of a maximal independent set.

In [3] Fink and Jacobson defined a generalization of the concept of independent sets. A set $S$ of $V$ is a $k$-independent set if the maximum degree of the subgraph induced by the vertices of $S$ is less or equal to $k - 1$. The lower $k$-independence number $i_k(G)$ is the minimum cardinality of a maximal $k$-independent set in $G$ and the $k$-independence number $\beta_k(G)$ is the maximum cardinality of a $k$-independent set. We notice that the 1-independent sets are the classical independent sets, so $i_1(G) = i(G)$, and $\beta_1(G) = \beta(G)$.

For any parameter $\mu$ associated to a graph property $\mathcal{P}$, we refer to a set of vertices with Property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set. For a comprehensive treatment of domination in graphs and its variations, see [4].

In this paper we investigate $k$-independence in trees and bipartite graphs, where lower and upper bounds are presented. For several bounds extremal trees are characterized. When no confusion can arise, we often abbreviate $n(G)$, $d_G(v)$, $N_G(v)$, $s(G)$, $i_k(G)$, $\beta_k(G)$, $\ldots$ to $n$, $d(v)$, $N(v)$, $s$, $i_k$, $\beta_k$, $\ldots$.

2. Preliminary properties of $i_k$ and $\beta_k$

For any graph and any $k \geq 1$, every set of at most $k$ vertices induces a $k$-independent set and thus $k \leq i_k \leq \beta_k$. The sequence $(\beta_k)$ is weakly increasing since every $k$-independent set is $(k + 1)$-independent, while the sequence $(i_k)$ is not necessarily monotone. Finally, $i_{\Delta+1} = \beta_{\Delta+1} = n$ for any graph and we suppose henceforth $k \leq \Delta$.

To prepare the following sections, we give three lemmas relative to the behavior of the two parameters under some graph operations.

Lemma 2. For $k \geq 1$, let $w$ be a vertex of a graph $G''$ such that every neighbor of $w$ has degree at most $k$, at least $w$ or one of its neighbors has degree $k$ or more, and every vertex in $V(G'') \setminus N[w]$, if any, has degree less than $k$ in $G''$. Let $G'$ be any graph and $G$ the graph constructed from $G'$ and $G''$ by adding an edge between $w$ and any vertex of $G'$. Then $\beta_k(G) = \beta_k(G') + |V(G'')| - 1$.

Proof. The set $V(G'')$ is not $k$-independent but there exists a $\beta_k(G')$-set $S$ containing $V(G'') \setminus \{w\}$. Thus $S \setminus (V(G'') \setminus \{w\})$ is a $k$-independent set of $G'$. Hence $\beta_k(G') = \beta_k(G) - |(V(G'') \setminus \{w\})|$. Conversely, every $\beta_k(G')$-set can be extended to a $k$-independent set of $G$ by adding $V(G'') \setminus \{w\}$ and so $\beta_k(G) = \beta_k(G') + |V(G'')| - 1$, implying the equality. □

Lemma 3. Let $dcb$ and $def$ or $defg$ be two pendant paths of a graph $G$, and let $H = G - \{a, b, c\}$. Then $i_2(G) = i_2(H) + 2$.

Proof. Let $S$ be a $i_2$-set of $H$. Then $S \cup \{a, b\}$ is a maximal 2-independent set of $G$ and thus $i_2(G) \leq i_2(H) + 2$.

Conversely let $S$ be a $i_2$-set of $G$. Then $|S \cap \{a, b, c\}| = 2$ and the 2-independent set $S' = S \setminus N(H)$ of $H$ has order $i_2(G) - 2$. If $S'$ is maximal in $H$ then $i_2(H) \leq |S'| = i_2(G) - 2$. Suppose now $S'$ not maximal.

Case 1: The second pendant path at $d$ is $def$ of length 2.

If $d \notin S'$, then necessarily $\{e, f\} \subset S'$ which implies that $S'$ is maximal. Hence $d \in S'$ and the only possibility for $S'$ to be not maximal is that $S \cap \{a, b, c\} = \{a, c\}$, $S \cap \{e, f\} = \{f\}$, and $d$ has a neighbor $d'$ not in $S'$ such that $N_{S'}(d') = \{d\}$. In this case, $(S' \setminus \{f\}) \cup \{e\}$ is a maximal 2-independent set of $G$ of order $|S'| = i_2(G) - 2$.

Case 2: The second pendant path at $d$ is $defg$ of length 3.

As for the path $abc$, $|S' \cap \{e, f, g\}| = 2$. If $d \notin S'$ and $S'$ is not maximal in $H$, necessarily $c \in S$ and $S \cap \{e, f, g\} = \{f, g\}$ or $\{e, g\}$. Then $(S' \setminus \{g\}) \cup \{e\}$ or $(S' \setminus \{g\}) \cup \{f\}$ is a maximal 2-independent set of $H$ of order $|S'| = i_2(G) - 2$. If $d \in S'$ and $S'$ is not maximal in $H$, then $d$ has a neighbor $d'$ not in $S'$ such that $N_{S'}(d') = \{d\}$. Necessarily $S \cap \{a, b, c\} = \{a, c\}$ and $S \cap \{e, f, g\} = \{f, g\}$. Then $(S' \setminus \{f\}) \cup \{e\}$ is a maximal 2-independent set of $H$ of order $|S'| = i_2(G) - 2$.

In both cases, $i_2(H) \leq i_2(G) - 2$ which completes the proof. □

Lemma 4. Let $abcdef$ be a pendant path of a graph $G$ with $d(a) = 1$ and let $H$ be the graph $G - \{a, b, c, d\}$. Then $i_2(G) = i_2(H) + 2$.

Proof. Let $S$ be a $i_2$-set of $H$. Then $S \cup \{b, c\}$ is a maximal 2-independent set of $G$. Hence $i_2(G) \leq i_2(H) + 2$. 

Conversely let $S$ be an $i_2$-set of $G$. The set $S$ has two or three vertices in $\{a, b, c, d\}$, and if $d \in S$ then $|S \cap \{a, b, c, d\}| = 3$. The set $S' = S \cap V(H)$ is a 2-independent set of $H$, not necessarily maximal.

If $S'$ is maximal in $H$, then $i_2(H) \leq |S'| \leq |S| - 2 = i_2(G) - 2$.

If $S'$ is not maximal, there are three possibilities. First, $e \in S'$, $f \notin S'$, $N(f) \cap S' = \{e\}$ and $d \in S$, in which case $S' \cup \{f\}$ is a maximal 2-independent set of $H$. Second, $e \notin S'$, $f \in S'$ and $N(f) \cap S' = \emptyset$, in which case $d \in S$ and $S' \cup \{e\}$ is a maximal 2-independent set of $H$. Third, $e \notin S'$, $f \notin S'$ and $f$ is adjacent to a vertex of $S'$ of degree 1 in $S'$, in which case $\{c, d\} \subset S$ and $S' \cup \{e\}$ is a maximal 2-independent set of $H$.

In all cases $|S'| = |S| - 3$ since $S$ contains $d$ and thus exactly three vertices of $\{a, b, c, d\}$. Therefore, $i_2(H) \leq |S'| + 1 = (|S| - 3) + 1 = i_2(G) - 2$ which completes the proof. \[ \qed \]

3. Upper bound on $i_k$

In this section we generalize to $k \geq 2$ the well-known inequality $i(G) \leq n - \Delta$. \[ \text{Theorem 5. For } 2 \leq k \leq \Delta, \text{ every graph } G \text{ of order } n \text{ and maximum degree } \Delta \text{ satisfies } i_k(G) \leq n - \Delta + k - 1. \text{ If moreover } G \text{ is connected and } \Delta < n - 1, \text{ then } i_2(G) \leq n - \Delta. \text{ The two bounds are sharp.} \]

\textbf{Proof.} Let $x$ be a vertex of degree $\Delta$ and $A$ a set of $k - 1$ neighbors of $x$. The set $A \cup \{x\}$ is $k$-independent. If $\Delta = n - 1$, then $A \cup \{x\}$ is a maximal $k$-independent set of $G$ of order $k$ and since $i_k \geq k$ for any $G$ and $k$, we have $i_k = k = n - \Delta + k - 1$ for all $k \geq 2$. If $\Delta < n - 1$, let $S$ be a maximal $k$-independent set of $G$ containing $A \cup \{x\}$. Since $d_S(x) \leq k - 1$, $S \cap (N(x) \setminus A) = \emptyset$. Hence $|S| + |N(x) \setminus A| \leq n$ implying $i_k \leq |S| \leq n - (\Delta - (k - 1)) = n - \Delta + k - 1$. For $k \geq 3$, generalized stars are examples of extremal trees. In the particular case $k = 2$, the set $A$ consists of one neighbor $a$ of $x$. If moreover $G$ is connected with $\Delta < n - 1$, then $a$ can be chosen with a neighbor $a'$ in $V \setminus \{x\}$. Then $S$ cannot contain $a'$ and $i_2 \leq |S| \leq n - |N(x) \setminus \{a\}| - 1 = n - \Delta$. Theorem 6 below shows that this bound is sharp too. \[ \qed \]

By Theorem 5, trees different from stars satisfy $i_2 \leq n - \Delta$. We characterize the extremal ones.

\[ \text{Theorem 6. A tree } T \text{ of order } n \geq 4 \text{ and maximum degree } \Delta < n - 1 \text{ satisfies } i_2(T) = n - \Delta \text{ if and only if } T \text{ is a generalized star } GS_{p_0,p_1,p_2} \text{ with } p_0 + p_1 \geq 1 \text{ and } p_1 + p_2 \geq 1 \text{ or } GS_{p_0,0,0,1} \text{ with } p_0 \geq 1, \text{ or is equal to } S_{p_2}^* \text{ with } p \geq 2. \]

\textbf{Proof.} We leave the reader check that when $p_0 + p_1 \geq 1$ and $p_1 + p_2 \geq 1$, then $i_2(GS_{p_0,p_1,p_2}) = p_1 + 2p_2 + 1 = n - \Delta$; that when $p_0 \geq 1$, $i_2(GS_{p_0,0,0,1}) = 4 = n - \Delta$; and that when $p_2 \geq 2$, $i_2(S_{p_2}^*) = 2 = n - \Delta$. Hence all the trees mentioned in the theorem satisfy $i_2 \leq n - \Delta$. Conversely consider the tree $T$ rooted at a vertex $x$ of maximum degree. For each neighbor $u$ of $x$ let $u(h)$ be the distance from $u$ to its farthest descendant. Let $y$ be a neighbor of $x$ with the highest degree among all such neighbors. If there is more than one, choose a $y$ such that $h(y)$ is minimum. Let $S$ be a maximal 2-independent set of $T$ containing $\{x, y\}$. Then $S \cap ((N(x) \cup N(y)) \setminus \{x, y\}) = \emptyset$ and thus $|S| + \Delta + d(y) - 2 < n$ with $d(y) \geq 2$ since $T$ is not a star. Therefore, $i_2(T) \leq |S| \leq n - \Delta - d(y) + 2$. If $i_2(T) = n - \Delta$, then $d(y) = 2$ and $V(T) \setminus (N(x) \cup N(y))$ is a 2-independent set of $T$. Thus $d(z) \leq 2$ for each neighbor $z$ of $x$ from the choice of $y$, and $I$ consists of paths $p_2$ and isolated vertices. Let $N(y) = \{x, y'\}$.

If some vertices of $N(x) \setminus \{y\}$ have degree 2, let $z$ be such a vertex with $h(z)$ maximum and let $N(z) = \{x, z'\}$. If $h(z) = 1$, that is $z'$ is an isolate of $I$, then $h(y) = 1$ and $h(t) = 0$ for all $t \in N(x) \setminus \{y, z\}$. The tree $T$ is a generalized star $GS_{p_0,p_1}$ with $p_1 \geq 2$ which is of type $GS_{p_0,p_1,p_2}$ with $p_0 + p_1 \geq 1$ and $p_2 = 0$. If $h(z) = 2$, that is if $z$ is adjacent to a path $z''$ of $I$, then $h(y) = 1$ or 2 and $h(t) = 0$ or 1 or 2 for all $t \in N(x) \setminus \{y, z\}$. Each of these vertices $t$ is a leaf, or is adjacent to an isolate $t'$ or to a path $t''$ of $I$. Finally, if $h(y) = 2$ then $y'$ is adjacent to isolates $w_1, w_2, \ldots, w_r$. Then $r \geq 2$. $(S \setminus \{w_1, \ldots, w_r, z', y'\}) \cup \{y', z\}$ is a maximal 2-independent set of $T$ smaller than $S$, a contradiction. Therefore, $r = 1$ and $T$ is a generalized star $GS_{p_0,p_1,p_2}$.

All the vertices of $N(x) \setminus \{y\}$ have degree 1, then either $y'$ is a leaf or $y'$ is adjacent to every component of $I$, say to $r$ isolates $w_1, w_2, \ldots, w_r$, and to the extremity $u_1$ of $q$ paths $u_1v_1, \ldots, u_qv_q$. In the first case, $T = GS_{p_0,1}$ which is of type $GS_{p_0,p_1,p_2}$ with $p_0 + p_1 \geq 1$. In the second case, $q \geq 1$ and $\Delta \geq q + 1 \geq 2$. Let $z \in N(x) \setminus \{y\}$ and $I' = (I \setminus \{w_2, \ldots, w_r, v_1, u_2, \ldots, u_q, y\}) \cup \{y', z\}$ if $q \neq 0$, $I' = (I \setminus \{w_2, \ldots, w_r, y\}) \cup \{y', z\}$ if $q = 0$. The set $I'$ is a maximal 2-independent set of $T$. This implies either $1 \leq r \leq 2$ and $q = 0$, or $r = 0$ and $q \geq 1$. Thus $T$ is a generalized star $GS_{p_0,0,1}$ (of type $GS_{p_0,p_1,p_2}$ with $p_0 + p_1 \geq 1$) or $GS_{p_0,0,0,1}$, or $S_{p_2}^*$ with $p = \Delta - 1 \geq 2$. \[ \qed \]
4. Lower bounds on $\beta_k$

The first bound, which generalizes for trees the trivial bound $\beta(T) \geq n/2$, was already obtained by Maddox in [7]. We give in Theorem 7 a proof which also determines the family of extremal trees.

**Definition.** The family $\mathcal{F}(k)$ is the set of trees which can be constructed recursively from $T_1 = K_{1,k}$ by using the operation $\mathcal{F}_1$ described below. Let $H$ be the star $K_{1,k}$.

**Operation $\mathcal{F}_1$:** Add a copy of $H$ attached by an edge from any vertex of $H$ to any vertex of $T_i$.

**Theorem 7.** Let $T$ be a tree of order $n$ and maximum degree $\Delta$. Then for every integer $k$ with $2 \leq k \leq \Delta$, $\beta_k(T) \geq kn/(k+1)$, with equality if and only if $T \in \mathcal{F}(k)$.

**Proof.** Let $Y(T)$ be the set of vertices of degree at least $k$ of a tree $T$. We proceed by induction on $|Y(T)|$. If $Y(T) = \emptyset$, then $\beta_k = n > kn/(k+1)$. If $|Y(T)| = 1$ then $\beta_k = n - 1 = (nk + (n-k-1))/(k+1) \geq kn/(k+1)$ since $k < \Delta < n - 1$. Moreover $\beta_k = kn/(k+1)$ if and only if $k = \Delta = n - 1$, that is $T = K_{1,k} \in \mathcal{F}(k)$. Suppose the property of the theorem true for all trees with $|Y(T)| < p$ with $p \geq 2$ and let $T$ be a tree of order $n$ such that $|Y(T)| = p$. Root $T$ at a leaf $r$. Let $u$ be a vertex of degree at least $k$ from $r$ and under this condition, of maximum degree. Since $|Y(T)| \geq 2$, $u$ is at distance at least 2 from $r$. Let $v$ and $t$ be the father and grandfather of $u$ in the rooted tree.

If $d_T(u) > k$, let $T'$ and $T''$ be the components of $T - uv$, respectively, containing $v$ and $u$. Then $d_{T''}(u) \geq k$ and from the first condition in the choice of $u$, all the vertices of $V(T'') \setminus \{u\}$ have degree less than $k$. Hence $u$ fulfills the conditions on the vertex $w$ in Lemma 2.

If $d_T(u) = k$, let $T'$ and $T''$ be the components of $T - vt$, respectively, containing $t$ and $v$. Then from the two conditions in the choice of $u$, $d_{T''}(u) = k$, the vertices of $N_{T''}(u) \setminus \{u\}$ have degree at most $k$ and all the vertices of $V(T'') \setminus N_{T''}[v]$ have degree less than $k$. Hence $u$ fulfills the conditions on the vertex $w$ in Lemma 2.

In both cases we can use Lemma 2 and apply the inductive hypothesis to $T'$ since $|Y(T')| < |Y(T)|$. Moreover, $|V(T'')| \geq k + 1$ and thus

$$\beta_k(T) = \beta_k(T') + |V(T'')| - 1 \geq k(n - |V(T'')|)/(k+1) + |V(T'')| - 1$$

$$\geq (kn + |V(T'')| - k - 1)/(k+1) \geq kn/(k+1).$$

Finally, $\beta_k(T) = kn/(k+1)$ if and only if we have equality throughout this inequality chain, that is if and only if $\beta_k(T') = kn/(k+1)$ and $|V(T'')| = k + 1$. So by the inductive hypothesis, $T' \in \mathcal{F}(k)$, and $T''$ is a star $K_{1,k}$ attached by its center $u$ in the first case, by a leaf $v$ in the second case. Since $T$ is obtained from $T'$ by using Operation $\mathcal{F}_1$, it follows that $T \in \mathcal{F}(k)$. Therefore the property is true for $T$, which completes the proof. $\Box$

Note that Theorem 7 still holds for $k = 1$. In this case $Y(T) = V$ and one can make a similar proof by induction on $n$.

To appreciate the interest of the bound $B_k = nk/(k+1)$ on $\beta_k$ in trees, we compare it to the best two lower bounds on $\beta_k$ which are known in general graphs. The first one, due to Hopkins and Staton [5], is $\text{HS}_k=n/(1 + \lfloor \Delta/k \rfloor)$. If $\Delta/2 < k \leq \Delta$ then $\text{HS}_k = n/2 \leq nk/(k+1)$. If $k \leq \Delta/2$, let $\Delta = kq + r$ with $0 \leq r < k$. Then $\text{HS}_k = nk/((\Delta + k - r) < nk/(k+1)$. In any case, $B_k \geq \text{HS}_k$. The second bound, due to Jelen, is the $k$-residue $R_k$ constructed from the degree sequence of the graph by a process of successive reductions (see [6] for the definition of $R_k$). The bounds $B_k$ and $R_k$ are not comparable. For instance for $q \geq 5$, the subdivided star $S_{q}^2$ of order $n = 2q + 1$ satisfies $R_3 = 5q/3 > 3(2q + 1)/4 = B_3$. On the other hand, let $t$ be any non-negative integer and let $T$ be the tree of order $n = 5(15t + 10)$ of $\mathcal{F}(4)$ consisting of a chain of $15t + 10$ stars $K_{1,4}$ as shown in Fig. 1 for $t = 0$. Then $R_4(T) = (3n + 1)/4 < 4n/5 = B_4(T)$ and $B_4(T) - B_4(T) = (n-5)/20$. This construction, which can be generalized to any value of $k$, shows that the bound $B_k$ can be arbitrarily larger than $R_k$.

![Fig. 1. The tree $T_i$ for $t = 0$.](image-url)
Theorem 8. If $G$ is a connected bipartite graph of order $n \geq 2$, with $\ell(G)$ leaves and $s(G)$ support vertices, then $\beta_k(G) \geq \beta(G) \geq (n + \ell(G) - s(G))/2$, and the bound is sharp even for $\beta_k$.

Proof. Let $G$ be a bipartite graph of order $n$. It is a routine matter to check the result if $\text{diam}(G) \in \{1, 2\}$. So assume that $\text{diam}(G) \geq 3$. Then the bipartite graph $G'$ obtained from $G$ by removing all its leaves is connected, has order at least two and admits a bipartition $A, B$. Let $A' = S(G) \cap A$ and $B' = S(G) \cap B$. We assume without loss of generality that $|A \setminus A'| \geq |B \setminus B'|$, and so $|A \setminus A'| \geq (n - \ell(G) - s(G))/2$. No vertex of $A \setminus A'$ is adjacent to any leaf of $L(G)$, and the set $L(G) \cup (A \setminus A')$ is independent. Thus $\beta_k(G) \geq \beta(G) \geq \ell(G) + |A \setminus A'| \geq (n + \ell(G) - s(G))/2$.

That this bound is sharp for any value of $k$ may be seen by the caterpillar formed by a path $P_q$ where each vertex of the path is adjacent to exactly $k$ leaves. Then $n = (k + 1)q$, $\ell(G) = kq$, $s(G) = q$ and $\beta_k(G) = kq = (n + \ell(G) - s(G))/2$. □

5. Bounds on $i_2$ and $\beta_2$

Theorem 9. Let $G$ be a connected bipartite graph of order $n \geq 2$ with $s(G)$ support vertices. Then $\beta_2(G) \geq (n + s(G))/2 \geq i_2(G)$, and these bounds are sharp.

Proof. The result can be easily checked if $\text{diam}(G) \in \{1, 2\}$. Thus assume that $\text{diam}(G) \geq 3$ and let $C$ be a set of leaves defined as follows: for each support vertex of $G$ we put in $C$ exactly one of its leaves. Clearly $|C| = s(G)$. Let $A$ and $B$ be the bipartition of the subgraph induced by the vertices of $V(G) - C$, with $|A| \leq |B|$. Since $\text{diam}(G) \geq 3$, $A \neq \emptyset$, $B \neq \emptyset$ and $|B| \geq (n - s(G))/2 \geq |A|$. Every leaf of $A$ is adjacent to a support vertex of $B$, every support vertex of $A$ is adjacent to a vertex of $B$ and a vertex of $C$, and every vertex of $A$ different from a leaf and a support vertex is dominated twice by $B$. Thus $B \cup C$ is a maximal $2$-independent set of $G$ and similarly, $A \cup C$ is a maximal $2$-independent set of $G$. Hence

$$i_2(G) \leq |A \cup C| \leq (n - s(G))/2 + s(G),$$

and

$$\beta_2(G) \geq |B \cup C| \geq (n - s(G))/2 + s(G).$$

That these bounds are sharp may be seen for trees by the following two theorems and for bipartite graphs different from trees by the graph $G_k$ ($k \geq 1$) obtained from a path $P_k$ and $k$ cycles $C_4$ by identifying a vertex of each cycle with a vertex of the path so that the resulting cycles are vertex disjoint. Then $n = 4k$, $s(G) = 0$ and $i_2(G) = \beta_2(G) = 2k$. □

Corollary 10. If $G$ is a connected bipartite graph of degree at least two, then $i_2(G) \leq n/2$.

We are interested in characterizing trees that attain the bound of Theorem 9 for each parameter.

Definition. The family $\mathcal{G}$ is the set of trees which can be constructed from $T_0 = P_2$ or $T'_0 = P_3$ by recursively performing Operations $\mathcal{O}_1, \mathcal{O}_2$ or $\mathcal{O}_3$ listed below.

- **Operation $\mathcal{O}_1$:** Add a pendant path $abc$ of length 2 attached by an edge $cd$ at a vertex $d$ of a graph already containing a pendant path $def$ of length 2.
- **Operation $\mathcal{O}_2$:** Add a pendant path $abc$ of length 2 attached by an edge $cd$ at a vertex $d$ of a graph already containing a pendant path $def$ of length 3.
- **Operation $\mathcal{O}_3$:** Add a pendant path $abcd$ of length 3 attached by an edge $de$ at a leaf $e$ of a graph such that $e$ is the only leaf of its support vertex.

Note that whatever the initial graph is, $T_0$ or $T'_0$, at the first step $T_1 = P_6$ obtained from $P_2$ by $\mathcal{O}_3$ or from $P_3$ by $\mathcal{O}_1$. 


Lemma 11. Every tree $T$ of $\mathcal{G}$ satisfies $i_2(T) = (n + s)/2$.

Proof. We make an induction on the number of operations $c_i$ performed to construct $T$. The property is true for $T_0 = P_2$ and $T_0' = P_3$. Suppose the property true for all trees of $\mathcal{G}$ constructed with $k - 1 \geq 0$ operations and let $T$ be a tree of $\mathcal{G}$ constructed with $k$ operations.

If the last operation, performed on a tree $T'$ obtained by $k - 1$ operations, is $c_1$ or $c_2$, then $n(T) = n(T') + 3$. If it is obtained by $c_1$, then $e$ is a support vertex in $T'$ and in $T$. If it is obtained by $c_2$, then $T'$ is not reduced to $defg$ since $P_4 \notin \mathcal{G}$, and $e$ is not a support vertex in $T'$ nor in $T$. In both cases $b$ is a new support vertex in $T$ and $s(T) = s(T') + 1$. Hence $(n(T) + s(T))/2 = (n(T') + s(T'))/2 + 2$. By the inductive hypothesis, $i_2(T') = (n(T') + s(T'))/2$ and by Lemma 3, $i_2(T) = i_2(T') + 2$. Therefore $i_2(T) = (n(T) + s(T))/2$.

If the last operation performed on $T'$ is $c_3$, then $n(T) = n(T') + 4$, $s(T) = s(T')$, and thus again $(n(T) + s(T))/2 = (n(T') + s(T'))/2 + 2$. Applying the inductive hypothesis to $T'$ and Lemma 4 gives as previously $i_2(T) = (n(T) + s(T))/2$. $\Box$

Theorem 12. A non-trivial tree $T$ satisfies $i_2(T) = (n(T) + s(T))/2$ if and only if it belongs to $\mathcal{G}$.

Proof. The part “if” is proved in Lemma 11. We prove the converse by induction on the order of $T$. The property is true for trees of order two or three. Suppose it is true for all trees of $\mathcal{G}$ constructed with $k - 1 \geq 0$ operations and let $T$ be a tree of $\mathcal{G}$ constructed with $k$ operations.

If $T$ is obtained by $c_1$ or $c_2$, then $n(T) = n(T') + 3$. If $T'$ is a star and $i_2(K_{1,p}) = (n + s)/2$ if and only if $p = 1$ or $2$, then when $T'$ is a subdivided star $S_{p,q}$ with $p > q = 1$ and $i_2(T') < (n + s)/2$. Hence we can suppose henceforth $diam(T') \geq 4$. With the notation of Theorem 9, $i_2(T') = (n + s)/2$ implies $A = (n - s)/2 = B$ and thus $A \cup C$ and $B \cup C$ are two $i_2(T)$ sets. For each leaf in $A$ or in $B$, its support vertex has another leaf in $C$. Hence there exists a longest path in $T$ whose endvertices are in $C$. Let $v_1$ be an endvertex in $C$ of such a path and $v$ its support vertex. By the symmetry between $A \cup C$ and $B \cup C$, we can suppose without loss of generality that $v$ is in $B$. The other leaves $v_2, v_3, \ldots, v_p$ of $v$, if any, are in $A$ and the vertex $v$ has exactly one non-leaf neighbor $u$.

Claim 1. $u$ is not a support vertex and $d(v) = 2$.

Proof of Claim 1. If $u$ is a support vertex, let $u_1$ be its leaf in $C$. The set $(A \cup C) \setminus \{u_1, v_1, v_2, \ldots, v_p\} \cup \{v\}$ is another maximal 2-independent set of $T$ and must be as large as $A \cup C$. Hence $u_1, v_2, \ldots, v_p$ do not exist. Therefore, $u$ is not a support vertex and $v_1$ is the only leaf of $v$. $\Box$

Claim 2. $d(u) = 2$ and the neighbor $w$ of $u$ different from $v$ is not a support vertex.

Proof of Claim 2. Since $diam(T) \geq 4$, $d(u) > 1$. From Claim 1, no neighbor of $u$ is a leaf. Moreover, since $v_1$ is the endvertex of a longest path, at most one such neighbor lies along a path longer than 1. Let $N(u) = \{v, w, w_1, \ldots, w_p\}$ where for $1 \leq i \leq p$, all the neighbors different from $u$ of $w_i$ are leaves among them one, say $w'_1$, is in $C$. Each vertex $w'_1$ is as $v_1$ the end of a longest path and from Claim 1, $d(w') = 2$ for $1 \leq i \leq p$. A subdivided star different from $P_3$ does not satisfy $i_2 = (n + s)/2$ since $i_2(S_{q,p}) = q + 1$ and $(n + s)/2 = (3q + 1)/2$. Hence $G$ is not a subdivided star and $w$ has at least one neighbor in $A \setminus \{u\}$, and possibly an attached leaf $w_1$ in $C$. In this case, $(B \cup C) \setminus \{w_1, w_1, w_2, \ldots, w_p, v\} \cup \{u\}$ is a maximal 2-independent set of $T$. Hence $\{w_1, w_1, w_2, \ldots, w_p\} = \emptyset$, implying that the only neighbor of $u$ different from $v$ is $w$ and that $w$ is not a support vertex. This completes the proof of Claim 2. $\Box$

Let $H$ be the tree $T - \{u, v, v_1\}$ considered as rooted at $w$. By the longest path argument, at most one edge incident to $w$ is the beginning of a path of $H$ longer than 3. Since $w$ is not a support vertex, the other edges of $H$ incident to $w$ belong to paths of length 2 or 3 from $w$.

Suppose $w$ has a neighbor $e$ whose all other neighbors $f_1, \ldots, f_p$ are leaves. Then $(B \cup C) \setminus \{f_1, \ldots, f_p\} \cup \{e\}$ is a maximal 2-independent set of $T$. Hence $p = 1$, $d(e) = 2$ and $wef_1$ is a pendant path of $H$. Moreover $s(T) = s(H) + 1$ and thus $(n(T) + s(T))/2 = (n(H) + s(H)))/2 + 2$. By Lemma 3, $i_2(H) = i_2(T) - 2 = (n(T) + s(T))/2 - 2 = (n(H) + s(H))/2$. By the inductive hypothesis the graph $H$ is in $\mathcal{G}$. Therefore $T$, which is obtained from $H$ by performing $c_1$, is also in $\mathcal{G}$.

Suppose $w$ has a neighbor $e$ such that every path of $H$ beginning by $we$ has length at most 3 and that at least one of these paths, say $wefg$, has length 3. Then $g$ is the endvertex of another longest path of $T$. By analogy with $v_1$,
Theorem 14. Let $d(e) = d(f) = 2$, and $wefg$ is a pendant path of $H$. Since $P_7$ does not satisfy $i_2 = (n + s)/2$, $T$ is not reduced to the path $v_1vwefg$ and $d(w) > 2$. This implies $s(T) = s(H) + 1$. As previously, by Lemma 3, $i_2(H) = i_2(T) = (n(T) + s(T))/2 - 2 = (n(H) + s(H))/2$. By the inductive hypothesis, the graph $H$ is in $\mathcal{G}$. Therefore, $T$, which is obtained from $H$ by performing $\mathcal{C}_2$, is also in $\mathcal{G}$.

So we can now suppose that $d(w) = 2$ and the diameter of $T$ is at least 7. Let $N(w) = \{u, t\}$. If $d(t) > 2$ or if $N(t) = \{w, x\}$ and $x$ is a support vertex, then $((B \cup C) \setminus \{w, v_1\}) \cup \{u\}$ is a maximal 2-independent set of $T$ smaller than $i_2(T)$, a contradiction. Hence $d(t) = 2$ and $x$ is not a support vertex in $T$. In the graph $H' = T - \{v_1, v, u, w\}$, $t$ is thus the only leaf attached at $x$. Hence $s(T) = s(H')$, $n(T) = n(H') + 4$ and $(n(T) + s(T))/2 = (n(H') + s(H'))/2 + 2$. By Lemma 4, $i_2(H') = i_2(T) - 2 = (n(T) + s(T))/2 - 2 = (n(H') + s(H'))/2$. By the inductive hypothesis the tree $H'$ is in $\mathcal{G}$. Therefore the tree $T$, obtained from $H'$ by performing $\mathcal{C}_2$, is also in $\mathcal{G}$. □

Now we turn our attention to characterize the trees achieving equality in the lower bound for the 2-independence number in Theorem 9.

Definition. The family $\mathcal{H}$ is the set of trees which can be constructed from a tree $T_1$ that consists in a path $P_3$ or $P_4$ by recursively performing operations $\mathcal{H}_1$ or $\mathcal{H}_2$. Let $H$ be a path $P_3$.

- Operation $\mathcal{H}_1$: Add a copy of $H$ attached by an edge between any vertex of $H$ and a vertex $r$ of $T_1$, with the condition that if $r$ is a leaf of $T_1$ then it must be adjacent to a strong support vertex.
- Operation $\mathcal{H}_2$: Add a path $P_4$ of support vertices $u, v$ attached by an edge $uz$ at a vertex $z$ of $T_1$, with the condition that if $z$ is a leaf of $T_1$ then $z$ is adjacent to a strong support vertex.

Lemma 13. If $T = P_2$ or $T \in \mathcal{H}$, then $\beta_2(T) = (n + s(T))/2$.

Proof. Clearly if $T = P_2$, $\beta_2(T) = (n + s(T))/2$. So let $T$ be any tree of $\mathcal{H}$. We proceed by induction on the number of operations $\mathcal{H}_i$ performed to construct $T$. The property is true for $T_1 = P_3$ or $P_4$. Suppose the property true for all trees of $\mathcal{H}$ constructed with $k - 1 \geq 0$ operations and let $T$ be a tree of $\mathcal{H}$ constructed with $k$ operations. Let us consider the following two cases depending on whether the last operation performed to obtain $T$ is $\mathcal{H}_1$ or $\mathcal{H}_2$.

If the last operation, performed on a tree $T'$ obtained by $k - 1$ operations, is $\mathcal{H}_1$, then $n(T) = n(T') + 3$ and $s(T) = s(T') + 1$. By Lemma 2 and the inductive hypothesis applied to $T'$,

$$\beta_2(T) = \beta_2(T') + 2 = (n(T') + s(T'))/2 + 2 = (n(T) + s(T))/2.$$

If the last operation, performed on a tree $T'$ obtained by $k - 1$ operations, is $\mathcal{H}_2$, then $n(T) = n(T') + 4$ and $s(T) = s(T') + 2$. By Lemma 2 and the inductive hypothesis applied to $T'$,

$$\beta_2(T) = \beta_2(T') + 3 = (n(T') + s(T'))/2 + 3 = (n(T) + s(T))/2.$$

Theorem 14. Let $T$ be a non-trivial tree. Then $\beta_2(T) = (n + s(T))/2$ if and only if $T = P_2$ or $T \in \mathcal{H}$.

Proof. The sufficient condition follows from Lemma 13. Conversely, if $n = 2$ then $T = P_2$. So assume that $n \geq 3$. We proceed by induction on the order $n$ of $T$. If $n = 3$, then $T = P_3$ and so $T \in \mathcal{H}$, establishing the base case. Assume that every tree $T'$ of order $n'$ with $3 \leq n' < n$, satisfying $\beta_2(T') = (n' + s(T'))/2$ is in $\mathcal{H}$. Let $T$ be a tree of order $n$ such that $\beta_2(T) = (n + s(T))/2$. The tree $T$ is not a star since $\beta_2(K_{1, p}) = p > (n + s)/2$. For a double star $S_{p, q}$ with $p \geq q$, $\beta_2(S_{p, 1}) = p + 2$ and $\beta_2(S_{p, q}) = p + q$ if $q \geq 2$. Hence if $\text{diam}(T) = 3$, then $T = P_4$ or $T = S_{2, 2}$ and thus $T$ belongs to $\mathcal{H}$. Suppose henceforth $\text{diam}(T) \geq 4$ and root $T$ at a vertex $r$. Let $v$ be a support vertex at maximum distance from $r$ and $u$ its parent. We distinguish between two cases:

Case 1: $|Lv| \geq 2$. Let $T' = T - T_v$. Then $n' = n - |Lv| - 1 \geq 3$ and $s(T') \geq s(T') \geq s(T) - 1$. Moreover, $s(T') = s(T)$ if and only if $u$ is the unique leaf of a support vertex of $T'$. Now by Lemma 2, $\beta_2(T) = \beta_2(T') + |Lv|$, and by Theorem 9 we have:

$$(n + s(T))/2 = \beta_2(T) = \beta_2(T') + |Lv| \geq (n' + s(T'))/2 + |Lv| \geq (n + s(T) + |Lv| - 2)/2 \geq (n + s(T))/2.$$
The equality between the extremal two members implies that \( \beta_2(T') = (n' + s(T'))/2 \), \(|L_v| = 2\) and \( s(T') = s(T) - 1 \). Thus \( u \) is either a leaf of a strong support vertex in \( T' \) or different from a leaf in \( T' \). Now by induction on \( T', T' \in \mathcal{H} \). Since \( T \) is obtained from \( T' \) by using operation \( \mathcal{H}_1, T \in \mathcal{H} \).

**Case 2:** \(|L_v| = 1\). Let \( T' = T - Tu \). Seeing the above case, we may assume that every descendant of \( u \) has degree at most two. For a subdivided star \( SS_q \) with \( q \geq 2 \), \( \beta_2(SS_q) = 2q > (n + s)/2 = (3q + 1)/2 \). Hence \( T \) is not a subdivided star and thus \( n(T') \geq 3 \). Suppose that \( u \) is adjacent to \( q \geq 0 \) leaves and has \( p \geq 1 \) children as support vertices. By Lemma 2 and Theorem 9 we have

\[
(n + s(T))/2 = \beta_2(T) = \beta_2(T') + 2p + q \geq (n' + s(T'))/2 + 2p + q.
\]

Now \( n' = n - 2p - q - 1 \) and by looking at the situations related to the value of \( q \) and the position of the parent \( w \) of \( u \) in \( T' \), one can check that \( s(T) - p \geq s'(T) \geq s(T) - p - 1 \) if \( q \geq 1 \), \( s(T) - p + 1 \geq s'(T) \geq s(T) - p \) if \( q = 0 \). Moreover, if we write \( s(T') \geq s(T) - p - i \) where \( i = 1 \) if \( q \geq 1 \), \( i = 0 \) if \( q = 0 \), then \( s(T') = s(T) - p - i \) if and only if \( w \) either is not a leaf of \( T' \) or is a leaf of a strong support vertex of \( T' \). Since \( p \geq 1 \) and \( q \geq 1 \) we get

\[
(n + s(T))/2 \geq (n + s(T) + p + q - 1 - i)/2 \geq (n + s(T))/2.
\]

The equality between the extremal two members implies that \( \beta_2(T') = (n' + s(T'))/2 \), and thus \( T' \in \mathcal{H} \) by the induction hypothesis, \( p + q - 1 - i = 0 \) and \( s(T') = s(T) - p - i \). It follows from \( p + q - 1 - i = 0 \) that \( p = 1 \) and \( q = i \), that is either \( p = q = 1 \) or \( p = 1 \) and \( q = 0 \). Moreover, \( w \) is either a vertex of degree at least two in \( T' \) or a leaf adjacent to a strong vertex. In both cases, \( T \) can be obtained from \( T' \) by using operation \( \mathcal{H}_2 \) if \( p = q = 1 \), or \( \mathcal{H}_1 \) if \( p = 1 \) and \( q = 0 \). Therefore \( T \in \mathcal{H} \) which completes the proof. \( \Box \)

**Remarks.** 1. Different bounds on \( i_2 \) and \( \beta_2 \) have been obtained in Sections 3 and 4 by letting \( k = 2 \) and in Theorem 9. That one of these bounds is better than the other one(s) depends on the structure of the considered graph or tree.

2. Going back to the definition of Families \( \mathcal{G} \) and \( \mathcal{H} \) in Section 5, we can observe that the trees of \( \mathcal{G} \cap \mathcal{H} \) different from \( P_2 \) are recursively obtained from a path \( P_3 \) by adding a pendant \( P_3 \) at any vertex already admitting a pendant path \( P_2 \). This shows that the non-trivial trees satisfying \( i_2 = \beta_2 \) are the 2-coronas of trees, which was already known by Favaron and Hartnell [2].

**References**