On the Concept of Number

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1 Translator's Introduction

In the 1887 article "Über den Zahlbegriff" [4] Leopold Kronecker set himself the task (among other things) of demonstrating that talk of algebraic numbers is unnecessary, in the sense that it can be eliminated in favor of (somewhat elaborate) talk of natural numbers. In Section 2 below, we present a translation of the portion of Kronecker's article where he carries out this demonstration. But first, in Section 1.1 we place our translated excerpt in the context of the article as a whole, and indicate the relationship between two versions of the article. Then, in Section 1.2 we preview some of the mathematical details from the excerpt in order to aid the reader's comprehension of the notoriously inscrutable Kronecker.

1.1 The Versions of "Über den Zahlbegriff"

Kronecker's "Über den Zahlbegriff" was originally a short piece written as part of a *Festschrift* for Eduard Zeller, which has already appeared in English translation in Ewald's collection *From Kant to Hilbert* [5]. That early version begins with a discussion of finite ordinals and cardinals (§1 and §2), and goes on to present the commutative law for addition (§3) and multiplication (§4) of natural numbers. Then §5 is titled "Calculations with Variables," and in it Kronecker briefly indicates how he intends to reduce all higher algebraic talk to talk only of natural numbers:

> [The laws of addition and multiplication] had to be assumed as authoritative as soon as one began to apply letters to designate numbers whose determination can or should remain in reserve. But with the introduction *in principle* of 'indeterminates', which is due to Gauss, the special theory of whole numbers widened into the general arithmetical theory of total whole-numbered functions of indeterminates. This general theory allows us to discard all the concepts that, properly speaking, are foreign to arithmetic – for instance, that of irrational algebraic numbers. ([5], 954)

*Translated by Edward T. Dean. Wilfried Sieg provided helpful comments.

In this early version of the paper, however, Kronecker illustrates the idea at hand only for *negative* numbers.

If we suppose, for the sake of argument, that someone has a comfortable mastery of the basic arithmetic of the natural numbers $\mathbb{N}$ (including subtraction to the degree that it can occur in that domain), yet has never encountered the notion of negative numbers, then even such a simple statement as

$$7 - 9 = 3 - 5$$

will make no *a priori* sense to the subject. From his knowledge of subtraction, he understands that of course one cannot subtract 9 from 7 (or 5 from 3) and obtain a natural number. If the subtraction operation is to return some "thing" in this and similar cases, we must have some new sort of objects outside of $\mathbb{N}$; moreover, from the point of view of our subject, this is a new and different use of the equality sign.

Kronecker's approach allows us to avoid conceiving of negative numbers as a new sort of object that enlarges our domain. Using Gauss' modular arithmetic notion of congruence, Kronecker recasts the preceding equation as

$$7 + 9x \equiv 3 + 5x \pmod{x + 1},$$

where x is indeterminate. We can recapture the other way of thinking, namely that we are enlarging our domain, if we conceive of the indeterminate x as an actual element determined by the equation $x + 1 = 0$, but from Kronecker's perspective it is important that we need not do so.

This earlier version of the article abruptly ends at that point with a concluding paragraph, leaving Kronecker's stated goal of discarding the concept of irrational algebraic numbers untouched. However, an extended version of "Über den Zahlbegriff" appeared in Crelle's journal in 1887 [4]. Therein, Kronecker expands §5 by inserting, before the concluding paragraph, first a quick demonstration that rational numbers can be eliminated via indeterminates, and second the promised elimination of algebraic numbers in general. As far as we are aware, this later version of the article has so far not appeared in English; our translation in Section 2 below contains only the later addition to §5.

1.2 Mathematical Overview of the Translated Excerpt

Recall that an algebraic number ξ is precisely a root of a polynomial with integer coefficients, say

$$p(x) := x^n - c_1 x^{n-1} + c_2 x^{n-2} - \cdots \pm c_n.$$

We can think of two different scenarios in which we might want to work with such a ξ:

1. We do not care about ξ in particular, but really *just the fact that it is a root of* $p(x)$.

2. We really do care about the specific value of ξ.

In the text, Kronecker points to the fact that he had already shown how to deal with the first scenario in previous work [2, 3]. In the present article, Kronecker tackles the second scenario, showing how the distinct roots of such a $p(x)$ – so-called *conjugates* of each other – can be isolated from one another using solely talk of natural numbers.

Kronecker's argument for the eliminability of talk of algebraic numbers is not too involved, but his presentation is not ideal. Kronecker is notoriously difficult to read, and even though the mathematics here is not particularly complicated, the text can still be tougher going than it need be. For these reasons, we present here a reconstruction of some of the mathematical argument, which is an offshoot of Sturm's theorem [6], as Kronecker indicates.

We follow Kronecker's notation for the sake of easy comparability with his text. Throughout this section, we fix $f(x)$ as denoting some polynomial in the indeterminate x whose coefficients are *integers*:

$$f(x) := a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \qquad (a_0, \ldots, a_n \in \mathbb{Z}).$$

We can summarize Kronecker's overall goal as being to find explicitly a natural number s sufficiently large so that:

1. We can isolate the real roots of f via intervals of size $1/s$, in the sense that no such interval contains more than one real root of f;

2. "Calculation with the roots of f" can also be replaced by approximating the roots (and the value of f at such approximations) to an arbitrary precision.

While Kronecker does emphasize certain points typographically and provide a helpful summary of the work done so far at one point, there is nothing like a bold demarcation of lemmas and theorems in Kronecker's text, so we will here take the liberty of singling out certain significant "stopping points" from Kronecker's development, giving an outline of Kronecker's result.

As $x \to \infty$ or $x \to -\infty$, either $f(x) \to \infty$ or $f(x) \to -\infty$. The starting point for Kronecker's development is Cauchy's calculation of an explicit bound, depending only on the polynomial's coefficients, of a "critical strip" around the origin within which all roots of the polynomial must lie:[1]

[1] Cauchy's bound is easy to obtain; Kronecker proves that it works in just a couple of lines. However, he makes no mention of Cauchy. This occasional inattention to citation (of others' sources and often also his own) sometimes contributes to the difficulty of reading Kronecker.

Theorem (Cauchy's bound, 1829 [1]). *All real roots of f lie in the interval $[-\mathfrak{r}, \mathfrak{r}]$, where*

$$\mathfrak{r} := \max_{0 \le i \le n-1} \left\{ \frac{|a_i| + |a_n|}{|a_n|} \right\}.$$

(See Figure 1 for an illustration of this bound for a particular polynomial.)

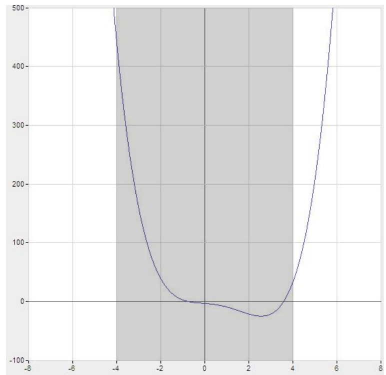

Figure 1: Cauchy's bound tells us that all real roots of the polynomial $g(x) = x^4 - 3x^3 - x^2 - 3x - 3$ lie in the interval $[-4, 4]$.

Kronecker then explicitly determines his intended s in terms of various other values: the coefficients a_0, ..., a_n of f, the coefficients of certain other integral polynomials, the bound $\mathfrak{r}$, the absolute value D of the discriminant of f. All of these, however, are completely determined from the coefficients a_i of f, and thus so too is s. The first important property Kronecker establishes is that his s is so large that

$$|f(x')| + \left| \frac{f(x'') - f(x')}{x'' - x'} \right| > \frac{1}{s(s-1)} \qquad (\mathfrak{C}.)$$

holds for any x' and x'' in the critical strip that are no further than $1/s$ apart. We will see this come into play in a moment.

The second important property that Kronecker establishes is as follows. Suppose we have some interval in the critical strip that is of size no larger than $1/s$, and suppose that f takes on opposite signs at the endpoints of the interval. Kronecker proves that *for any positive integer r*, there is a subinterval of size $1/rsD$ in the interval such that: f takes on opposite signs at the endpoints of this subinterval, and

$$|f(x)| < \frac{1}{r} \qquad (\mathfrak{E}.)$$

on the entire subinterval, as in Figure 2.

This property clearly moves us toward goal (2) above, but it is also instrumental for goal (1). An interval for which f takes opposite signs at the endpoints contains at least one real root, by the intermediate value

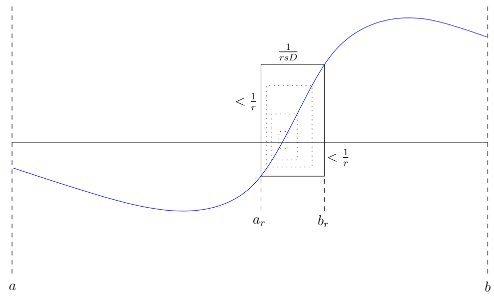

Figure 2: Given an interval $[a, b]$ of size $\leq 1/s$ in the critical strip for which f takes on opposite signs at the endpoints, as $r \to \infty$ there are subintervals $[a_r, b_r]$ of size $1/rsD$ that pin down a root of f with greater and greater precision.

theorem. But Kronecker wants to establish that s is so large that no interval of size $\leq 1/s$ can contain more than one root; this is after all the sense in which intervals of this size are supposed to "isolate" the real roots of f. Kronecker shows this by showing – as is clearly sufficient – that for any interval of size $\leq 1/s$ for which f takes on the same sign at the endpoints, f has that same sign throughout the interval.

Suppose not; so we have such an interval $[x_0, x_4]$ and a point $x_2 \in [x_0, x_4]$ where f takes the opposite sign. If one sets

$$x_1 := x_2 - \frac{|f(x_2)|}{(s-1)D}, \qquad x_3 := x_2 + \frac{|f(x_2)|}{(s-1)D},$$

then Kronecker shows that x_1, x_3 are within the interval $[x_0, x_4]$, and $f(x_1)$, $f(x_3)$ share the same sign as $f(x_2)$. But then for an arbitrary r, we can find by $(\mathfrak{C}.)$ some $x' \in [x_0, x_1]$ and $x'' \in [x_3, x_4]$ such that $|f(x')|, |f(x'')| < 1/r$; using $(\mathfrak{C}.)$ it is not hard to show that

$$r < s(s-1)\left(1 + \frac{(s-1)D}{|f(x_2)|}\right).$$

The right-hand side is a constant, though, and r could be arbitrarily large; contradiction. Thus no interval of size $\leq 1/s$ can encompass multiple changes of sign. That is, any interval of size $\leq 1/s$ either contains no roots, or just one.

So Kronecker sets t to be the least integer $\geq \mathfrak{r}s$, and then the $2t$ intervals of size $1/s$, starting out from the origin, cover the entire critical strip. All we need to do is calculate the sign of f at the (rational) endpoints of these intervals in order to tell which ones contain a root. And as we have seen, within such an interval, Kronecker has given a way to get arbitrarily close to said root, and to get values of f arbitrarily close to 0. Thus Kronecker has met his goals (1) and (2).

2 Kronecker's Text

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II. The concept of *fractional* numbers is to be avoided by replacing the factor $\frac{1}{m}$ in formulas with an indeterminate x_m, and replacing the equality sign with the GAUSSian congruence sign *modulo* $(mx_m - 1)$. The three fraction rules, namely that of addition:

$$\frac{a}{m} + \frac{b}{n} = \frac{an + bm}{mn},$$

that of multiplication:

$$\frac{a}{m} \cdot \frac{b}{n} = \frac{ab}{mn},$$

and that of division:

$$\frac{a}{m} : \frac{b}{n} = \frac{an}{bm},$$

are then completely substantiated by the three corresponding congruences:

$$ax_m + bx_n \equiv (an + bm)x_{mn} \,(\text{modd. } mx_m - 1, nx_n - 1, mnx_{mn} - 1), \tag{1}$$

$$ax_m \cdot bx_n \equiv abx_{mn} \,(\text{modd. } mx_m - 1, nx_n - 1, mnx_{mn} - 1), \tag{2}$$

$$ax_m \cdot x_{bx_n} \equiv anx_{bm} \,(\text{modd. } mx_m - 1, nx_n - 1, bmx_{bm} - 1, bx_n x_{bx_n} - 1). \tag{3}$$

These three congruences themselves result, however, from the following three identities:

$$\text{(I.)} \quad \begin{cases} ax_m + bx_n = (an + bm)x_{mn} + anx_{mn}(mx_m - 1) + bmx_{mn}(nx_n - 1) \\ \qquad -(ax_m + bx_n)(mnx_{mn} - 1), \end{cases}$$

$$\text{(II.)} \quad \begin{cases} ax_m \cdot bx_n = abx_{mn} + abnx_n x_{mn}(mx_m - 1) + abx_{mn}(nx_n - 1) \\ \qquad -abx_m x_n(mnx_{mn} - 1), \end{cases}$$

$$\text{(III.)} \quad \begin{cases} ax_m \cdot x_{bx_n} = anx_{bm} + anx_{bm}(mx_m - 1) - abmx_m x_{bm} x_{bx_n}(nx_n - 1) \\ \qquad -ax_m x_{bx_n}(bmx_{bm} - 1) + amnx_m x_{bm}(bx_n x_{bx_n} - 1). \end{cases}$$

"Greater than" and "less than" for fractions can be seen as given through the addition rule, in that the fraction arising from the addition of two fractions can be pronounced greater than each of the two summands. In this way the succession of rational fractions is not merely defined, but rather is justified as well.[i] [2]

[i] In the preface to his work: "Introduction à la théorie des fonctions d'une variable" [7] Mr. JULES TANNERY says, p. VIII: "One can entirely constitute analysis with the notion of whole number and the notions related to the addition of whole numbers; it is useless to make appeal to any other postulate, to any other data from experience; ... a fraction, from the point of view that I indicate, cannot be regarded as the union of equal parts of the unit; these words 'parts of the unit' no longer have sense; a fraction is a set [ensemble] of two whole numbers, arranged in a determinate order; for this new kind of number, it is necessary to reformulate the definitions of equality, inequality and the arithmetic operations." How this last can indeed occur – albeit in another order – has been demonstrated above.

[2] In Kronecker's German original, he left the quotation from Tannery's textbook that appears in footnote i in French. We saw no reason not to put it into English as well.

III. I have shown[ii] in an earlier article that the introduction and application of *algebraic* numbers is expendable whenever the isolation of conjugates among themselves is not necessary. However, this isolation can also be done without the introduction of new concepts, and the essence of the matter can clearly emerge only if it is carried out in this way, which will be presented here in the same manner as I have done it for ten years in my university lectures. Thereby, the "more precise analysis of the concept of real roots of algebraic equations" which I have announced at the end of the first part of "Grundzüge einer arithmetischen Theorie der algebraischen Grössen"[iii] will simultaneously be given.

If $f(x)$ is an entire integral function of x which has no common factor with its derivative $f'(x)$, then there are entire integral functions $\varphi(x)$, $\varphi_1(x)$ for which the equation:

$$\varphi(x)f(x) + \varphi_1(x)f'(x) = D \tag{$\mathfrak{A}$.}$$

holds. Here D denotes the absolute value of the discriminant of $f(x)$, hence a positive whole number. Now let:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

and let a_g be the absolute greatest of the n coefficients a_0, a_1, $\dots$ a_{n-1}. If one then designates the rational fraction $\frac{|a_g|+|a_n|}{|a_n|}$ with $\mathfrak{r}$, then:

$$\left|\frac{f(x)}{a_n} - x^n\right| < (\mathfrak{r}-1)\frac{|x|^n-1}{|x|-1},$$

hence for any value of x not lying between $-\mathfrak{r}$ and $\mathfrak{r}$:

$$|f(x) - a_nx^n| < |a_nx^n| \quad \text{and consequently:} \quad \text{sgn.}f(x) = \text{sgn.}a_nx^n.$$

So then $f(x)$ changes its sign only within the interval $(-\mathfrak{r}, \mathfrak{r})$.

If one sets the abbreviation:

$$f(x+\sigma) - f(x) = \sigma f_1(x,\sigma), \qquad (f_1(x,\sigma) - f'(x))\varphi_1(x) = \sigma\psi(x,\sigma),$$

then $f_1(x,\sigma)$ and $\psi(x,\sigma)$ are entire integral functions of x and σ, and if one understands $\overline{f}_1(x,\sigma)$, $\overline{\varphi}(x)$, $\overline{\varphi}_1(x)$, $\overline{\psi}(x,\sigma)$, respectively, to be those functions which develop from $f_1(x,\sigma)$, $\varphi(x)$, $\varphi_1(x)$, $\psi(x,\sigma)$ by replacing the coefficients with their absolute values, then the inequalities:

$$|f_1(x,\sigma)| < \overline{f}_1(\mathfrak{r},1), \quad |\varphi(x)| < \overline{\varphi}(\mathfrak{r}), \quad |\varphi_1(x)| < \overline{\varphi}_1(\mathfrak{r}), \quad |\psi(x,\sigma)| < \overline{\psi}(\mathfrak{r},1)$$

[ii] "Ein Fundamentalsatz der allgemeinen Arithmetik" vol. 100, p. 490 of this journal. [3] Cf. especially the end of this essay, op. cit., p. 510. In addition to what is said there, in certain domains of algebra the application of modules and module-systems in place of the algebraic numbers is not only permissible, but is even necessary. Thus the question, whether an irreducible integral function $F(x)$ becomes reducible under the adjunction of a root of an irreducible integral equation $\Phi(y) = 0$, can only be decided in the form, whether $F(x)$ can be represented *modulo* $\Phi(y)$ as a product of entire functions of x and y with rational coefficients.

[iii] Vol. 92, p. 44 of this journal. [2]

obviously hold, so long as the value of x lies between $-\mathfrak{r}$ and $\mathfrak{r}$ and σ lies between -1 and 1. If s denotes now a whole number which exceeds the greatest of the four rational values:

$$\frac{\overline{f}_1(\mathfrak{r},1)}{D}, \quad \frac{\overline{\varphi}(\mathfrak{r})}{D}, \quad \frac{\overline{\varphi}_1(\mathfrak{r})}{D}, \quad \frac{\overline{\psi}(\mathfrak{r},1)}{D},$$

by at least one unit, and if one then sets:

$$\varphi(x) = (s-1)D\theta(x), \qquad \varphi_1(x) = (s-1)D\theta_1(x), \qquad \psi(x,\sigma) = (s-1)DH(x,\sigma),$$

then the equation ($\mathfrak{A}$.) is transformed into the following:

$$\theta(x)f(x) + \theta_1(x)\cdot\frac{f(x+\sigma)-f(x)}{\sigma} = \sigma H(x,\sigma) + \frac{1}{s-1}, \qquad (\mathfrak{B}.)$$

and the values of the functions $\theta(x)$, $\theta_1(x)$, $H(x,\sigma)$ are absolutely less than 1 for the values of x and σ which are restricted by the inequalities:

$$-\mathfrak{r} < x < \mathfrak{r}, \qquad -1 < \sigma < 1.$$

If σ is absolutely less than $\frac{1}{s}$, then the inequality:

$$|f(x)| + \left|\frac{f(x+\sigma)-f(x)}{\sigma}\right| > \frac{1}{s(s-1)}$$

follows from the equation ($\mathfrak{B}$.), and the inequality:

$$|f(x')| + \left|\frac{f(x'')-f(x')}{x''-x'}\right| > \frac{1}{s(s-1)} \qquad (\mathfrak{C}.)$$

thence obtains for any two values x', x'' lying in the interval $(-\mathfrak{r},\mathfrak{r})$ whose difference, taken absolutely, is less than $\frac{1}{s}$.

It will be shown now that while x remains in an interval of size $\frac{1}{s}$, the function $f(x)$ will change its sign either not at all or only *one* time, i.e. if:

$$x' < x'' < x''' \quad \text{and} \quad x''' - x' \leq \frac{1}{s},$$

then it cannot be that:

$$\text{sgn.}f(x') = -\text{sgn.}f(x'') = \text{sgn.}f(x''').$$

If the value of $f(x)$ has at the beginning of an interval the opposite sign as it does at the end of the interval, which interval we would like to designate with (J) and which is no larger than $\frac{1}{s}$, then for any division of (J) into subintervals, the same must also be the case for at least one of the subintervals. Now let r be an arbitrary whole number, and think of the interval (J) as divided into rD equal parts. Then let (J')

be one such subinterval in which the initial and final values of $f(x)$ have opposite signs. Finally let x', x'' be two arbitrary values of x lying in the interval (J'), for which:

$$x' < x'', \qquad \text{sgn.} f(x') = -\text{sgn.} f(x'').$$

Since now:

$$f(x'') - f(x') = (x'' - x') f_1(x', x'' - x')$$

and also:

$$|f(x'') - f(x')| < (x'' - x')\overline{f}_1(\mathfrak{r}, 1) \leq (x'' - x')(s-1)D, \tag{$\mathfrak{D}$.}$$

with consideration of the inequality: $x'' - x' \leq \frac{1}{rsD}$, it follows that:

$$|f(x'') - f(x')| < \frac{1}{r},$$

and therefore, since $f(x')$ and $f(x'')$ have opposite signs, also:

$$|f(x')| < \frac{1}{r}, \qquad |f(x'')| < \frac{1}{r} \tag{$\mathfrak{E}$.}$$

must hold. *In any interval of size $\frac{1}{s}$, whose initial and final points have $f(x)$ with opposite signs, if one selects an arbitrary whole number r, then one can find at least one interval of size $\frac{1}{rsD}$ whose initial and final points likewise have $f(x)$ with opposite signs, and in which all values of $f(x)$ are absolutely less than $\frac{1}{r}$.*

If $f(x)$ has the *same* sign at the initial point as at the end point of an interval which is no larger than $\frac{1}{s}$, then $f(x)$ retains just this sign within the entire interval.

In particular, if one designates the interval with (J^0), its initial point with x_0, its final point with x_4, and one assumes that for a value x_2 lying between x_0 and x_4 the function $f(x)$ has a different sign as $f(x_0)$ than as $f(x_4)$, then there would also be two values x_1 and x_3 on either side of x_2 and still lying within the interval (J^0), determined by the equations:

$$x_1 = x_2 - \frac{|f(x_2)|}{(s-1)D}, \qquad x_3 = x_2 + \frac{|f(x_2)|}{(s-1)D}, \tag{$\mathfrak{F}$.}$$

for which:

$$\text{sgn.} f(x_0) = -\text{sgn.} f(x_1) = \text{sgn.} f(x_4) = -\text{sgn.} f(x_3).$$

First of all, that the values x_1 and x_3 still lie within the interval (J^0), i.e. that the inequalities:

$$x_2 - x_0 > \frac{|f(x_2)|}{(s-1)D}, \qquad x_4 - x_2 > \frac{|f(x_2)|}{(s-1)D}$$

obtain, can be inferred from the inequalities:

$$|f(x_2) - f(x_0)| < (x_2 - x_0)(s-1)D, \qquad |f(x_4) - f(x_2)| < (x_4 - x_2)(s-1)D,$$

which follow from the above inequality ($\mathfrak{D}$.), by further considering that according to the hypothesis:

$$\text{sgn.} f(x_2) = -\text{sgn.} f(x_0) = -\text{sgn.} f(x_4).$$

Secondly, we now have according to the above inequality ($\mathfrak{D}$.):

$$|f(x_2) - f(x_1)| < (x_2 - x_1)(s-1)D, \qquad |f(x_3) - f(x_2)| < (x_3 - x_2)(s-1)D,$$

therefore as a consequence of the equations ($\mathfrak{F}$.):

$$|f(x_2) - f(x_1)| < |f(x_2)|, \qquad |f(x_3) - f(x_2)| < |f(x_2)|,$$

and these inequalities necessitate that both $f(x_1)$ and $f(x_3)$ have the same sign as $f(x_2)$, thus the opposite of the function values $f(x_0)$ and $f(x_4)$. After this, both the interval (x_0, x_1) and the interval (x_3, x_4) would be such that $f(x)$ has opposite signs at the start and finish, and so according to that, as was proven above, values x', x'' could be determined for which:

$$x_0 < x' < x_1, \qquad x_3 < x'' < x_4, \qquad |f(x')| < \frac{1}{r}, \qquad |f(x'')| < \frac{1}{r},$$

where r is hypothetically arbitrary. Now, however, according to the inequality ($\mathfrak{C}$.):

$$|f(x')| + \left| \frac{f(x'') - f(x')}{x'' - x'} \right| > \frac{1}{s(s-1)}$$

would have to hold, and thus, since:

$$|f(x')| < \frac{1}{r}, \qquad |f(x'') - f(x')| < |f(x'')| + |f(x')| < \frac{2}{r},$$

also:

$$\frac{1}{r} + \frac{2}{r(x'' - x')} > \frac{1}{s(s-1)}$$

holds, and finally, since:

$$x'' - x' > x_3 - x_1 = \frac{2|f(x_2)|}{(s-1)D}$$

holds, also:

$$\frac{1}{r} + \frac{(s-1)D}{r|f(x_2)|} > \frac{1}{s(s-1)},$$

or:

$$r < s(s-1)\left(1 + \frac{(s-1)D}{|f(x_2)|}\right).$$

Since, however, the number r can be selected arbitrarily, this inequality cannot hold, and thus it is indeed to be concluded that in an interval which is no larger than $\frac{1}{s}$, as soon as one knows the function $f(x)$ has only one sign at both of the endpoints, the same is the case throughout the interval.

By now it follows immediately that in an interval of size $\frac{1}{s}$, $f(x)$ cannot change sign *more* than once. For, were it the case that

$$\text{sgn}.f(x_0) = -\text{sgn}.f(x_1) = \text{sgn}.f(x_2)$$

for three values x_0, x_1, x_2, with $x_0 < x_1 < x_2$, lying in the interval, then the interval (x_0, x_2) would be one such whose size is less than $\frac{1}{s}$, and at whose initial and final points $f(x)$ would have the same sign. However, as was just proven, in such an interval $f(x)$ cannot change its sign; thus it cannot be that

$$\text{sgn}.f(x_0) = -\text{sgn}.f(x_1).$$

The result developed in the preceding can be formulated as follows:

First let $a_0+a_1x+a_2x^2+\cdots+a_nx^n$ be an integral function of x, which we would like to designate with $f(x)$; let D be the absolute value of the discriminant of the function $f(x)$ and let $f'(x)$ be its derivative.

Second let $\varphi(x)$, $\varphi_1(x)$ be integral functions of x, of degrees $n-2$ and $n-1$ respectively, for which the equation:

$$\varphi(x)f(x) + \varphi_1(x)f'(x) = D$$

holds, and let:

$$\varphi(x) = \sum_{k=0}^{k=n-2} \alpha_k x^k, \qquad \varphi_1(x) = \sum_{k=0}^{k=n-1} \alpha'_k x^k.$$

Third let the functions $f_1(x,y)$, $\psi(x,y)$ be defined by means of the equations:

$$f(x+y) - f(x) = yf_1(x,y), \qquad (f_1(x,y) - f'(x))\varphi(x) = y\psi(x,y),$$

so that in the expansions:

$$\begin{aligned} f_1(x,y) &= \sum_{h,k} b_{h,k}x^hy^k \qquad (h,k = 0,1,\ldots n-1), \\ \psi(x,y) &= \sum_{h,k} c_{h,k}x^hy^k \qquad (h,k = 0,1,\ldots 2n-4), \end{aligned}$$

the coefficients b and c denote whole numbers.

Fourth let $|a_g|$ be the greatest of the values $|a_0|$, $|a_1|$, $\ldots |a_{n-1}|$, and let s be the least positive whole number which satisfies the inequality conditions:

$$\begin{aligned}
(s-1)D &\geq \sum_h |\alpha_h| \cdot \left(\frac{|a_g|+|a_n|}{|a_n|}\right)^h & \qquad (h=0,1,\ldots n-2),\\
(s-1)D &\geq \sum_h |\alpha'_h| \cdot \left(\frac{|a_g|+|a_n|}{|a_n|}\right)^h & \qquad (h=0,1,\ldots n-1),\\
(s-1)D &\geq \sum_{h,k} |b_{h,k}| \cdot \left(\frac{|a_g|+|a_n|}{|a_n|}\right)^h & \qquad (h,k=0,1,\ldots n-1),\\
(s-1)D &\geq \sum_{h,k} |c_{h,k}| \cdot \left(\frac{|a_g|+|a_n|}{|a_n|}\right)^h & \qquad (h,k=0,1,\ldots 2n-4).
\end{aligned}$$

Then it cannot be that $\text{sgn.} f(x') = -\text{sgn.} f(x'') = \text{sgn.} f(x''')$, if

$$x' < x'' < x''' \qquad \text{and} \qquad x''' - x' \leq \frac{1}{s};$$

the function $f(x)$ thus retains its sign in any interval of size $\frac{1}{s}$ in which the signs at the initial and final points are equal, and it changes its sign only a single time in any interval of size $\frac{1}{s}$ in which the signs at the initial and final points are different. Furthermore, in an interval of the latter sort, if r denotes an arbitrary positive whole number, then a subinterval of size $\frac{1}{rsD}$ can be determined, such that the function $f(x)$ has different signs at the initial and final points and its absolute value remains less than $\frac{1}{r}$ throughout the subinterval. Finally, the function $f(x)$ retains the sign of $a_n x^n$ as soon as the absolute value of x becomes greater than $\frac{|a_g|+|a_n|}{|a_n|}$.

After this, if the whole number t is determined by the inequality condition:

$$s(|a_g|+|a_n|) \leq t|a_n| < |a_n| + s(|a_g|+|a_n|),$$

the function $f(x)$ can change sign only in an interval: $\left(\frac{k-1}{s}, \frac{k}{s}\right)$, in which k has one of the values: $-t+1$, $-t+2$, …, $t-1$, t. Therefore one needs to determine only the signs of the $2t$ values:

$$f\left(\frac{k}{s}\right) \qquad (k=-t+1, -t+2, \ldots t-1, t)$$

in order to determine in which of those $2t-1$ intervals of size $\frac{1}{s}$ the function $f(x)$ changes its sign – and then only one time. The Anzahl of these intervals is at the same time that which one designates as the Anzahl of the real roots of the equation $f(x) = 0$, and the former will then totally replace the latter via the specified procedure, which STURM's theorem provides. But also the so-called calculation with the real roots itself will be replaced through the given procedure; for if it is shown for a particular number k that:

$$\text{sgn.} f\left(\frac{k-1}{s}\right) f\left(\frac{k}{s}\right) = -1,$$

then one needs only calculate the initial and final values of $f(x)$ in the subintervals of size $\frac{1}{rsD}$, i.e. thus the $rD+1$ values:

$$f\left(\frac{k}{s}-\frac{h}{rsD}\right) \qquad (h=0,1,\ldots rD)$$

and to determine that number h for which:

$$\text{sgn.} f\left(\frac{k}{s}-\frac{h}{rsD}\right) f\left(\frac{k}{s}-\frac{h-1}{rsD}\right) = -1,$$

in order to consequently infer that, in the interval:

$$\frac{k}{s}-\frac{h}{rsD} \leq x < \frac{k}{s}-\frac{h-1}{rsD},$$

the function $f(x)$ changes its sign and remains absolutely less than $\frac{1}{r}$ throughout.

The so-called existence of the real irrational roots of algebraic equations is grounded solely in the existence of intervals with the specified quality; the legitimacy of calculating with the individual roots of an algebraic equation is based completely upon the possibility of isolating them, hence upon the possibility of determining a number, which we designated with s above. If such a number s is determined which has the property that the intervals of size $\frac{1}{s}$ are sufficiently small to isolate the distinct roots of the same equation, then "greater than" and "less than" will be defined simply through the succession of the relevant isolation intervals. "Greater than" and "less than" for arbitrary irrational algebraic numbers is also determined after this, if – as is obviously permissible – one thinks of the two algebraic numbers, which are to be ordered, as two roots of one and the same equation. The true essence of the matter, however, becomes completely clear in the above deduction only when one also avoids the use of fractions therein and makes use exclusively of whole numbers.

If, for this purpose, the *homogeneous* entire function:

$$a_0 y^n + a_1 y^{n-1} z + a_2 y^{n-2} z^2 + \cdots + a_n z^n$$

is introduced in place of $a_0 + a_x + a_2 x^2 + \cdots + a_n x^n$ and designated with $F(y,z)$, then

$$f\left(\frac{z}{y}\right) = \frac{1}{y^n} F(y,z).$$

Thus:

$$\text{sgn.} F(rsD, krD-h) \cdot F(rsD, krD-h+1) = -1,$$

and if q denotes an indeterminate positive whole number, then:

$$|F(qrsD, z)| < r^{n-1}(qsD)^n$$

holds for all integral values of z which lie between:

$$(krD - h)q \qquad \text{and} \qquad (krD - h + 1)q,$$

while the *sign* of $F(qrsD, z)$ for $z = (krD - h)q$ opposes that for $z = (krD - h + 1)q$.

The number s is determined in the manner specified above from the coefficients of the function $F(x, y)$. Then the distinct integral values of k which characterize the distinct real roots of the equation $f(x) = 0$ are determined by the condition:

$$\text{sgn.} F(s, k - 1) \cdot F(s, k) = -1.$$

Supposing now that r is an arbitrary number, then the number h will be defined by the condition:

$$\text{sgn.} F(rsD, krD - h) \cdot F(rsD, krD - h + 1) = -1$$

for some particular relevant value of k that is positive and does not exceed rD, and then:

$$\begin{aligned} |F(rsD, krD - h)| &< r^{n-1}(sD)^n, \\ |F(rsD, krD - h + 1)| &< r^{n-1}(sD)^n. \end{aligned}$$

Each of the real roots of the equation $f(x) = 0$ is thus totally characterized by a particular number k; but then to any posited arbitrary number r there belongs this particular number h, and one can thus conceive of the numbers h as "functions of the indeterminate whole numbers r" which are defined via the integral function $F(y, z)$.

⋮ ⋮ ⋮

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