

H. Steck, M. Naraschewski, and H. Wallis

Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, D-85748 Garching, Germany

and

Sektion Physik, Ludwig-Maximilians-Universität München, Theresienstraße 37, D-80333 München, Germany

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We study the output properties of a pulsed atom laser consisting of an interacting Bose-Einstein condensate (BEC) in a magnetic trap and an additional rf field transferring atoms to an untrapped Zeeman sublevel. For weak output coupling we calculate the dynamics of the decaying condensate population, of its chemical potential and the velocity of the output atoms analytically.

The experimental breakthrough to Bose-Einstein condensation with small numbers of atoms in magnetic traps [1] has raised much interest in the properties of mesoscopic quantum gases. Bose-Einstein condensates with atoms in a single magnetic sublevel have been studied experimentally and theoretically. The recent experimental and theoretical investigations of interference between two independent Bose-Einstein condensates convincingly proved their macroscopic coherence [2,3]. Moreover, the laser-like coherence of the atoms is preserved in the presence of a matter-wave splitter based on rf-transitions pumping the atoms into untrapped magnetic sublevels [4]. These states are either strong-field seeking or have no magnetic moment at all, and leave the trap. Alternatively, optical Raman transitions can be used for the transfer [5,6]. Such schemes provide controllable output couplers for coherent atom lasers.

In analogy to a laser one can distinguish between a cw laser, based on continuous refilling of the condensate, and a pulsed atom laser where the condensate is periodically refilled and slowly released, similar to [4]. Whereas a continuous wave atom laser has been studied only theoretically [6–8], current Bose-Einstein condensation experiments are limited to the pulsed mode of operation. The closest approximation of a cw atom laser by a pulsed one can be reached in the limit of a weak coupling rf field. In this case we are able to describe the decay of the trapped condensate and its energy width analytically. Previous calculations addressed the opposite limit of strong coupling by numerical calculations [9] or neglected the important influence of atom-atom interactions [10].

The output coupler consists of a monochromatic resonant rf field of frequency  $\omega_{\text{rf}}$  transferring  $^{23}\text{Na}$  atoms in the  $F = 1$  hyperfine state from the trapped  $m = -1$  into the untrapped  $m = 0$  and the repelled  $m = 1$  magnetic sublevels. For simplicity an isotropic harmonic trap

potential  $V_{-1}(\mathbf{r}) = V_{\text{off}} + M\omega_{\text{T}}^2\mathbf{r}^2/2$ ,  $V_{+1}(\mathbf{r}) = -V_{-1}(\mathbf{r})$  and  $V_0(\mathbf{r}) \equiv 0$  are assumed while effects of gravity are neglected.

The three coupled coherent matter waves are described by a three-component Gross-Pitaevskii equation (GPE) with resonant excitation in rotating wave approximation first studied for a generic two-level system in Ref. [9].

In the following we adopt the point of view of *spontaneously* broken gauge symmetry for a Bose gas initially at zero temperature. The system of equations for the macroscopic wave function  $\tilde{\psi}_m(t) = e^{-im\omega_{\text{rf}}t}\langle\hat{\psi}_m(t)\rangle$  in rotating wave approximation for  $m, m' \in \{-1, 0, +1\}$  now reads

$$i\hbar\frac{\partial}{\partial t}\tilde{\psi}_m(\mathbf{r}, t) = \left(-\frac{\hbar^2\nabla^2}{2M} + V_m(\mathbf{r}) + \hbar m\omega_{\text{rf}} + U\|\tilde{\psi}(\mathbf{r}, t)\|^2\right)\tilde{\psi}_m(\mathbf{r}, t) + \hbar\Omega\sum_{m'}(\delta_{m, m'+1} + \delta_{m, m'-1})\tilde{\psi}_{m'}(\mathbf{r}, t). \quad (1)$$

Here, we have replaced the atomic density by the modulus of the wave function

$$\langle\hat{n}(\mathbf{r}, t)\rangle = \|\tilde{\psi}(\mathbf{r}, t)\|^2 = \sum_m\|\tilde{\psi}_m(\mathbf{r}, t)\|^2. \quad (2)$$

The coupling constant  $\hbar\Omega = g\mu_{\text{Bohr}}|B|/\sqrt{2}$  refers to the Rabi frequency due to the rf field. At zero magnetic field the symmetrized s-wave scattering matrix elements  $U_{mm'} = 4\pi\hbar^2a_{mm'}/M$  for an elastic collision of a pair of atoms in the sublevels  $m, m' \in \{-1, 0, +1\}$  are all nearly equal to  $\bar{a} = 53a_0$ , ( $a_0$  is the Bohr radius), according to preliminary calculations of E. Tiesinga and P. S. Julienne [11]. Since the steady state operation depends mainly on the initial condensate mean field, we assume in the following a diagonal scattering matrix  $a_{mm'} = \delta_{mm'}\bar{a}$  for simplicity. Consequently, the Hartree mean field potential for each spin component is equal to the total atom density  $\langle\hat{n}(\mathbf{r}, t)\rangle$  multiplied by  $U = 4\pi\hbar^2\bar{a}/M$ .

The initial condition is chosen as the solution of the stationary GPE for the trapped condensate in the absence of the rf field, i.e  $\Omega = 0$  in Eq. (1). In the Thomas-Fermi approximation it reads

$$|\tilde{\psi}_{-1}(\mathbf{r}, 0)|^2 = \max\left[\frac{\mu + V_{\text{off}} - V_{-1}(\mathbf{r})}{U}, 0\right]. \quad (3)$$

For a small coupling strength ( $\Omega \ll \omega_{\text{T}}$ ) the process of atoms leaking out of the resonance points is faster than

the Rabi oscillations. Therefore the coupling into state  $m = +1$  can be neglected, since it is proportional to  $\Omega^4$ . In the following only the states  $m = -1$  and  $m = 0$  are considered. After switching on the coupling due to the rf field, initial oscillations die out quickly, because the untrapped atoms leak out of the trap within less than one Rabi cycle. Other condensate atoms move into the resonance area replacing the leaving ones. Eventually a quasi-stationary state is reached, i.e. the  $m = -1$  condensate wave function decays slowly without oscillations while the atoms coupled out of the condensate are expelled due to the mean-field potential and form a steady current. A numerical solution of the two-component GPE shows the uniform decay of the trapped condensate (cf. Fig. 1). The sum of external and mean field potentials is in a good approximation spatially independent inside the condensate so that a description of the decay in terms of the chemical potential  $\mu(t)$  is appropriate. In the following we present an analytical calculation of the time dependent output intensity in this quasi-stationary regime, assuming a three dimensional isotropic harmonic trap.

We first calculate the rate  $\Gamma$  of transitions from the condensate into the output. In the spirit of the Thomas-Fermi approximation we neglect the kinetic energy in a two-component GPE corresponding to Eq. (1) and solve for the output density distribution

$$|\tilde{\psi}_0(r, t)|^2 = \frac{4\Omega^2 \sin^2[\frac{1}{2}\sqrt{\Delta^2(r) + 4\Omega^2 t}]}{\Delta^2(r) + 4\Omega^2} |\tilde{\psi}_{-1}(r, 0)|^2. \quad (4)$$

The maximum amplitude of the Rabi oscillations is located at  $r_{\text{res}}$  determined by the resonance condition

$$\hbar\Delta(r_{\text{res}}) = 0 \quad (5)$$

where  $\hbar\Delta(r) = \hbar\omega_{\text{rf}} - V_{-1}(r)$ . Thus, the main contribution to the output coupling stems from a small shell around that resonance radius.

The time-derivative of the density gives the density transition rate. The total transition rate is obtained by integrating the position dependent rate over the condensate volume. The transition rate is negligible outside a minute resonance shell and strongly peaked within that shell. Expanding the position dependent detuning  $\Delta(r)$  to first order one obtains the time dependent transition probability

$$\begin{aligned} \Gamma(t) &\equiv \frac{1}{|\tilde{\psi}_{-1}(r_{\text{res}}, 0)|^2} \int_0^\infty 4\pi r^2 \frac{\partial}{\partial t} |\tilde{\psi}_0(r, t)|^2 dr \\ &\approx 4\pi r_{\text{res}}^2 \int_{-\infty}^\infty 2\Omega^2 \frac{\sin[(\sqrt{\Delta'(r_{\text{res}})^2 r^2 + 4\Omega^2 t}]}{\sqrt{\Delta'(r_{\text{res}})^2 r^2 + 4\Omega^2}} dr \\ &= 8\pi^2 r_{\text{res}}^2 \frac{\Omega^2}{\Delta'(r_{\text{res}})} J_0(2\Omega t). \end{aligned} \quad (6)$$

The calculation of the above rate  $\Gamma(t)$  does not account for the losses due to the leaving atoms. However, a rate equation allowing for these losses can be derived in the limit of weak coupling by using the perturbational rate ( $t \ll \pi/\Omega$ )

$$\Gamma = 8\pi^2 \hbar \Omega^2 \frac{\sqrt{2\hbar\Delta(0)}}{(M\omega_{\text{T}}^2)^{3/2}} \quad (7)$$

with  $\Delta'(r_{\text{res}}) = 2\Delta(0)/r_{\text{res}}$ .

Due to the spatial localization of the output coupling, the decay of the condensate population

$$N(t) \equiv \int d^3r |\tilde{\psi}_{-1}(r, t)|^2, \quad N(0) = N_0 \quad (8)$$

depends solely on the density of the atoms around the resonance shell with radius  $r_{\text{res}}$

$$\frac{dN(t)}{dt} = -\Gamma |\tilde{\psi}_{-1}(r_{\text{res}}, t)|^2. \quad (9)$$

In the quasi-stationary regime we assume the shape of the condensate density being equal to the Thomas-Fermi solution of the stationary GPE (cf. Fig. 1) with a slowly varying atom number  $N(t)$ ,

$$N(t) = \frac{4\pi}{15U} [2\mu(t)]^{5/2} / (M\omega_{\text{T}}^2)^{3/2}. \quad (10)$$

The condensate density at the resonance points is then given by

$$|\tilde{\psi}_{-1}(r_{\text{res}}, t)|^2 = [\mu(t) - \hbar\Delta(0)]/U. \quad (11)$$

Inserting this into the decay law (9) we obtain a nonlinear differential equation for the decay of the chemical potential

$$\frac{d}{dt}\mu + \alpha \frac{\mu - \hbar\Delta(0)}{\mu^{3/2}} = 0, \quad (12)$$

where  $\alpha = 3\Gamma(M\omega_{\text{T}}^2)^{3/2} 2^{-7/2} \pi^{-1}$ . Integration yields

$$\left[ 2\hbar\Delta(0)\mu^{1/2} + \frac{2}{3}\mu^{3/2} - 2[\hbar\Delta(0)]^{3/2} \text{artanh}\sqrt{\mu/\hbar\Delta(0)} \right]_{\mu(t)}^{\mu(0)} = \alpha t. \quad (13)$$

With Eq. (10) this yields additionally the time evolution for the number of trapped atoms  $N(t)$  as well as the flux and the velocity of the untrapped atoms.

During the depopulation of the condensate the chemical potential and the spatial extension of the condensate decrease until the resonance points lie on the surface of the shrank condensate. At this point the flux out of the condensate vanishes, because  $|\tilde{\psi}_{-1}(r_{\text{res}}, t)|^2 = 0$ , and the

chemical potential and the number of atoms remaining in the trap become constant in time

$$\mu(\infty) = \hbar\Delta(0). \quad (14)$$

As an example we chose a small value of  $\Delta(0)$  and a trap frequency  $\omega_T = 2\pi \times 106$  Hz, the geometric mean of the values given in Ref. [4]. The resulting time evolution of the system variables according to the nonlinear differential equation is shown in Figs. 2(a) and (b). The chemical potential  $\mu(t)$  and the number of trapped atoms reach their steady state after roughly 25 seconds. Correspondingly the flux and the velocity of the untrapped atoms decrease to zero (cf. Figs. 2(c) and (d)). The corresponding calculations were also carried out in one dimension in order to compare the analytical expressions with numerical simulations of the full coupled GPE's. The results showed excellent agreement (less than 5 % deviation).

The finite duration of the atom pulse ejected from the trap leads to a finite energy width of both the condensate and the output beam that is called here the *natural energy width*  $\delta E$  of a pulsed atom laser, in analogy to the natural linewidth of spontaneously emitted photons. It can be calculated approximately by describing the initial stage of the output coupling process by an exponential decay of the condensate population

$$\frac{dN}{dt}(t) = -\Gamma_{\text{pop}} [N(t) - N(\infty)]. \quad (15)$$

The rate  $\Gamma_{\text{pop}}$  results in an energy uncertainty

$$\delta E = \hbar\Gamma_{\text{pop}} \approx \frac{15}{2} \pi \hbar^{5/2} \frac{\Omega^2 \sqrt{\Delta(0)}}{\mu(0)^{3/2}} \quad (16)$$

Whereas the transition rate  $\Gamma$  does not depend on  $\mu$  and  $N_0$ , the population decay rate  $\Gamma_{\text{pop}}$  does so due to its dependence on  $\psi_{-1}(r, 0)$ . We thus find a dependence of the energy width on the condensate number

$$\delta E \propto \frac{\Omega^2 \sqrt{\Delta(0)}}{N_0^{3/5}}. \quad (17)$$

The natural energy width becomes narrower for weaker coupling strength  $\Omega$ , i.e. a slower output coupling process. The same effect can be achieved by choosing a smaller detuning  $\Delta(0)$  which causes the sphere of the resonance points to shrink towards the center of the condensate. The energy width can further be reduced by starting out from a condensate with a large population  $N_0$ .

The energy width  $\delta E$  can also be understood as a velocity width  $\delta v = \delta E / (Mv)$  of the untrapped atoms leaving the condensate. The parameters given in Fig. 2 result in  $\langle v \rangle = 1.31$  cm/s for the velocity outside the condensate and  $\delta v / \langle v \rangle \approx 10^{-6}$  for the relative velocity width.

An additional energy width is imposed by the temporal decay of the chemical potential  $\mu(t)$ . It leads to a decrease of the output velocity according to  $Mv(t)^2/2 = \mu(t) - \hbar\Delta(0)$  thereby implying a frequency chirp of the output beam. This frequency chirp can be compensated, however, by imposing a chirp on the frequency  $\omega_{\text{rf}}$  of the rf field such that the output velocity rather than the detuning  $\Delta(0)$  becomes constant.

The output pulse can last for up to the order of 100 seconds such that causes of phase fluctuation which would also influence a cw-atom laser have to be considered as well. Additional broadening of the energy width of the output beam is caused by thermal excitations of the condensate wavefunction and technical noise of the output coupling mechanism such as fluctuations of the confining magnetic field. Thus, the *natural energy width* calculated above has to be understood as a lower limit. A comprehensive theoretical description of the fluctuations of the condensate wavefunction, particularly of its phase, is beyond the scope of this Letter.

It has been shown in [6] that the temporal decay of phase correlations transforms into a decay of the spatial coherence of the output beam along the mean classical trajectory that is being passed during the according correlation time of the condensate phase.

At present, random variations of the bias magnetic field at the mGauss level are among the most important experimental limitations to the coherence properties of the output, prior to fluctuations of the condensate wavefunction. They might contribute to a phase diffusion of the output beam in the 100 Hz range [12].

Pulsed atom lasers will play an important role in creating coherent matter waves as the size of the condensate may be largely increased in future experiments. The results of this Letter are based on the solutions of the coupled Gross-Pitaevskii equations and allow us to extract the relevant properties of the output analytically. The velocity of the untrapped atoms depends on the slowly decaying population of the trapped condensate fraction, leading to a slow chirp of the output frequency. While this frequency chirp might be compensated by a variation of the rf-field frequency, other sources of spectral width like fluctuations of the confining magnetic field are surely more difficult to overcome.

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FIG. 1. Uniform decay of the condensate density in terms of the effective potential  $V_{-1}(\mathbf{r}) - V_{\text{off}} + U|\tilde{\psi}_{-1}(\mathbf{r})|^2$  for small field strength  $\Omega = 12 \text{ s}^{-1}$ ,  $N_0 = 5 \times 10^6$  and  $\Delta(0) = 3100 \text{ s}^{-1}$  in a one-dimensional situation. The lower density is reached after 670 ms.

FIG. 2. Time evolution in the Thomas-Fermi approximation: (a) chemical potential, (b) number of particles in the trapped  $m = -1$  state, (c) particle flux in the untrapped  $m = 0$  state, (d) velocity of atoms leaving the condensate. The parameters are  $N_0 = 5 \times 10^6$ ,  $\Delta(0) = 3500 \text{ s}^{-1}$ ,  $\Omega = 20 \text{ s}^{-1}$ .