Optimal Dividend Payment Problems in Piecewise-Deterministic Compound Poisson Risk Models

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Abstract—This work deals with an optimal dividend payment problem for a piecewise-deterministic compound Poisson insurance risk model. The objective is to maximize the expected discounted dividend payout up to the time of ruin. When the dividend payment rate is restricted, the value function is shown to be a solution of the corresponding Hamilton-Jacobi-Bellman equation, which in turn leads to a tractable methodology to find an optimal threshold dividend payment policy. For the case of unrestricted payment rate, the value function and an optimal barrier strategy are determined explicitly with exponential claim size distributions. A comparison of two examples is provided to illustrate the main results.

Key Words. Piecewise-deterministic compound Poisson model, Hamilton-Jacobi-Bellman equation, quasi-variational inequality, threshold strategy, barrier strategy.

AMS subject classifications. 93E20, 60J75

I. INTRODUCTION

Since the seminal work [de Finetti, 1957], the optimization of dividend payments in insurance risk models has attracted growing attention. The optimality is often considered to be a strategy which maximizes the expected present value of dividends received by the shareholders. [Jeanblanc-Picqué and Shiryaev, 1995] and [Asmussen and Taksar, 1997] investigated in diffusion models the dividend problems where the dividends are permitted to be paid out up to a maximal constant rate or a ceiling. (We shall refer to such a type of dividend problem as restricted payment scheme.) Both papers demonstrated that the dividends should be paid out at the maximal admissible rate as soon as the surplus exceeds a certain threshold. Interestingly, it turns out that such a threshold strategy is the optimal restricted payment scheme in a variety of other risk models. For example, [Gerber and Shiul, 2006] discussed the threshold strategy in the compound Poisson risk model and solved the problem explicitly when the claim size is exponentially distributed. [Fang and Wu, 2007] studied a similar problem in the compound Poisson risk model with constant interest and showed that the threshold strategy is optimal when the claim sizes are exponentially distributed.

On the other hand, thanks to by the fact that dividends are not usually paid out continuously in time in practice, there has also been a significant amount of literature on unrestricted payment problems, where there is no such restriction of maximal rate imposed on dividend payments. For instance, insurance companies may distribute dividends on discrete time points, in theory allowing for unbounded payment rates. In such a scenario, the surplus level changes drastically on a dividend payday. In other words, abrupt or discontinuous changes occur due to “singular” dividend distribution policy. This gives rise to a singular stochastic control problem. Such problems are studied in [Choulli et al., 2003], [Paulsen, 2007], [Paulsen, 2008], [Paulsen and Gjessing, 1997], and the references therein when the surplus dynamics is modeled by a controlled diffusion. But to the best of our knowledge, related work in the setting of piecewise-deterministic compound Poisson risk model with interest is relatively scarce. One exception is [Schmidli, 2008, Section 2.4], which formulates and solves an optimal unrestricted payment problem when the surplus process follows a classical Crámer-Lundberg risk model. See also [Albrecher and Thonhauser, 2008] for a viscosity solution characterization of the value function for optimal dividend payments problems in the setup of a compound Poisson risk model with interests.

Since the classical Crámer-Lundberg risk model and the compound Poisson risk model with interest are all special cases of piecewise-deterministic compound Poisson (PDCP) risk model, one naturally asks whether there exist unifying optimal solutions to both dividend payment schemes in piecewise-deterministic compound Poisson risk models. If so, can we confirm in general that the threshold strategy is the optimal restricted dividend policy whereas the barrier strategy is the optimal unrestricted dividend policy? We provide affirmative solutions to both questions in this paper under certain conditions. While the results are presented here, the proofs are referred to our recent work [Feng et al., 2012].

The contributions and novelty of this work can be summarized as follows. First, we formulate and solve the problem within the framework of stochastic control theory in the specific setting of piece-wise deterministic compound Poisson risk model. Roughly, the idea is to pay out the dividend at a dynamic rate in such a way that a certain reward function is optimized. Compared with the aforementioned work in the setup of controlled diffusions, the associated Hamilton-Jacobi-Bellman (HJB) equation in our work contains a nonlocal term (the integral with respect to the claim size distribution term), resulting in a nonlinear integral-differential
equation and hence substantial difficulty and technicality in the analysis. Nevertheless, we use renewal type arguments to overcome this difficulty and establish the HJB equation. Furthermore, if the claim sizes are exponentially distributed, we obtain general sufficient conditions for the optimality of the threshold and barrier strategies in an arbitrary piecewise-deterministic compound Poisson model. Both restricted and unrestricted payment schemes are presented and directly compared in this paper. Finally, it is worth mentioning that the solution methods presented in this paper can be more efficient alternatives of the approaches used in the existing literature.

The rest of the paper is organized as follows. The optimality of dividend strategies is formulated as a stochastic control problem in Section 2. We consider in Section 3 the restricted dividend payment schemes. We derive some properties of the value function and show that the value function is a classical solution to the HJB equation (3.4). As a result, we establish the optimality of the threshold strategy and propose a realistic way to identify the threshold. In Section 4, we formulate the optimal unrestricted payment scheme problem as a singular stochastic control problem. A verification theorem is established. Furthermore, under some fairly general conditions, we provide a tractable procedure to obtain an optimal dividend barrier and the corresponding value function. When the claims are exponentially distributed, we obtain explicit solutions for both the restricted and unrestricted dividend payment schemes in Sections 3 and 4, respectively. Finally, the paper is concluded with several remarks in Section 5.

2. THE MATHEMATICAL MODEL AND PROBLEM FORMULATION

We assume that in the absence of dividends, the surplus level is modeled by a piecewise-deterministic compound Poisson process.

Definition 2.1: A piecewise-deterministic compound Poisson (PDCP) process is a real-valued stochastic process \( X = \{ X(t), 0 \leq t < \infty \} \) defined on a given probability space \( \{ \Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P} \} \) satisfying the following properties:

(i) \( X(0) = x \geq 0 \).
(ii) Let \( 0 = T_0 < T_1 < T_2 < \ldots \) denote a sequence of jump points of the process \( X \). Assume that \( T_{i+1} - T_i \) has exponential distribution with mean \( 1/\lambda > 0 \) for every \( i = 0, 1, \ldots \). Then the adapted counting process defined by \( N(t) = \sum_{i=1}^{\infty} I\{T_i \leq t\} \) follows a homogeneous Poisson process with intensity rate \( \lambda \).
(iii) The jump sizes \( Y_k = \Delta X(T_k) = X(T_k) - X(T_k^-) \) for \( k = 1, 2, \ldots \) are independent and identically distributed nonnegative random variables with common distribution function \( \mathbb{P}\{Y_i \leq y\} = \mathbb{P}\{Y_1 \leq y\} \), \( 0 \leq y < \infty \).
(iv) The process between any two consecutive jumps is deterministic and given by \( X_t = \phi_{X(T_k)}(t), \ t \in [T_k, T_{k+1}), \ k = 0, 1, 2, \ldots \), where \( \phi_z(t) \) is determined by
\[
\frac{d\phi_z(t)}{dt} = g(\phi_z(t)) dt, \ t > 0,
\]
satisfying \( \phi_z(0) = z \) and \( \lim_{t \to \infty} \phi_z(t) = L \in [-\infty, \infty] \). The function \( g : B \to (0, \infty) \) satisfies the linear growth condition and is Lipschitz continuous on its domain \( B \).

By virtue of [Davis, 1993], the generator of the PDCP is defined as
\[
\mathcal{A}h(x) = g(x)h'(x) - \lambda h(x) + \lambda \int_0^\infty h(x-y) dQ(y), \ x \in B,
\]
where \( h \) is continuously differentiable.

As pointed out in [Cai et al., 2009], the class of PDCP processes includes many interesting risk models which appeared in the literature such as the compound Poisson risk models with interest, absolute ruin, dividend, and their respective dual models. The corresponding expressions for \( g(x) \) and \( \phi_x(t) \) for these specific models are as follows.

- In the classical compound Poisson model, the deterministic piecewise between any two consecutive claims is given by \( g(x) = c, \ x \geq 0 \). Hence, \( \phi_x(t) = x + ct, \ x \geq 0, \) and \( L = \infty \).
- In the modification of the classical compound Poisson model where all positive surplus earns interest at rate \( \rho > 0 \), \( g(x) = \rho x + c, \ x \geq 0 \). Hence, \( \phi_x(t) = (x + ct)/\rho e^\rho t - c/\rho, \ x \geq 0 \).

We now enrich the model by considering dividend payout. We denote by \( D(t) \) the aggregate dividends by time \( t \). We assume that \( D = \{D(t), t \geq 0\} \) is càdlàg (right continuous with left limits), nondecreasing, and \( \mathcal{F}_t \)-adapted with \( D(0-) = 0 \). Moreover, we require that at any time \( t \), the dividend payment should not exceed the current surplus level, i.e.,
\[
\Delta D(t) := D(t) - D(t-) \leq X(t-).
\]

Any dividend payment scheme \( D = \{D(t), t \geq 0\} \) satisfying the above conditions is called an admissible control and the collection of all admissible controls is denoted by \( \Pi \). The dynamics of the controlled surplus process under the admissible control \( D \) is
\[
X^D(t) = x + \int_0^t g(X^D(s)) ds - \sum_{i=1}^{N(t)} Y_i - D(t), \ t \geq 0.
\]

The time of ruin is denoted by
\[
\tau = \tau(x, D) := \inf \{ t \geq 0 : X^D(t) < 0 \},
\]
where \( x \geq 0 \) is the initial surplus. Throughout the paper, we use the conventions \( \sup \emptyset = -\infty \) and \( \inf \emptyset = +\infty \).

The performance functional or the expected present value (EPV) of dividends up to ruin is defined as
\[
J(x, D) = \mathbb{E}_x \int_0^\tau e^{-\delta t} dD(t),
\]
where \( \delta > 0 \) is the force of interest. The objective is to find an admissible control \( D^* \in \Pi \) that maximizes the
performance functional. That is
\[ V(x) := \sup_{D \in \Pi} \{ J(x, D) \} = J(x, D^*). \] (2.4)
Note that \( V(x) = 0 \) for all \( x < 0 \). Also, depending on the parameters of the model, \( V \) can be \( \infty \). In the rest of the paper, we will assume that \( V(x) < \infty \) for all \( x \geq 0 \). See Section 4 for a sufficient condition for the finiteness of the value function.

3. Restricted Payment Scheme

We first consider problem (2.4) for the case when the dividend payment scheme \( D = D_R \) is absolutely continuous with respect to time. That is, there exists some \( u = \{u(t), t \geq 0\} \) such that \( D_R(t) = \int_0^t u(s) \, ds \). Moreover, we assume that \( u \) is \( F_t \)-adapted and that there exists some positive constant \( u_0 < \inf \{g(x), x \geq 0\} \) such that \( 0 \leq u(t) \leq u_0, \quad \forall t \geq 0 \). Denote the collection of all such dividend payment schemes by \( \Pi_R \). The EPV corresponding to the initial surplus \( x \geq 0 \) under the dividend payment policy \( D_R = \{D_R(t), t \geq 0\} \) is given by
\[ J(x, D_R) = \mathbb{E}_x \int_0^\infty e^{-\delta t} \, dD_R(t) = \mathbb{E}_x \int_0^\infty e^{-\delta t} u(t) \, dt. \] (3.1)
The goal is to find an admissible policy \( D^*_R \in \Pi_R \) such that
\[ V^*_R(x) := \sup_{D_R \in \Pi_R} J(x, D_R) = J(x, D^*_R). \] (3.2)

Apparently, we have \( V^*_R(x) \leq V(x) \) for all \( x \geq 0 \), where \( V(x) \) is the value function defined in (2.4).

A. Properties of the Optimal Value Function and the HJB Equation

We first derive some elementary properties of the value function (3.2), which will help us establish the HJB equation in Theorem 3.2.

**Lemma 3.1:** The function \( V_R(x) \) is bounded by \( u_0/\delta \), increasing, and Lipschitz continuous on \( [0, \infty) \), and therefore absolutely continuous, and converges to \( u_0/\delta \) as \( x \to \infty \).

With Lemma 3.1 at our hands, together with the following dynamic programming principle ([Fleming and Soner, 2006]),
\[ V_R(x) = \sup_{D_R \in \Pi_R} \mathbb{E}_x \left[ \int_0^{\tau \wedge \theta} e^{-\delta s} u(s) \, ds + e^{-\delta (\theta \wedge \tau)} V_R(X(\theta \wedge \tau)) \right], \] (3.3)
where \( x \geq 0 \) and \( \theta \) is an \( F_t \)-stopping time, we can show that

**Theorem 3.2:** The function \( V_R(x) \) is differentiable and fulfills the HJB equation
\[ \sup_{0 \leq s \leq s_0} \left\{ [g(x) - u] V_R(x) - (\lambda + \delta) V_R(x) + u + \lambda \int_y^x V_R(x-y) \, dQ(y) \right\} = 0, \quad x \geq 0. \] (3.4)

Note that the HJB equation (3.4) is linear in \( u \). The maximum value of the expression on the left hand side of (3.4) is achieved when \( u = 0 \) or \( u = 1 \), corresponding to whether \( V'_R(x) > 1 \) or \( V'_R(x) < 1 \), respectively. If \( V'_R(x) = 1 \), then \( u \) can be any value in \( [0, 1] \). In view of this observation, it follows that the strategy \( D^*_R = \{ \int_0^s u_R(s) \, ds, s \geq 0 \} \) given by
\[ u^*_R(t) = \begin{cases} 0, & \text{if } V'_R(X^*_R(t)) > 1, \\ u_0, & \text{if } V'_R(X^*_R(t)) \leq 1, \end{cases} \] (3.5)
is optimal, where \( X^*_R(t) \) is the corresponding surplus process under the strategy (3.5). Moreover, one can directly verify that \( J(x, D^*_R) = V_R(x) \).

B. Exponential Claims

In order to obtain an explicit solution to the HJB equation (3.4) and an optimal dividend payment policy, we assume that the claims \( Y_1, Y_2, \ldots \) are independently and exponentially distributed with mean \( 1/\alpha \), where \( \alpha \) is some positive constant.

Note that under the assumption that \( g(x) > 0 \) for \( x \geq 0 \), the integro-differential equation
\[ g(x) h''(x) - (\lambda + \delta) h(x) + \lambda \int_0^x h(x-y) \, dQ(y) = 0, \] (3.6)
has a strictly increasing solution. In fact, consider the solution \( \phi(x) \) with \( \phi(0) = 1 \). Then \( \phi'(0) > 0 \) so the solution is increasing for small \( x > 0 \). Let \( x_0 := \inf \{x > 0 : \phi'(x) = 0\} \). Then clearly \( x_0 > 0 \). Suppose that \( x_0 < \infty \). Then \( \phi(x) \) is increasing for \( 0 \leq x \leq x_0 \) and so \( \int_0^{x_0} \phi(x_0 - y) \, dQ(y) \leq \phi(x_0) \). Therefore it follows from (3.6) that,
\[ 0 = (\lambda + \delta) \phi(x_0) - \lambda \int_0^{x_0} \phi(x_0 - y) \, dQ(y) \geq (\lambda + \delta - \lambda) \phi(x_0) = \delta \phi(x_0) > 0. \]
This is a contradiction. Therefore \( x_0 = \infty \) and the solution \( \phi(x) \) is strictly increasing for all \( x \geq 0 \).

We make the following assumption.

**Hypothesis A** The differential equation
\[ \phi''(x) + [\alpha g(x) - \alpha u_0 + \phi'(x) - (\lambda + \delta)] \phi'(x) - \alpha \delta \phi(x) = 0, \quad x > 0, \] (3.7)
has a bounded concave solution \( \psi_2(x) \).

**Theorem 3.3:** Let \( \psi_1 \) be a strictly increasing solution to (3.6). Assume hypothesis A and that there exists a unique number \( d > 0 \) such that \( \psi_1 \) is concave on \( (0, d) \), \( \psi_2(d) > 0 \), with
\[ \frac{\psi_1(d)}{\psi'_2(d)} > \frac{u_0}{\delta}. \] (3.8)
Then the value function \( V_R(x) \) is given by
\[ V_R(x) = \begin{cases} \frac{\psi_1(x)}{\psi'_2(x)}, & \text{if } 0 \leq x < d, \\ \frac{u_0}{\delta} + \frac{\psi_2(x)}{\psi'_2(d)}, & \text{if } x \geq d. \end{cases} \] (3.9)

Moreover, the optimal dividend payment policy is the threshold strategy
\[ u^*(t) = \begin{cases} 0, & \text{if } 0 \leq X^*(t) < d, \\ u_0, & \text{if } X^*(t) \geq d, \end{cases} \] (3.10)
where \( X^* \) is the corresponding controlled surplus process.
The interpretation of such an optimal strategy is as follows. First, a threshold \( d \) is determined so that dividends are paid according to whether or not the surplus exceeds the threshold. Whenever the threshold is attained, the dividends are paid out continuously at the maximal rate \( u_0 \) per time unit. Otherwise, no dividend is paid at all.

**Remark 3.4:** Assume that hypothesis A holds. If \( \psi_0'(0) > 0 \) and
\[
[g(0) - u_0]\psi_0'(0) - (\lambda + \delta)\psi_0(0) + u_0 = 0,
\]
then one can show that \( \psi_2(x)/\psi_0'(0) + u_0/\delta \) solves the HJB equation (3.4). Moreover, thanks to Hypothesis A, we must have \( \psi_2(x)/\psi_0'(0) < 1 \). Thus the value function is given by
\[
V(x) < 1 \quad \text{for any } x \geq 0.
\]
and the optimal restricted dividend payment scheme is to pay dividends continuously at the maximal rate \( u_0 \) per time unit until ruin.

**Example 3.5:** To illustrate our results, let us consider the special case when \( g(x) \equiv c > 0 \) and the claim size distribution \( Q(y) = 1 - e^{-\alpha y}, \ y \geq 0 \). As argued in [Gerber and Shiu, 1998], the unique (up to a constant multiple) strictly increasing solution to (3.6) (with \( g(x) \equiv c \)) is \( \psi_1(x) = (r + \alpha)e^{\gamma x} - (s + \alpha)e^{\gamma x} \), where \( -\alpha < s < 0 < r \) are the roots of
\[
e^{\xi^2} - (\lambda + \delta - \alpha c)\xi - \alpha \delta = 0.
\]
Similarly, with \( g(x) \equiv c \), the differential equation (3.7) has a unique (up to a constant multiple) bounded concave solution \( \psi_2(x) = -e^{\gamma x} \), where \( t \) is the negative root of
\[
(c - u_0)e^{\gamma x} - (\lambda + \delta - \alpha c + \alpha u_0)\xi - \alpha \delta = 0.
\]
By virtue of condition (3.8), we obtain
\[
\frac{(r + \alpha)e^{\gamma x} - (s + \alpha)e^{\gamma x}}{r(r + \alpha)e^{\gamma x} - s(s + \alpha)e^{\gamma x}} = \frac{1}{t} + \frac{u_0}{\delta}.
\]
Solve the above equation for \( d \)
\[
d = \frac{1}{r - s} \ln \left[ \frac{(s + \alpha)(\delta t - \delta s - su_0)}{(r + \alpha)(\delta t - \delta r - ru_0)} \right] = \frac{1}{r - s} \ln \left[ \frac{s(s - t)}{r(r - t)} \right],
\]
which agrees with (9.15) of [Gerber and Shiu, 2006]. However, our approach is considerably simpler than their method of optimizations.

Assume that \( d > 0 \), or equivalently,
\[
(s + \alpha)(\delta t - \delta s - su_0) > 0.
\]
We claim that \( \psi_1 \) is concave on the interval \((0, d)\). In fact, it is straightforward to verify that the function \( \psi_1'(x) = (r + \alpha)e^{\gamma x} - (s + \alpha)e^{\gamma x} \) is decreasing on \((-\infty, b)\) and increasing on \([b, \infty)\), where
\[
b = \frac{1}{r - s} \ln \left( \frac{s^2(s + \alpha)}{r^2(r + \alpha)} \right) = \frac{1}{r - s} \ln \left( \frac{s(\lambda + \delta)s + \alpha \delta}{r(\lambda + \delta)r + \alpha \delta} \right).
\]
Therefore the desired concavity will follow if we can show that \( d \leq b \). Recall that \(-\alpha < s < 0 < r \). Thus a comparison between (3.13) and (3.15) reveals that it suffices to prove
\[
\frac{s(s - t)}{r(r - t)} - \frac{s(\lambda + \delta)s + \alpha \delta}{r(\lambda + \delta)r + \alpha \delta} < 0,
\]
which follows by straightforward calculations and the fact that \( t \) is the negative root of (3.12). Thus it follows that \( \psi_1(x) \) is indeed concave on \((0, d)\).

Therefore, according to Theorem 3.3, if \( d > 0 \), then the value function is
\[
V(x) = \frac{(r + \alpha)e^{\gamma x} - (s + \alpha)e^{\gamma x}}{r(r + \alpha)e^{\gamma x} - s(s + \alpha)e^{\gamma x}}, \quad \text{if } 0 \leq x < d;
\]
\[
\frac{u_0}{\delta} + \frac{1}{\delta} x(e^{\gamma x} - d), \quad \text{if } x \geq d,
\]
and the optimal restricted dividend payment scheme is the threshold strategy given in (3.10).

4. **Unrestricted Payment Scheme**

In Section 3, we considered the case when the dividend payment rate is bounded. Consequently, the surplus level changes continuously in time \( t \) in response to the dividend payment policy. However, as we discussed in Section 1, in many applications, the boundedness of the dividend payment rate seems rather restrictive. For instance, insurance companies are more likely to distribute the dividend at discrete time points as opposed to continuously pay dividend with a ceiling of dividend payout rate. In light of these discussions, we remove the restriction on the maximal dividend rate at any instant and consider the (singular) optimal dividend payment policy for the piecewise deterministic Poisson risk model introduced in Section 2. Thus, throughout this section, \( D(t) \), the total amount of dividends paid out up to time \( t \), is not necessarily absolutely continuous with respect to \( t \).

Recall that for a given admissible dividend strategy \( D = \{D(t), t \geq 0\} \), the associated EPV is given by (2.3) and the goal is to find an admissible dividend strategy \( D^* = \{D^*(t), t \geq 0\} \) that achieves the value function \( V \) given by (2.4). The following proposition indicates that the value function \( V \) is nondecreasing. It can be proved using exactly the same arguments as those in [Song et al., 2011].

**Proposition 4.1:** For any \( 0 \leq y \leq x \), we have
\[
V(x) \geq (x - y) + V(y).
\]
Using Itô’s formula, we can also prove that

**Proposition 4.2:** Suppose that there is a continuously differential function \( \phi : \mathbb{R} \rightarrow \mathbb{R}_+ \) such that \( \phi'(x) \geq 0 \) for all \( x \geq 0 \). Then
\[
V(x) \leq \frac{1}{\kappa} \phi(x) + \frac{1}{\kappa} \sup_{D \in \Pi} \mathbb{E}_x \int_0^\theta e^{-rs}(L - \delta)\phi(X(s)) \, ds,
\]
for any \( x \geq 0 \).

**Remark 4.3:** Proposition 4.2 establishes an upper bound for the value function \( V \). In particular, it follows that if there is a function \( \phi \) satisfying the conditions in Proposition 4.2 and that \( \sup_{D \in \Pi} \mathbb{E}_x \int_0^\theta e^{-rs}(L - \delta)\phi(X(s)) \, ds \leq \infty \), then \( V(x) < \infty \) for any \( x \geq 0 \). In the rest of the paper, we assume that \( V \) is finite.
A. The Verification Theorem

The following verification theorem will help us find the value function and an optimal dividend payment strategy.

**Theorem 4.4:** Suppose there exists a continuously differentiable function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \), satisfying \( \varphi(y) = 0 \) for \( y < 0 \) and it solves the following quasi-variational inequality:

\[
\max \left\{ (\mathcal{L} - \delta)\varphi(x), 1 - \varphi'(x) \right\} = 0, \quad x > 0. \tag{4.2}
\]

(i) Then \( \varphi(x) \geq V(x) \) for every \( x \geq 0 \).

(ii) Define the continuation region

\[ C = \{ x \geq 0 : 1 - \varphi'(x) < 0 \}. \]

Assume there exists a dividend payment scheme \( \pi^* = \{ \mathcal{D}^*(t) : t \geq 0 \} \in \Pi \) and corresponding process \( X^* \) satisfying (2.1) such that,

\[
X^*(t) \in \mathcal{C} \text{ for Lebesgue almost all } 0 \leq t \leq \tau, \tag{4.3}
\]

\[
\lim_{\Delta \to \infty} \mathbb{E}_x \left[ e^{-r(\tau \wedge N)} \varphi(X^*(\tau \wedge N)) \right] = 0, \tag{4.5}
\]

and if \( X^*(s) \neq X^*(s-), \) then

\[
\varphi(X^*(s)) - \varphi(X^*(s-)) = -\Delta D^*(s), \tag{4.6}
\]

where \( D^{cc}(s) := D^*(s) - \sum_{0 \leq s \leq t} \Delta D^*(s) \) denotes the continuous part of \( D^* \). Then \( \varphi(x) = V(x) \) for every \( x \geq 0 \) and \( \varphi' \) is an optimal dividend payment strategy.

B. Exponential Claims

In order to obtain an explicit solution to the quasi-variational inequality (4.2) and an optimal dividend payment policy, as in Section 3-B, we again assume that the claims \( Y_1, Y_2, \ldots \) are independently and exponentially distributed with mean \( 1/\alpha \) for some \( \alpha > 0 \).

In what follows, we construct an explicit solution to (4.2), and verify that the solution is the value function defined in (2.4). To this end, we make the following assumption.

**Hypothesis B.** The integral-differential equation (3.6) has a continuously differentiable solution \( \psi(x) \) and that \( \psi'(x) \) achieves its minimum value at \( b > 0 \) and \( \psi'(x) \) is non-decreasing on \( (b, \infty) \).

**Theorem 4.5:** Under hypothesis B, the solution to (4.2) is

\[
\Phi(x) = \begin{cases} 
\frac{\psi(x)}{\psi(b)}, & \text{if } x \leq b, \\
\frac{x - b + \psi(b)}{\psi(b)}, & \text{if } x > b.
\end{cases} \tag{4.7}
\]

Moreover, the barrier strategy given by continuous part

\[
dD^*(t) = \begin{cases} 
0, & \text{if } X(t) < b, \\
g(b) dt, & \text{if } X(t) = b,
\end{cases} \tag{4.8}
\]

and singular part

\[
\Delta D^*(t) = X(t) - b, \quad \text{if } X(t) > b,
\]

with \( D^*(0-) = 0 \) is the optimal control that corresponds to \( \Phi(x) \) given in (4.7), that is, \( V(x) = \Phi(x) = J(x, D^*) \).

**Example 4.6:** Similar to Example 3.5, we consider a controlled piecewise-deterministic compound Poisson surplus process. As in Example 3.5, we take \( g(x) = c > 0 \) and \( Q(y) = 1 - e^{-\alpha y}, y \geq 0 \). But in contrast to Example 3.5, here we allow the optimal dividend payment policy to be singular.

Recall that \( \psi(x) = (r + \alpha)e^{rx} - (s + \alpha)e^{sx} \) solves the integral-differential equation (3.6) and that \( \psi(x) = r(r + \alpha)e^{rx} - s(s + \alpha)e^{sx} \) achieves its unique minimum value at

\[
b = \frac{1}{r - s} \ln \left( \frac{s^2 + \alpha^2}{r^2 + \alpha^2} \right) \quad \text{and that } \psi' \text{ is non-decreasing on } (b, \infty). \]

Therefore in view of Theorem 4.5, if \( b > 0 \), then the dividend payment strategy defined in (4.8) is optimal and the value function is

\[
V(x) = \begin{cases} 
\frac{(r + \alpha)e^{r(x-b)} - (s + \alpha)e^{s(x-b)}}{r(r + \alpha)e^{rx} - s(s + \alpha)e^{sx}}, & \text{if } x < b, \\
\frac{x - b + (r + \alpha)e^{r(x-b)} - (s + \alpha)e^{s(x-b)}}{r(r + \alpha)e^{rx} - s(s + \alpha)e^{sx}}, & \text{if } x \geq b.
\end{cases}
\]

On the other hand, if \( b \leq 0 \), then one can verify that the function \( x \mapsto x + \frac{c}{x + \delta} \) solves the quasi-variational inequality (4.2). Hence Theorem 4.4 implies that \( W(x) \leq x + \frac{c}{x + \delta} \). Moreover, the EPV of dividends from the strategy of paying all surplus immediately until the first claim is equal to \( x + \frac{c}{x + \delta} \). Hence it follows that

\[
V(x) = x + \frac{c}{x + \delta}, \quad \text{if } b \leq 0.
\]

Using the fact that \( r \) and \( s \) are the roots of (3.11), we can verify that \( b > 0 \) if and only if

\[
\alpha \lambda c > (\lambda + \delta)^2. \tag{4.9}
\]

Hence we can summarize the value function as

\[
V(x) = \begin{cases} 
\frac{(r + \alpha)e^{r(x-b)} - (s + \alpha)e^{s(x-b)}}{r(r + \alpha)e^{rx} - s(s + \alpha)e^{sx}}, & \text{if } \alpha \lambda c > (\lambda + \delta)^2 \text{ and } x < b, \\
\frac{x - b + (r + \alpha)e^{r(x-b)} - (s + \alpha)e^{s(x-b)}}{r(r + \alpha)e^{rx} - s(s + \alpha)e^{sx}}, & \text{if } \alpha \lambda c > (\lambda + \delta)^2 \text{ and } x \geq b,
\end{cases}
\]

\[
x + \frac{c}{x + \delta}, \quad \text{if } \alpha \lambda c \leq (\lambda + \delta)^2.
\]

Note that our result agrees that of [Schmidli, 2008, p. 94]. But our approach is much simpler than theirs.

Finally we demonstrate the comparison of restricted and unrestricted payment schemes through a numerical example, in which \( \alpha = 1, \delta = 0.1, c = 4, \lambda = 2, \) and \( u_0 = 3 \). Note that both (3.14) and (4.9) are satisfied. The resulting unrestricted and restricted value functions \( V(x) \) and \( V_R(x) \) are shown in Figure 1. Note that the plot of \( V_R(x) \) also demonstrates the limit result of \( V_R(x) \) presented in Lemma 3.1.

5. Conclusions and Remarks

This work is devoted to the optimal dividend payment problem for piecewise-deterministic compound Poisson risk models. Under certain conditions, it is shown that the optimal restricted dividend payment scheme is the threshold strategy, in which dividends are paid only at the maximal
Fig. 1. Comparison of the unrestricted and restricted value functions

(a) Value functions $V(x)$ and $V_R(x)$

(b) The difference $V(x) - V_R(x)$

References


