EMBEDDING RIEMANN SURFACES PROPERLY INTO $\mathbb{C}^2$

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ABSTRACT. For certain bordered submanifolds $M \subset \mathbb{C}^2$ we show that M can be embedded properly and holomorphically into $\mathbb{C}^2$. An application is that any subset of a torus with two boundary components can be embedded properly into $\mathbb{C}^2$.

1. Introduction, Main Results and Notation

In this paper we consider the problem of embedding bordered Riemann surfaces properly into $\mathbb{C}^2$. A bordered Riemann surface is obtained by taking a compact Riemann surface $\mathcal{R}$, and removing a finite set of disjoint closed connected components $D_1, ..., D_m$, i.e. the bordered surface is $\tilde{\mathcal{R}} = \mathcal{R} \setminus (\cup_{i=1}^m D_i)$. The case $\mathcal{R} = \mathbb{C} \cup \{\infty\}$ and m finite, was settled in [11]. The only case known thus far when the genus of $\mathcal{R}$ is greater than zero, is the case where $\tilde{\mathcal{R}}$ is a surface that is hyperelliptic (meaning that its double is hyperelliptic), due to Černe and Forstnerič [9]. An example is the torus take away one disc. This was proved by adjusting an embedding of $\tilde{\mathcal{R}}$ into the polydisc (see Gouma [5] and Rudin [6]), and composing with a certain Fatou-Bieberbach map (as constructed by Stensønes [7] and Globevnik [4]). This naturally suggests the following splitting of the general embedding problem into the following two problems:

(a) Find some conditions on a bordered submanifold M of $\mathbb{C}^2$ that enables you to embed it properly into $\mathbb{C}^2$.
(b) Embed $\tilde{\mathcal{R}}$ onto a surface $M \subset \mathbb{C}^2$ satisfying the conditions from (a).

In [9] the authors established a condition that M is a certain kind of closed submanifold of the polydisc as sufficient, and they also suggested polynomial convexity as a condition on M.

In proving Theorem 1 we establish another condition on the submanifold M (it follows from Proposition 3.1 in [9] that a surface satisfying the condition established in [9] can be perturbed to satisfy our condition), and in proving Theorem 2 we solve the problem (b) for subsets of the torus with two boundary components. Such a subset is not typically hyperelliptic, so the result does not follow from [9]. In proving Theorem 3, we show that by a linear change of coordinates, the condition is always satisfied for a bounded submanifold of $\mathbb{C}^2$ whose boundary is a smooth Jordan curve (such a surface is always polynomially convex). In particular, this implies the embedding theorem for any subset of a surface with one boundary component whose double is hyperelliptic (via the embedding provided by Rudin). Lastly we show that any bounded submanifold of $\mathbb{C}^2$ that is Runge can be exhausted by surfaces admitting proper embeddings.

Theorem 1. *Let $M \subset \mathbb{C}^2$ be a Riemann surface whose boundary components are smooth Jordan curves $\partial_1, ..., \partial_m$. Assume that there are points $p_i \in \partial_i$ such that*

$$\pi_1^{-1}(\pi_1(p_i)) \cap \overline{M} = p_i.$$

Assume that $\overline{M}$ is a smoothly embedded surface, and that all p_i are regular points of the projection π_1. Then M can be properly holomorphically embedded into $\mathbb{C}^2$.

Theorem 2. *Any subset of a torus with two boundary components can be properly holomorphically embedded into $\mathbb{C}^2$.*

We note that one or both boundary components may reduce to a point. The precise regularity of the boundary of D_i is not so important, as it is always possible to embed $\tilde{T}$ onto a subset of a torus such that the boundary consists of two real analytic components.

Theorem 3. *Let $M \subset \mathbb{C}^2$ be a bounded Riemann surface such that ∂M is a smooth Jordan curve. Then M can be properly holomorphically embedded into $\mathbb{C}^2$.*

In the following theorem we say that a surface $M \subset \mathbb{C}^2$ is Runge if for any compact set $K \subset\subset M$, we have that $\widehat{K} \cap \overline{M} \subset\subset M$, where $\widehat{K}$ denotes the polynomially convex hull of K.

Theorem 4. *Let $M \subset\subset \mathbb{C}^2$ be a Riemann surface whose boundary is a finite collection of smooth Jordan curves, and assume that M is Runge. Then M has an exhaustion M_i of Riemann surfaces that are diffeomorphic to M, and each M_i embeds properly holomorphically into $\mathbb{C}^2$.*

As usual we will let $\mathbb{B}$ and $\triangle$ denote the unit ball in $\mathbb{C}^2$ and the unit disk in $\mathbb{C}$ respectively, and we let B_R $(\triangle_R)$ denote the ball (disk) centered at the origin with radius R. We let π_i denote the projection on the ith coordinate axis, and we let $\|\cdot\|_K$ denote the euclidian sup-norm over a compact set K. We let $Aut_0(\mathbb{C}^2)$ denote the group of holomorphic automorphisms fixing the origin. Recall the definition of a basin of attraction: If $\{F_j\} \subset Aut_0(\mathbb{C}^2)$ is a countable sequence of automorphisms fixing the origin, we will let $F(j)$ denote the composition map $F_j \circ F_{j-1} \circ \cdots F_1$, and we define the basin of attraction:

$$\Omega^0_{\{F_j\}} = \{x \in \mathbb{C}^2; \lim_{j \to \infty} F(j)(x) = 0\}$$

A Fatou-Bieberbach domain is a proper sub-domain $\Omega \subset \mathbb{C}^2$, with a biholomorphic map $\psi : \Omega \to \mathbb{C}^2$ that is onto.

2. Fatou-Bieberbach domains and Proper Holomorphic Embeddings

Let us start with a surface $M \subset \mathbb{C}^2$ for which we want to construct a holomorphic map

$$\psi : M \to \mathbb{C}^2$$

that embeds M properly into $\mathbb{C}^2$. It is sometimes convenient to achieve this by constructing a Fatou-Bieberbach domain $\psi \colon \Omega \to \mathbb{C}^2$. It is clear that ψ embeds Ω properly into $\mathbb{C}^2$, so if $M \subset \Omega$ and if $\partial M \subset \partial \Omega$, then ψ will embed M properly. Let us generalize this situation.

Assume that we are in the following setting: We have two disjoint sets $V, M \subset \mathbb{C}^k$ and we want to construct a Fatou-Bieberbach domain Ω such that $M \subset \Omega \subset \mathbb{C}^k \setminus V$. This is of course not always possible, but we will give some conditions that are sufficient for the construction to be possible. The conditions will of course be such that we may apply them to prove Theorem 1.

Condition 1: Let $K \subset \mathbb{C}^k \setminus V$ be an arbitrary polynomially convex compact set. For any $R \in \mathbb{R}^+$ and any $\epsilon > 0$ there exists an automorphism $\phi \in \mathrm{Aut}(\mathbb{C}^k)$ such that the following are satisfied:

(i) $\|\phi(x) - x\| < \epsilon$ for all $x \in K$,

(ii) $\phi(V) \subset \mathbb{C}^k \setminus B_R$.

Condition 2: The set M can be written as an increasing union of compact sets $M = \cup_{i=1}^\infty K_i$ such that if $K \subset \mathbb{C}^k \setminus V$ is an arbitrary polynomially convex compact set, then

$$\widehat{K \cup K_j} \cap V = \emptyset$$

for all $j \in \mathbb{N}$.

We have the following theorem:

Theorem 5. *Let $M, V \subset \mathbb{C}^k$ be disjoint sets satisfying the two conditions above. Then there exists a Fatou-Bieberbach domain Ω such that*

$$M \subset \Omega \subset \mathbb{C}^k \setminus V$$

Proof. We may assume that $\overline{\mathbb{B}} \cap V = \emptyset$. Let $A \colon \mathbb{C}^k \to \mathbb{C}^k$ be the linear automorphism defined by

$$A(z_1, ..., z_k) = \left(\frac{z_1}{2}, ..., \frac{z_k}{2}\right).$$

By Theorem 4 in [10] there exists a $\delta > 0$ such that if $\{\sigma_j\}_{j=1}^\infty \subset \mathrm{Aut}_0(\mathbb{C}^k)$ is a sequence of automorphisms satisfying

$$(*) \quad \|\sigma_i(x) - A(x)\| < \delta \text{ for all } x \in \mathbb{B},$$

then the basin of attraction of the sequence σ_j is biholomorphic to $\mathbb{C}^k$.

We will prove the result by an inductive argument, and we let the following be our induction hypothesis I_j: We have a collection of automorphisms $\{F_i\}_{i=1}^j \subset \mathrm{Aut}_0(\mathbb{C}^k)$ with the following satisfied:

(a) Each F_i is a finite composition of automorphisms σ_k satisfying $(*)$,

(b) $F(j)(K_j) \subset \mathbb{B}$,

(c) $F(j)(V) \cap \overline{\mathbb{B}} = \emptyset$.

We may assume that I_1 is satisfied with $F_1 = A$. By Condition 2 it follows that for $K = \overline{\mathbb{B}} \cup F(j)(K_{j+1})$ we have that

$$\widehat{K} \cap F(j)(V) = \emptyset$$

Let $s \in \mathbb{N}$ such that $A^s(K) \subset \mathbb{B}$. For any $\epsilon > 0$, by Condition 1 there exists an automorphism $\varphi \in \mathrm{Aut}_0(\mathbb{C}^k)$ such that

(i) $\|\varphi(x) - x\| < \epsilon$ for all $x \in K$,

(ii) $\varphi(F(j)(V)) \cap A^{-s}(B_2) = \emptyset$.

If ϵ is chosen small enough we have that

$$\|A \circ \varphi(x) - A(x)\| < \delta \text{ for all } x \in \overline{\mathbb{B}},$$

so the composition map $F_{j+1} = A^s \circ \varphi$ clearly satisfies (a). It is easy to see that (b) and (c) are also satisfied, so F_{j+1} gives us I_{j+1}. It follows that $\Omega^0_{\{F_j\}}$ satisfies the claims of the theorem. $\qquad\square$

The reason for proving this theorem is that Condition 1 and Condition 2 can be proved to be satisfied for certain Riemann surfaces $M \subset \mathbb{C}^2$, and their boundaries $V = \partial M$. Establishing this for a set of surfaces containing (up to biholomorphism) all finitely connected subsets of $\mathbb{C}$ was indeed the content of the paper [11].

In what follows, the following lemma from [11] will be an essential ingredient in establishing Condition 1:

Lemma 1. *Let $K \subset \mathbb{C}^2$ be a polynomially convex compact set, let $\epsilon > 0$, and let $\Gamma = \{\gamma_j(t); j = 1, ..m, t \in [0, \infty)\}$ be a collection of disjoint smooth curves in $\mathbb{C}^2 \setminus K$ without self-intersection, such that $\lim_{t \to \infty} |\pi_1(\gamma_j(t))| = \infty$ for all j. Assume that there exists an $N \in \mathbb{R}$ such that $\mathbb{C} \setminus (\overline{\triangle}_R \cup \pi_1(\Gamma))$ does not contain any relatively compact components for $R \geq N$. Let $p \in K$. Then for any $R \in \mathbb{R}$ there exists an automorphism $\phi \in Aut(\mathbb{C}^2)$ such that the following is satisfied:*

(i) $\|\phi(x) - x\| < \epsilon$ for all $x \in K$,
(ii) $\phi(\Gamma) \subset \mathbb{C}^2 \setminus B_R$,
(iii) $\phi(p) = p$.

3. Proof of Theorem 1

Proposition 1. *Let M be a bordered submanifold of $\mathbb{C}^2$ whose boundary is a set of smooth curves $\partial_1, ..., \partial_m$ that are all unbounded. Then Condition 2 is satisfied for the pair M and $V = \partial M$.*

Proof. Let $\{K_j\}$ be an exhaustion of M by compact sets such that the boundary of each K_i is a finite collection of smooth Jordan curves, and such that each K_i is holomorphically convex relative to M. Choose an $m \in \mathbb{N}$ such that $K \cap M \subset K_i$ for all $i \geq m$. We claim that

$$\widehat{K \cup K_i} = \widehat{K \cup \partial K_i} = K \cup K_i$$

for all $i \geq m$. By [8] we have that $\widehat{K \cup \partial K_i} \setminus (K \cup \partial K_i)$ is a closed subvariety of $\mathbb{C}^2 \setminus (K \cup \partial K_i)$. Observe that this variety does not contain isolated points.

Assume that this set contains a variety X different from $K_i \setminus (K \cup \partial K_i)$, such that $\widehat{K \cup K_i} = K \cup K_i \cup X$. We will show that this implies $X = \emptyset$.

We claim first that $\overline{X} \cap (K_i \setminus K)$ is at most a discrete set of points in $K_i \setminus K$. Let $x_0 \in \partial K_i \cap \overline{X}$, and observe that $Y := \widehat{K_{i+1} \cup K}$ is a variety at x_0 (we may assume that $K_i \subset\subset K_{i+1}$), and that $X \subset Y$. Choose a small ball $B_\epsilon(x_0)$ such that $Y_{x_0} := Y \cap B_\epsilon(x_0)$ may be written

$$Y_{x_0} = S_0 \cup S_1 \cup \cdots \cup S_m;$$

a union of irreducible components such that Y_{x_0} is a submanifold of $B_\epsilon(x_0)$ except possibly at x_0. We arrange this so that $S_0 = B_\epsilon(x_0) \cap K_{i+1}$. We have that $X \cap S_0 = \emptyset$, for if not we would have $X \cap S_0 = S_0 \setminus K_i$, which would imply that

X is unbounded, by the hypothesis on M and the fact that K_i is holomorphically convex. This follows from Proposition 1, page 61 in [1]. Now we have that

$$\overline{X} \cap \partial K_i \cap B_\epsilon(x_0) \subset \overline{\cup_{j=1}^m S_j} \cap \partial K_i = \{x_0\},$$

and it follows that x_0 is an isolated point in the intersection. For a point $x_0 \in K_i \setminus K$ not in the boundary, repeat the same argument, this time using the assumption that $X \cap K_i = \emptyset$, and it follows that x_0 is isolated. We have thus shown that $\overline{X} \cap (K_i \setminus K)$ at most is a set $P := \{p_j\}_{j=1}^\infty$ which is discrete in $K_i \setminus K$. In other words; the only accumulation points for the set P in $\mathbb{C}^2$ has to be contained in K.

By the local maximum modulus principle we have that

$$\widehat{K \cup K_i} = (K \cup K_i) \cup (\overline{X} \cap \widehat{(K \cup K_i)}),$$

and we have just shown that

$$\overline{X} \cap (K \cup K_i) \subset K \cup P.$$

Now we claim that the set $K \cup P$ is polynomially convex. Since K is polynomially convex, it has a Runge and Stein neighborhood basis $\{\Omega_j\}$. For any j there is an $N \in \mathbb{N}$ such that $P_N := \cup_{i=N}^\infty \{p_i\} \subset \Omega_j$. Since $\widehat{K \cup P_N} \subset \Omega_j$, and since the union of a polynomially convex set and a finite set of points is polynomially convex, it follows that $\widehat{K \cup P} \subset \Omega_j \cup P$. Now let $j \to \infty$, and we have shown that

$$(\overline{X} \cap \widehat{(K \cup K_i)}) \subset \widehat{K \cup P} = K \cup P \Rightarrow \widehat{K \cup K_i} = K \cup K_i \ (\Rightarrow X = \emptyset).$$

$\square$

Proposition 2. *Let M be a surface as in Theorem 1. There exists an embedding*

$$\varphi \colon M \to \mathbb{C}^2$$

such that Conditions 1 and 2 are satisfied for the pair $\varphi(M)$ and $\partial(\varphi(M))$.

Proof. Let $c_1, ..., c_m$ denote the derivatives $\frac{\partial z_i}{\partial t}(0)$, and define

$$\varphi(z, w) = (z, w + \sum_{i=1}^m \frac{a_i}{z - z_i(0)}).$$

We will show that we may choose the coefficients a_i such that Condition 1 follows from Lemma 1 (we look at π_2 instead of π_1). Let Γ_i denote $\varphi(\partial_i \setminus p_i)$ close to p_i, and let Γ_i^+ and Γ_i^- denote the parts where t is positive and negative. As t approaches 0, we have that $\pi_2(\Gamma_i^-)$ stays close to the line

$$w(0) + \frac{a_i}{c_i t}.$$

We also have that the absolute value of $\pi_2(\Gamma_i^-)$ is strictly increasing when t is close to and approaches zero. The same conclusion holds for Γ_i^+ - and we observe that the projection of the curve will point in the opposite direction. The projections of Γ_i^+ and Γ_i^- on the second coordinate axis are thus strictly increasing in absolute value, and they do not intersect each other. By choosing a_1 through a_m appropriately we see that we may direct the projections such that none of the image curves intersect when t is close to zero, and they all increase in absolute value. That Condition 1 is satisfied now follows from Lemma 1. Condition 2 follows from Proposition 1 above, and the proposition is proved. $\square$

Proof of Theorem 1: The result is now an immediate consequence of Theorem 5, Proposition 2, and the discussion at the beginning of Section 2. $\square$

4. PROOFS OF THEOREMS 3 AND 4

To prove Theorem 3, we show that we may change coordinates such that the conditions in Theorem 1 are satisfied.

Lemma 2. *Let* $\mu_1, \mu_2\colon [0,1] \to \mathbb{C}$ *be* $\mathcal{C}^1$-*continuous curves, and assume that there is no* $t_0 \in [0,1]$ *such that* $\mu_1(t_0) = \mu_2(t_0) = 0$. *Then there exists a set* $E \subset \mathbb{C}$ *of measure zero such that for all* $c \in \mathbb{C} \setminus E$ *we have*

$$\mu_1(t) + c \cdot \mu_2(t) \neq 0$$

for all $t \in [0,1]$.

Proof. Let $\{\Omega_j\}$ be a decreasing sequence of open sets in $[0,1]$ such that

$$\cap_{j=1}^{\infty} \Omega_j = \{t \in [0,1]; \mu_2(t) = 0\}.$$

For each $i \in \mathbb{N}$ we let

$$E_i = \{-\frac{\mu_1(t)}{\mu_2(t)}; t \in [0,1] \setminus \Omega_i\}.$$

Then each E_i is a compact curve in $\mathbb{C}$ and has measure zero. It follows that the set

$$E = \cup_{i=1}^{\infty} E_i$$

has measure zero. Assume to get a contradiction that there is a $c \in \mathbb{C} \setminus E$ such that

$$\mu_1(t) + c \cdot \mu_2(t) = 0$$

for a $t \in [0,1]$. By assumption we have that $\mu_2(t) \neq 0$, so there is a Ω_i with $t \notin \Omega_i$. But then we have

$$\mu_1(t) + c \cdot \mu_2(t) = 0 \Rightarrow c = -\frac{\mu_1(t)}{\mu_2(t)} \Rightarrow c \in E_i \subset E.$$

$\square$

Lemma 3. *Let* M *be a surface as in Theorem 3, and let* $\gamma(t) = (z(t), w(t))$ *parametrize* ∂M. *There exist coordinates on* $\mathbb{C}^2$ *such that*

$$\frac{\partial z}{\partial t}(t) \neq 0, \frac{\partial w}{\partial t}(t) \neq 0$$

for all t. *Moreover, the new coordinates can be given by a linear map that is arbitrarily close to the identity.*

Proof. Define the following linear map $A\colon \mathbb{C}^2 \to \mathbb{C}^2$:

$$A(z, w) = (z + c_1 w, w + c_2 z).$$

Since $\frac{\partial z}{\partial t}(t)$ and $\frac{\partial w}{\partial t}(t)$ are never zero for the same t it follows from the previous lemma that the coefficients c_1 and c_2 can be chosen to prove the lemma. $\square$

Lemma 4. *Let* M *be a surface as in Theorem 3. There exist coordinates on* $\mathbb{C}^2$ *such that there is a point* $p \in \partial M$ *with*

$$\pi_1^{-1}(\pi_1(p)) \cap \overline{M} = p.$$

Also, if we let $z(t)$ *parametrize* $\pi_1(\partial M)$ *near* $\pi_1(p)$ *with* $z(0) = \pi_1(p)$, *we have* $\frac{\partial z}{\partial t}(0) \neq 0$. *Moreover, the new coordinates can be given by a linear map that is arbitrarily close to the identity.*

Proof. Choose coordinates as in the previous lemma. Since $\overline{M}$ is compact we have that the restriction of π_1 achieves its maximum absolute value at a point $p \in \partial M$. We may assume that $\pi_1(p) = 1$. Let P be the set

$$P = \{p_i\} = \{(1, w_i) \in \partial M\},$$

which is finite by assumption. We will show that an arbitrarily small $\mathbb{C}$-linear perturbation of M will yield the result.

First we may perturb the w-coordinate such that the set $\{Re(w_i)\}$ has a unique maximum, say $y_1 = Re(w_1)$. Let $\gamma(t) = (z(t), w(t))$ parametrize ∂M near p_1 with $\gamma(0) = p_1$. Make sure that $Re(w'(0)) \neq 0$ for the new w-coordinate. Notice that for small perturbations of the first coordinate we will still have that $z'(t) \neq 0$.

Let (x, y) denote the coordinates on the real space $\mathbb{R}^2 = \{(Re(z), Re(w)\}$ and let $\tilde{M}$ be the projection of M onto $\mathbb{R}^2$. Let $\tilde{\partial}$ denote the projection of ∂M and let $(x(t), y(t))$ be the parametrization of the projection near $(1, y_1)$. There exists an $N \in \mathbb{R}$ such that

$$\tilde{M} \subset \{(x, y); |x| < 1, |y| < N\}.$$

Since $y'(0) \neq 0$ and since $x'(t)$ is nonzero for $t \neq 0$ near the origin we have that $\tilde{\partial}$ near $(1, y_1)$ has tangent lines with arbitrarily small positive angles Θ to the line $l = \{1\} \times \mathbb{R}$. And since $\tilde{\partial}$ is compact and y_1 is the largest value for which $\tilde{\partial}$ intersects l, we may chose Θ so small that the tangent line does not intersect $\tilde{\partial}$ at other points than at a point $(\tilde{x}_1, \tilde{y}_1)$ close to $(1, y_1)$. Rotate both axes in $\mathbb{R}^2$ through the angle Θ and let the rotated vectors generate complex coordinates e_1 and e_2 on $\mathbb{C}^2$. The effect of the rotation is that in the new coordinates the function $Re(z)$ restricted to ∂ takes its unique maximum at a point $\tilde{p}_1$. So we may normalize to get $Re(\pi_1(\tilde{p}_1)) = 1$ and that $Re(\pi_1(q)) < 1$ for all other points $q \in \partial M$. Define a function $f \colon M \to \mathbb{C}$ by

$$f(z, w) = e^z.$$

It follows that $|f(q)| \leq e^1$ for all $q \in \partial M$, with $|f(q)| = e^1$ only for $q = \tilde{p}_1$. It follows then from the maximum principle that no other point of $\overline{M}$ projects onto $\pi_1(\tilde{p}_1)$, and the proof is finished. $\square$

Remark 1. The conclusion of the previous lemma is stable under small linear perturbations, and this will be used in the proof of Theorem 2. First we locate a point p for one boundary component, and then we perturb to get another point q for the other boundary component.

Proof of Theorem 3: The theorem follows from the previous lemma and Theorem 1. $\square$

Proof of Theorem 4: By scaling we may assume that $M \subset \triangle^2$. Let $\{K_i\}$ be an exhaustion of M by polynomially convex compact sets such that ∂K_i is a set of smooth Jordan curves and K_i° are all diffeomorhic to M. For a given K_i choose points p_j^i, one in each connected component of ∂K_i. By Theorem 2.3 in [3] there exists an automorphism $\phi \in \mathrm{Aut}(\mathbb{C}^2)$ such that the following are satisfied:

(i) $\phi(p_j^i) = q_j^i \in \partial \triangle^2$ for each p_j^i,

(ii) $|\pi_1(q_j^i)| = 1$ for each q_j^i, and $\pi_1(q_m^i) \neq \pi_1(q_j^i)$ for $m \neq j$,

(ii) $\phi(K_{i-1}) \subset \triangle^2$.

Let $\tilde{M}$ be the connected component of $\phi^{-1}(\phi(M) \cap \triangle^2)$ that contains K_{i-1}. It follows from the maximum principle that $\tilde{M}$ is homeomorphic to K_{i-1}°. We have

that $\tilde{M}$ contains a smoothly bounded domain U which is diffeomorphic to K_{i-1}°, such that $K_{i-1} \subset U$, and such that U satisfies the conditions of Theorem 1. $\square$

5. Bordered Riemann Surfaces - proof of Theorem 2

In this section we will apply the above results to embed subsets $\tilde{T}$ of the torus with two boundary components properly into $\mathbb{C}^2$. To do this, we will use the Weierstrass p-function to embed $\tilde{T}$ onto a submanifold of $\mathbb{C}^2$ satisfying the conditions in Theorem 1.

Let ω_1 and ω_2 be two non-zero complex numbers that are linearly independent over the real numbers, and let $X = \{\omega \in \mathbb{C}; \omega = n\omega_1 + m\omega_2, n, m \in \mathbb{Z}\}$. We obtain a torus T by introducing the following equivalence relation on $\mathbb{C}$:

$$z_1 \sim z_2 \Leftrightarrow z_1 - z_2 \in X,$$

and endowing the quotient space with the obvious complex structure. It is known that these are in fact all tori (see for instance [2]). Recall that we have the following meromorphic function defined on the tours (called the Weierstrass p-function):

$$\varrho(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

We have that ϱ is two-to-one (counted with multiplicity), and the ramification points are $p_1 \sim 0, p_2 \sim \frac{\omega_1}{2}, p_3 \sim \frac{\omega_2}{2}, p_4 \sim \frac{\omega_1 + \omega_2}{2}$. We claim that for most automorphisms of the torus $\phi \in \mathrm{Aut}(T)$, the functions ϱ and $\varrho \circ \phi$ separate points. For a complex number $c \in \mathbb{C}$, let ϕ be the automorphism $\phi(z) = z + c$, and let $z_0 \in \mathbb{C} \setminus X$. For any $N \in \mathbb{N}$ we let $\Omega_N = \{\omega \in X; |n|, |m| \leq N \in \mathbb{N}\}$, and we have that

$$\sum_{\omega \in \Omega_N} \frac{1}{(z_0 - \omega)^2} = \sum_{\omega \in \Omega_N} \frac{1}{(z_0 + \omega)^2} = \sum_{\omega \in \Omega_N} \frac{1}{(-z_0 - \omega)^2}.$$

By letting $N \to \infty$, we see that $\varrho(z_0) = \varrho(-z_0)$. We have that $z_0 \sim -z_0 \Leftrightarrow 2z_0 \in X$, which means that z_0 is a ramification point. Since ϱ is two-to-one, we then have that

$$\varrho(z_0) = \varrho(z_0') \Leftrightarrow z_0' \sim -z_0 \text{ or } z_0' \sim z_0.$$

Assume that we also have $\varrho(z_0 + c) = \varrho(z_0' + c)$. We get that

$$z_0' + c \sim -z_0 - c \Leftrightarrow z_0' + 2c \sim -z_0 \Leftrightarrow 2c \in X.$$

So let $p, q \in T$ such that $2(p - q) \in \mathbb{C} \setminus X$. If we let $f_p(z) = \varrho(z - p)$ and $g_q(z) = \varrho(z - q)$, we get that the function $\psi : T \setminus \{p, q\} \to \mathbb{C}^2$ defined by

$$\psi(z) = (f_p(z), g_q(z))$$

is one-to-one. Since f_p and g_q will not share the same ramification points, we have that ψ is an embedding into $\widehat{\mathbb{C}}^2$. We may sum this up in the following lemma:

Lemma 5. *Let $p, q \in T$ such that $2(p - q) \in \mathbb{C} \setminus X$. Then there exists a proper holomorphic embedding $\psi = (f_p, g_q) : T \setminus \{p, q\} \to \mathbb{C}^2$. Moreover, f and g are translations of the Weierstrass p-function.*

If we let $D \subset T$ be a disc, we may use this lemma and Theorem 3 to give a proof of the fact that $T \setminus D$ can be properly embedded into $\mathbb{C}^2$. Let $p, q \in D$ satisfy the conditions in the lemma, and Theorem 3 tells us that $\psi(T \setminus D)$ can be embedded properly. In a similar manner, we will now employ Lemma 5 to embed the surface in Theorem 2 onto a submanifold of $\mathbb{C}^2$ that satisfies the conditions of Theorem 1.

Proof of Theorem 2: We start with the case of two points. By Lemma 5, for a point $p \in T$ we have that most choices of $q \in T$ gives us that the map $\psi = (f_p, g_q) : T \setminus \{p, q\} \to \mathbb{C}^2$ is a proper holomorphic embedding. The only case not covered is the case where $2(p - q) \in X, p \neq q$. Let $x_0 = p + \frac{q-p}{2}$. Then $g_{x_0}(p) = g_{x_0}(q) = \alpha$, and $2(x_0 - p) \in \mathbb{C} \setminus X$. So the pair f_p, g_{x_0} separate points on T, which means that the pair $f_p, \frac{1}{g_{x_0} - \alpha}$ separate points. They do not share any ramification points, so they furnish the embedding we are looking for.

Now to the case of two discs. By choosing the point p close enough to ∂D_1, we may assume that

$$N = \|f_p\|_{\partial D_1} > \|f_p\|_{\partial D_2}.$$

Choose a point $x_0 \in \partial D_2$ such that $f_p(x_0) \neq 0$. There exists an $M \in \mathbb{N}$ such that $|g_{x_0}(z)| < M$ for all $z \in \partial D_1$, so for q close enough to x_0 we have that $|g_q(z)| < M$ for all $z \in \partial D_1$. If q is close enough to x_0 we have that $|g_q(x_0)| \cdot |f_p(x_0)| > M \cdot N$. We get that the function $f_p \cdot g_q$ takes its maximum on ∂D_2. Let $\tilde{T}$ denote the embedded bordered surface $\psi(T \setminus (D_1 \cup D_2))$.

Now, by an argument as in the proof of Lemma 4 we may assume that the conditions in Theorem 1 are satisfied at a point $p_1 \in \psi(\partial D_1)$, and by a change of coordinates given by a translation in the z-coordinate, we may assume that $\pi_1(p_1) = 0$. Let $F \colon \mathbb{C}^2 \to \mathbb{C}^2$ be the map defined by

$$F(z, w) = (ze^{h(zw)}, we^{-h(zw)}),$$

where $h \in \mathcal{O}(\mathbb{C})$. Then F is an automorphism of $\mathbb{C}^2$, and if $h(0) = 0$ it fixes the coordinate axes. Choose the function h such that $h(0) = h'(0) = 0$, and such that the function $f_p e^{h(f_p g_q)}$ restricted to $T \setminus (D_1 \cup D_2)$ takes its maximum on ∂D_2. Now the projection on the z-axis restricted to the surface $F(\tilde{T})$ will take its maximum at a point $p_2 \in F(\psi(\partial D_2))$, and again we may assume that the condition in Theorem 1 is satisfied at p_2. The theorem then follows.

Lastly we consider the case of D_1 being a disc, and D_2 being a point q. We will of course choose a point $p \in D_1$, and let $\psi(x) = (f_p(x), g_q(x))$ be our first embedding into $\mathbb{C}^2$. In this case however, we cannot change coordinates such that we get a point on the boundary where the projection on the first coordinate takes its maximum without having any other points in the fiber. The result will follow from Theorem 1 if we choose the point p such that f_p takes its maximum at a point $x_0 \in \partial D_1$, such that $f_p(x_0)$ has no other pre-image.

Let $z_0 \in \partial D_1$ such that ∂D_1 is convex near z_0, so that Imz has a local minimum at z_0. We will show that for small $\delta > 0$ we can let $p_\delta = z_0 + i\delta$, and f_{p_δ} will work.

Let $r_0 > 0$ such that we may write the Weierstrass p-function as

$$\varrho(z) = \frac{1}{z^2} + f(z),$$

for an $f \in \mathcal{O}(\overline{\triangle}_{r_0})$. Let $M = \|f\|_{\overline{\triangle}_{r_0}}$. We have that

$$|\varrho(z)| \leq \frac{1}{r^2} + M = \frac{1 + Mr^2}{r^2}$$

for all $|z| = r$ when $r < r_0$. And for any $k \in \mathbb{N}$ we have that

$$|\varrho(z)| \geq \frac{k^2}{r^2} - M = \frac{k^2 - r^2 M}{r^2}$$

for all $z \in \triangle_{r/k}$. So if k is chosen big enough, for all $r < r_0$ we have that

$$(*) \ z \in \triangle_{r/k} \Rightarrow |\varrho(z)| > \|\varrho\|_{T \setminus \triangle_r}.$$

With p_δ as above, we have that $\phi_\delta(z) = z - p_\delta$ is an automorphism of T such that $\gamma_\delta := \partial(\phi_\delta(D_1))$ has negative imaginary part near the point z_0 in the new coordinates. If γ_δ passes through $\triangle_{r/k}$, by $(*)$ we have that to study where f_{p_δ} takes its maximum on ∂D_1, we only have to look at where ϱ takes its maximum on $\gamma_\delta \cap \triangle_r$. Near the origin we may write $\varrho(z) = \frac{1}{z^2} \cdot g(z)$, where $g \in \mathcal{O}(\triangle_r)$, and $g(0) = 1$. So we may let $h^2 = g$ and write $\varrho(z) = (\frac{1}{z} \cdot h)^2$. It suffices to look at where $\frac{1}{z} \cdot h$ takes its maximum, or equivalently where $\varphi(z) = z \cdot \frac{1}{h} = z + O(|z|^2)$ takes its minimum. Define the set

$$U_r = \{z \in \triangle_r; \mathrm{Im} z \leq -\frac{r}{2k}\}.$$

There is a $C \in \mathbb{R}^+$ (independent of r) such that $|\varphi(z) - z| \leq C|z|^2$ for all $z \in \triangle_r$, so if r is chosen small enough we have that $\varphi(U_r)$ is contained in the lower half-plane.

To show how to choose δ, we will now express δ in terms of r, and we let $\delta(r) = \frac{r}{k}$. Near the origin, the curve $\gamma_{\delta(r)}$ can be expressed as a smooth graph over the x-axis:

$$y = s(x) - \delta(r),$$

where $s(x)$ is independent of r, and $s(0) = s'(0) = 0$. If r is small enough then $s'(x) < \frac{1}{2k}$ for all $x \in (-r, r)$, so we have that $s(x) < \frac{r}{2k}$ for all $x \in (-r, r)$. In other words: $\gamma_{\delta(r)} \cap \triangle_r \subset U_r$ if r is small enough. Now, φ restricted to $\gamma_{\delta(r)} \cap \triangle_r$ has to take its minimum at a point z_0, which means that $\frac{1}{\varphi}$ restricted to $\gamma_{\delta(r)} \cap \triangle_r$ takes its maximum at z_0. We have that $\varrho = (\frac{1}{\varphi})^2$ (near the origin), and the square function is 1-1 on U_r, so it follows that ϱ (restricted to the boundary) takes its maximum at z_0 and that no other point on the boundary gets mapped to $\varrho(z_0)$. Since the only ramification point for ϱ near the origin is the origin, the proof is finished. $\square$

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