Affine $A_3^{(1)}$ $N=2$ Monopole as the D Module and Affine ADHMN Sheaf

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Abstract

A Higgs-Yang Mills monopole scattering spherical symmetrically along light cones is given. The left incoming anti-self-dual α plane fields are holomorphic, but the right outgoing SD β plane fields are antiholomorphic, meanwhile the diffeomorphism symmetry is preserved with mutual inverse affine rapidity parameters μ and μ^{-1}. The Dirac wave function scattering in this background also factorized respectively into the (anti)holomorphic amplitudes. The holomorphic anomaly is realized by the center term of a quasi Hopf algebra corresponding to an integrable conform affine massive field. We find explicit Nahm transformation matrix(Fourier-Mukai transformation) between the Higgs YM BPS (flat) bundles (D modules) and the affinized blow up ADHMN twistors (perverse sheafs). Thus establish the algebra for the 't Hooft-Hecke operators in the Hecke correspondence of the geometric Langlands Program.

keywords: Affine BPS monopole, ADHMN sheaf, Nahm transformation, 't Hooft-Hecke operator, geometric Langlands Program.

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1 Introduction

The 't Hooft BPS monopole always plays important role in gauge field theory, for example, for the Seiberg Witten monopole condensation [1] and for the 't Hooft operator [2] . In this paper, we first give an affine monopole which twist affinize the twistor . The key points lies in that, when we do the transformation from the space time fixed frame to the comoving frame, in fact we have transform to the static κ symmetric Killing gauge in Green Schwarz theory. Meanwhile, we start from the BPS Higgs Yang-Mills bundle, which is flat over the self dual plane α and the antiself dual plane β, to the affinized ADHMN sheaf (i. e. twistor sheaf).

Then we solve the Dirac equation in this background. Here we covariantly transform to the same comoving frame. This will manifest the Nahm transformation to the twistor space. These are realized as the soliton solution of the conformal affine Toda field [3, 4]. Furthermore, this will give the elementary building block for the moduli space of exactly solvable worldsheet theory (Bena, Polchinski, Roiban [5], and [6, 7]). Our work shows the explicit links between Yang-Mills theory, quantum massive integrable field and quasi Hopf (Drinfeld [8]) twistors for the scaled elliptic algebra and the scaled W algebra [9, 10, 11, 12, 13, 14]. So it generalize the algebraic formulation in the Langlands program by E. Frenkel et al. [15, 16, 17] to the 4 dimensional case.

2 The affine monopole solution

As Kapustin and Witten [2] we consider an affine family of the N=2 supersymmetric gauge field in the 4d Minkowski space $\mathbb{M}$ which has been twisted relevant to the geometric Langlands program. We pick a homotopic homomorphism $\mathfrak{K}$ from the space time symmetry $Spin(4)_S$, which is the universal cover $SU(4)$ of the conformal $SO(4, 2)$, to the R symmetry $Spin(6)_R$, $\mathfrak{K}$: $Spin(4)_S \to Spin(4)_R \subset Spin(6)_R$. By the choice of these Bochner Martinelli kernel homotopy operator $\mathfrak{K}$, we get a family of N=2 loop supersymmetry $\widetilde{SU}(4)$ with affine parameter $t \in \mathbb{CP}_1$. To establish the Hecke correspondence, it should be further central extended to $A_3^{(1)}$. The time reverse and orientation reversal symmetry described by [2] implies, and the global Riemann Hecke correspondence requires that the coordinates $x_\mu(\mu = 0, 1, 2, 3)$ of $\mathbb{M}$ has to be complexifield, embedded into $\mathbb{C}^4$ by analytically continue to the upper and lower complex planes respectively, such that the connection of the D module becomes flat in the SD planes α and the ASD planes β.

$$[D_{i+}, D_{\perp i+}] + *[D_{i+}, D_{\perp i+}] = 0, \qquad (\alpha) \qquad (2.1)$$

$$[D_{\bar{i}-}, D_{\perp \bar{i}-}] - *[D_{\bar{i}-}, D_{\perp \bar{i}-}] = 0. \qquad (\beta) \qquad (2.2)$$

Remark: The flat connection introduced by C.N.Yang [30] is the $i = 1$ case of (2.1), and (2.5).

Here in the covariant derivative

$$D_\mu = \partial_\mu + A_\mu, \qquad (2.3)$$

we have

$$A_\mu = A_\mu^{\hat{a}} T^{\hat{a}}, \mu = 0, 1, 2, 3, \qquad (2.4)$$

where $T^{\hat{a}}$ is the $A_3^{(1)}$ generators, the $*$ denotes the 4-dimensional Hodge dual, and the tangent vectors ∂ in the **fixed null frame** for the α planes are

$$\partial_{i_+} \equiv \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^0}), \partial_{\perp i_+} \equiv \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x^j} + \mathrm{i}\frac{\partial}{\partial x^k}), \tag{2.5}$$

$$\partial_{i_-} \equiv \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^0}), \partial_{\perp i_-} \equiv \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x^j} - \mathrm{i}\frac{\partial}{\partial x^k}), \tag{2.6}$$

while the tangent vectors $\overline{\partial}$ for the β planes are

$$\overline{\partial}_{i_-} \equiv \frac{1}{\sqrt{2}}(\frac{\partial}{\partial \overline{x^i}} - \frac{\partial}{\partial \overline{x^0}}), \ \overline{\partial}_{\perp i_-} \equiv \frac{1}{\sqrt{2}}(\frac{\partial}{\partial \overline{x^j}} - \mathrm{i}\frac{\partial}{\partial \overline{x^k}}), \tag{2.7}$$

$$\overline{\partial}_{i_+} \equiv \frac{1}{\sqrt{2}}(\frac{\partial}{\partial \overline{x^i}} + \frac{\partial}{\partial \overline{x^0}}), \overline{\partial}_{\perp i_+} \equiv \frac{1}{\sqrt{2}}(\frac{\partial}{\partial \overline{x^j}} + \mathrm{i}\frac{\partial}{\partial \overline{x^k}}). \tag{2.8}$$

The x_μ ($\bar{x}_\mu$) are the complexified Cartesian coordinate of $\mathbb{M}$, analytical continued upperwise (lowerwise) respectively. We use $\perp$ to denote perpendicular in α (β) plane , i, j, k are the cyclic permutation of $1, 2, 3$. The tangential vectors ∂_{i_+} and $\partial_{\perp i_+}$ are left null, while ∂_{i_-} and $\partial_{\perp i_-}$ are right null. To establish the Hecke correspondence, we should have left (right) D operators flat on α (β) plane and act on left (right) Hilbert space $\mathcal{H}_{L,R}$ respectively (c.f. the section 3). In this paper we consider the level one case, i.e. $\mathcal{H}_{L,R}$ are $|\Lambda_i\rangle, \langle\bar{\Lambda}_i|$ ($i = 0, 1, 2, 3$) respectively (c.f. the section 4). Over the α plane (1), the gauge field $\mathcal{F}_{\mu\nu}$ of the left bundle is anti-self dual, while over the β plane (2), $\mathcal{F}_{\bar{\mu}\bar{\nu}}$ is self dual. Here $\mu\nu$ or $\bar{\mu}\bar{\nu}$ implies that x_μ approach the real slice from upper or lower half complex planes of x_μ. We will find the monopole solution separately for the α (β) null planes, such that it is incoming(outgoing) along the left(right) real null lines ∂_{i_+}($\bar{\partial}_{i_-}$) spherical symmetrically, i.e. we boost the static BPS monopole (Appendix A) along the incoming(outgoing) real null lines. The interaction between incoming and outgoing waves at all the scattering points yields the central extension (c.f the section 4).

The *level one*, i.e. the grade one in principle gradation of the affine algrbra, Lorentz covariant spherical symmetric ansatz of the nonzero components of **incoming** $A_\mu^{\hat{a}}$ in the **tensor product** form of spherical **comoving** spacetime and gauge **frames** are (cf. Appendix B)

$$K_{T^+} = \frac{iF(\mathbb{r})}{\sqrt{2}\mathbb{r}}\rho^{-1}E^{-1}, \tag{2.9}$$

$$K_{T^-} = -\frac{iF(\mathbb{r})}{\sqrt{2}\mathbb{r}}\rho^{-1}E, \tag{2.10}$$

$$H_{T^+} = \frac{-\partial\gamma/\partial\varphi + \cos\theta}{\mathbb{r}\sin\theta}\rho^{-2}, \tag{2.11}$$

$$H_{T^-} = \frac{\partial\gamma/\partial\varphi + \cos\theta}{\mathbb{r}\sin\theta}\rho^{-2}, \tag{2.12}$$

$$K_{r-} = A_{r-} = G(\mathbb{r})\rho^{-2}, \tag{2.13}$$

$$A^c_{r+} = \zeta(\mathbb{r}), \tag{2.14}$$

$$A^{\mathrm{d}}_\mu = 0, \tag{2.15}$$

$$D_\mu = \partial_\mu + A_\mu, \qquad A_\mu = H_\mu + K_\mu, D_\mu^{(H)} = \partial_\mu + H_\mu,$$

where: the cyclic element

$$E=\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ 1 & & & 0 \end{pmatrix}, \tag{2.16}$$

and the Coxeter ρ lies in the center of $U(4)$

$$\rho=\begin{pmatrix} 1 & & & \\ & i & & \\ & & -1 & \\ & & & -i \end{pmatrix}, \tag{2.17}$$

i.e. $\left(\rho^{-1}\right)_{i,i}=\left(\rho^{-1}E\right)_{i,i+1}=(-\omega)^{i},\left(\rho^{-1}E^{-1}\right)_{i+1,i}=(-\omega)^{i}, \omega\equiv e^{\frac{2\pi i}{4}}$, here $\rho\,E\,(E\rho=\omega E\rho)$ are the generators of the affine Heisenberg algebra (group), used for the construction of the vertex operators in the principle realization of the affine algebra [24]. The **spherical incoming space time null frames**

$$\mathfrak{e}^{T^{+}}=-\frac{1}{\sqrt{2}}\mathbb{r}e^{-i\gamma}\left(d\theta+\mathrm{i}\sin\theta d\varphi\right), \tag{2.18}$$

$$\mathfrak{e}^{T^{-}}=\frac{1}{\sqrt{2}}\mathbb{r}e^{i\gamma}\left(d\theta-\mathrm{i}\sin\theta d\varphi\right), \tag{2.19}$$

$$\mathfrak{e}^{r^{\pm}}=\frac{1}{\sqrt{2}}\mu^{\pm 1}(dr\pm dt),\qquad r^{2}=x_{1}^{2}+x_{2}^{2}+x_{3}^{2}. \tag{2.20}$$

here: $\mathbb{r}=\mu r^{+}+\bar{\mu}^{-1}\bar{r}^{-}$, $r^{\pm}=\frac{1}{2}(r\pm t)$, $\gamma(\theta,\varphi)=\mp\varphi$ in north (south) Wu-Yang gauge. We have decomposed the connection A_μ into the gauge connection H_μ, which lies in the center $U(1)$ of $U(4)$, and the covariant constant independent of (θ,φ) components K_μ. The connection H_μ turns to be the $U(1)$ Dirac monopole component for the 't Hooft monopole. Now the K_μ in eq. (2.9), (2.10), (2.13) are not connections, but are tensors with the basis elements $A_{\beta,j}$ (given by ρ, E) in lemma (14.6) [24]. For the level one representation $|\Lambda_i\rangle$, in the expression of the principle realization in affine algebra for the vertex operator Γ, only the level one $j=1$ term in the infinite sum of the following eq.(14.6.7) [24] contributes.

$$\Gamma^{\beta}=\langle\Lambda_0,A_{\beta,0}\rangle\exp\left(\sum_{j=1}^{\infty}\lambda_{\beta j'}z^{b_j}x_j\right)\exp\left(-\sum_{j=1}^{\infty}\lambda_{\beta,N+1-j'}^{b_j}b_j^{-1}z^{-b_j}\frac{\partial}{\partial x_j}\right). \tag{14.6.7}$$

The $j=1$ term usually write as $E_{ij}^{(1)}$, with the grade $''1''$ correspond to the exponent $''1''$ of the loop parameter z in (14.6.7).

The explicit expression of the central term ζ will be given later. which will contribute the holomorphic anomaly. However it is irrelevant for the field strength over $\alpha(\beta)$ plane here. Actually, the incoming and outgoing $K_\mu\gamma^\mu$ in reduced cup product form (section 3) turns to be the Affine Toda Lax connection generate by the grade $1(-1)$ part of the soliton generating vertex operator (section 4) in the principle realization [3, 4].

Our ansatz for level -1 nonzero component $A_{\hat{\mu}}^{\hat{a}}$ in **the outgoing comoving null**

frames over the β planes (2.2) are:

$$K_{\bar{T}^+} = \frac{iF(\mathbb{r})}{\sqrt{2}\mathbb{r}}\rho E, \tag{2.21}$$

$$K_{\bar{T}^-} = \frac{iF(\mathbb{r})}{\sqrt{2}\mathbb{r}}\rho E^{-1}, \tag{2.22}$$

$$H_{\bar{T}^+} = \frac{\partial\bar{\gamma}/\partial\bar{\varphi} - \cos\bar{\theta}}{\mathbb{r}\sin\bar{\theta}}\rho^2, \tag{2.23}$$

$$H_{\bar{T}^-} = \frac{-\partial\bar{\gamma}/\partial\bar{\varphi} - \cos\bar{\theta}}{\mathbb{r}\sin\theta}\rho^2, \tag{2.24}$$

$$K_{\bar{r}^+} = G(\mathbb{r})\rho^2, \tag{2.25}$$

$$A^c_{\bar{r}^-} = \zeta(\mathbb{r}), \tag{2.26}$$

$$A^d_{\bar{\mu}} = 0. \tag{2.27}$$

As in the case of the static BPS monopole (Appendix (A13) A(14)), to obtain the field strength from the comoving frame ansatz (2.9)-(2.15) for the incoming potential become very simple, by using the generalized Gauss-Codazzi equations(C9), (C10). Now, only the Dirac potential component is dependent on (θ,φ) to get its $F^{(H)}_{\mu\nu}$ involves differential calculation, all other K_μ terms becomes (θ,φ) independent tensor matrices, its contribution to the field strength are just the tensor products and r directional covariant derivatives. The holomorphic ASD fields strength over α plane are

$$F_{T^+,r^-} = \frac{-F(\mathbb{r})G(\mathbb{r})}{\mathbb{r}}\rho E^{-1}, \tag{2.28}$$

$$F_{T^-,r^-} = \frac{F(\mathbb{r})G(\mathbb{r})}{\mathbb{r}}\rho E, \tag{2.29}$$

$$F_{T^+,T^-} = -\frac{F^2(\mathbb{r})}{\mathbb{r}^2}\rho^{-2} - \frac{1}{\mathbb{r}^2}\rho^{-2}, \tag{2.30}$$

$$F_{T^+,r^+} = -\frac{F'(\mathbb{r})}{\mathbb{r}}\rho^{-1}E^{-1}, \tag{2.31}$$

$$F_{T^-,r^+} = -\frac{F'(\mathbb{r})}{\mathbb{r}}\rho^{-1}E, \tag{2.32}$$

$$F_{r^+,r^-} = G'(\mathbb{r})\rho^{-2}, \tag{2.33}$$

here $G'(\mathbb{r}) = \frac{d}{d\mathbb{r}}G(\mathbb{r})$, $F'(\mathbb{r}) = \frac{d}{d\mathbb{r}}F(\mathbb{r})$.

The anti-self-dual equation becomes

$$F'(\mathbb{r}) = G(\mathbb{r})F(\mathbb{r}), \tag{2.34}$$

$$G'(\mathbb{r}) = \frac{F^2(\mathbb{r}) - 1}{\mathbb{r}^2}. \tag{2.35}$$

Remark: the seemingly mismatch factor ρ^2 is contributed by the opposite chirality in the $\gamma_\mu D^{(H)}_\mu$ and $\gamma_\mu K^\mu$ (c.f. Section 3) under homotopy, which include a $\rho^2 \in \mathbb{Z}_2$ factor in the Hodge duality in the target space, moduli space, as the opposite rotation of δA_z, $\delta\phi_z$ under the action of the complex structure (e.g. Kapustin and Witten [2] (4.3))

The unique "normalizable" solution is:

$$F(\mathbb{r}) = \frac{\mathbb{r}}{\mathtt{sh}\mathbb{r}}, \tag{2.36}$$

$$G(\mathbb{r}) = \frac{1}{\mathbb{r}} - \mu \mathtt{cth}\mathbb{r}. \tag{2.37}$$

This solution is the unique solution such that the field strength has appropriate asymptotical property both at the infinity and at the origin. When we calculate the left(right) part of our solution, we keep $\bar{r}_-(r_+)$ fixed. This implies as in section 4, push down to the worldsheet by the static (with respect to "time" $r_-(\bar{r}_+)$) gauge, the dependence on worldsheet coordinates $z = r_-(\bar{z} = \bar{r}_+)$ turns to be (anti)holomorphic, respectively.

From the outgoing comoving null frames(equations (2.21)-(2.27)), the antiholomorphic SD field strength equals the Hermitian conjugate of ASD part, only the sign of $\bar{H}_{T^+}, \bar{H}_{T^-}$ changes, since the orientation of the basis is reverse.

$$\bar{\mathfrak{e}}^{T^+} = -\tfrac{1}{\sqrt{2}} e^{-i\bar{\gamma}} \mathbb{r} \left(d\bar{\theta} - \mathrm{i} \sin\bar{\theta} d\bar{\varphi} \right), \bar{\mathfrak{e}}^{r^\pm} = \tfrac{1}{\sqrt{2}} \bar{\mu}^{\mp 1} (d\bar{r} \pm d\bar{t}), \ \bar{\mathfrak{e}}^{T^-} = \tfrac{1}{\sqrt{2}} e^{i\bar{\gamma}} \mathbb{r} \left(d\bar{\theta} + \mathrm{i} \sin\bar{\theta} d\bar{\varphi} \right).$$

3 The Dirac equation in the affine monopole background

Now we turn to the Dirac equation in this affine monopole background:

$$\text{For the right D bundle} \quad \gamma^\mu D_\mu |\psi\rangle = 0, \tag{3.1}$$

$$\text{For the left D bundle } \langle\psi| \left(\gamma^{\bar{\mu}} D_{\bar{\mu}}\right)^\dagger = 0. \tag{3.2}$$

As in last section(c.f. appendix B), we decompose $A = A_\mu^{\hat{a}} \mathbf{T}^{\hat{a}} dx^\mu$ into the potential H_μ of Dirac monopole and the vector boson K_μ covariant with respect to both the spin S and the "R-spin" T.

$$A_\mu = H_\mu + K_\mu. \tag{3.3}$$

Let

$$D_\mu = D_\mu^{(H)} + K_\mu.$$

For the right D module, the outgoing spherical waves propagate along r^+ with $r^- = constant$. We introduce

$$\kappa \equiv -i\epsilon_{ijk}\sigma_i \hat{r}_j D_k^{(H)} + 1. \tag{3.4}$$

here $i, j, k = T^+, T^-, r^+$, then one can proof that

$$\gamma_i D_i^{(H)} = \gamma_{r^+} \left(\frac{\partial}{\partial r^+} + \frac{1}{r^+} \right) - \frac{1}{r^+} \gamma_{r^+} \kappa, \tag{3.5}$$

where $\gamma_{r^+} = \Sigma^2 \otimes \sigma_{r^+}$, $\sigma_{r^+} = \sum\limits_i \sigma_i \hat{r}_i^+$.

Let the fixed frame wave function in the tensor product form of "R-spin" $|I\rangle_\nu^\beta$ and the space time spin $|S\rangle_\lambda^\alpha$ be factorized into (t, r) and (φ, θ) dependent part[2]

$$|\psi\rangle = \psi_{\lambda\nu}^{\alpha\beta} |S\rangle_\lambda^\alpha |I\rangle_\nu^\beta, \qquad \alpha, \beta = +, -; \qquad \lambda, \nu = \pm\frac{1}{2}. \tag{3.6}$$

[2]we adopt the convention $|v\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum\limits_{i=1}^n v_i |e\rangle_i, \qquad |e\rangle_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots.$

Then the incoming Dirac equation (3.1) becomes

$$\gamma_\mu D_\mu^{(H)}|\psi\rangle + \sum_{a,j=1,2} \epsilon_{r^+aj}(\Sigma^2\otimes\sigma_j)^{\alpha\alpha'}_{\lambda\lambda'}\mathbb{F}(\mathbb{r})^{\alpha'\alpha'';\beta'\beta''}_{\lambda'\lambda'';\nu'\nu''}(\Sigma^2\otimes\sigma_a)^{\beta\beta'}_{\nu\nu'}\psi^{\alpha''\beta''}_{\lambda''\nu''}|S\rangle^{\alpha''}_{\lambda''}|I\rangle^{\beta''}_{\nu''}$$

$$+\eta_{r^-r^+}(\Sigma^{r^+}\otimes\sigma_{r^-})^{\alpha\alpha'}_{\lambda\lambda'}\mathbb{G}(\mathbb{r})^{\alpha'\alpha'';\beta'\beta''}_{\lambda'\lambda'';\nu'\nu''}(\Sigma^{r^-}\otimes\sigma_{r^+})^{\beta\beta'}_{\nu\nu'}\psi^{\alpha''\beta''}_{\lambda''\nu''}|S\rangle^{\alpha''}_{\lambda''}|I\rangle^{\beta''}_{\nu''}=0, \quad (3.7)$$

where

$$\mathbb{F}^{\alpha\alpha';\beta\beta'}_{\lambda\lambda';\nu\nu'} \equiv \left((\gamma_+)^{\alpha\alpha'}_{\lambda\lambda'}\underset{ST}{\otimes}(\rho)^{\beta\beta'}_{\nu\nu'}\right)\frac{F(\mathbb{r})}{\mathbb{r}}, \quad \mathbb{G}^{\alpha\alpha';\beta\beta'}_{\lambda\lambda';\nu\nu'} \equiv \left((\gamma_0)^{\alpha\alpha'}_{\lambda\lambda'}\underset{ST}{\otimes}(\rho)^{\beta\beta'}_{\nu\nu'}\right)G(\mathbb{r}).; \quad (3.8)$$

and the γ_μ matrices is decomposed as the direct product $\otimes$ of the $4d$ chirality $\sum$ and the spin σ,

$$\gamma_i=\Sigma^2\otimes\sigma_i, \gamma_0=\Sigma^1\otimes\sigma_{0,}\gamma_5=\Sigma^3\otimes\mathbf{1},$$

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad (3.9)$$

$$\sigma_{r^+}=\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \sigma_{r^-}=\sigma_0=-\begin{pmatrix}i&0\\0&i\end{pmatrix}, \quad (3.10)$$

$$\Sigma^{r^+}\equiv\Sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \Sigma^{r^-}\equiv\Sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \Gamma\equiv\Sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \Gamma^\pm=\frac{1}{2}(1\pm\Gamma). \quad (3.11)$$

and $\eta^{r_-r_+}$ is the light cone metric in real null direction.

We use the D function to transform from the fixed frame basis $|S\rangle=\begin{pmatrix}|S\rangle^+\\|S\rangle^-\end{pmatrix}, |I\rangle=\begin{pmatrix}|I\rangle^+\\|I\rangle^-\end{pmatrix}$ to the comoving frame $|s>^\pm_\lambda, |i>^\pm_\nu$

$$|s>^\pm_\lambda=\sum_\rho D^{\pm S}_{\rho\lambda}(\varphi,\theta,\gamma)|S>^\pm_\rho, |S\rangle_{\frac{1}{2}}=\begin{pmatrix}1\\0\end{pmatrix}, |S\rangle_{-\frac{1}{2}}=\begin{pmatrix}0\\1\end{pmatrix}, \quad (3.12)$$

$$|i>^\pm_\lambda=\sum_\rho D^{\pm T}_{\rho\lambda}(\varphi,\theta,\gamma)|I>_\rho, |I\rangle_{\frac{1}{2}}=\begin{pmatrix}1\\0\end{pmatrix}, |I\rangle_{-\frac{1}{2}}=\begin{pmatrix}0\\1\end{pmatrix}, \quad (3.13)$$

which satisfy for the spin part: $\sigma_r|s>^\pm_\lambda=2\lambda|s>^\pm_\lambda$, $\sigma_r|i>^\pm_\nu=2\nu|i>^\pm_\nu$, for the chirality part: $\Gamma|i>^\pm=\pm|i>^\pm, \Gamma|s>^\pm=\pm|s>^\pm$. The superscript "S" and "T" denote the finite rotation matrix function $D^{\frac{1}{2}}_{\lambda\rho}(\varphi,\theta,\gamma)$ act in the spin space and the isospin space, respectively, with $S=\frac{1}{2}, T=\frac{1}{2}$.

Then the fixed frame eq. (3.6) (3.7) is transformed to the following comoving frame eq.

$$\left(\gamma_{r^+}(\frac{\partial}{\partial r^+}+\frac{1}{r^+})|\psi\rangle\right) \quad (3.14)$$

$$+\{\sum_{a,j=1,2}\epsilon_{3aj}(\hat{\Sigma}^2(x)^{\alpha\alpha'}\otimes\hat{\sigma}_j(x)_{\lambda\lambda'})(D^{\alpha S}_{\lambda''\lambda'})\underset{ST}{\otimes}\mathbb{F}^{\alpha'\beta';\alpha''\beta''}_{\lambda''\lambda''';\nu''\nu'''}(\mathbb{r})(D^{\beta T}_{\nu''\nu'})(\hat{\Sigma}^2(x)^{\beta\beta'}\otimes\hat{\sigma}_a(x)_{\nu\nu'})$$

$$+(\hat{\Sigma}^{r^+}(x)^{\alpha\alpha'}\otimes\hat{\sigma}_{r^-}(x)_{\lambda\lambda'})(D^{\alpha S}_{\lambda''\lambda'})\eta_{r^-r^+}\underset{ST}{\otimes}(D^{\beta T}_{\nu''\nu'})\mathbb{G}^{\alpha'\beta';\alpha''\beta''}_{\lambda''\lambda''';\nu''\nu'''}(\mathbb{r})(\hat{\Sigma}^{r^-}(x)^{\beta\beta'}\otimes\hat{\sigma}_{r^+}(x)_{\nu\nu'})\}$$

$$\times f^{\alpha''\beta''}_{\lambda'''\nu'''}|s\rangle^{\alpha''}_{\lambda'''}|i\rangle^{\beta''}_{\nu'''}=0, \quad (3.15)$$

$$\hat{\psi}^{\alpha\beta}_{\lambda\nu} = f^{\alpha\beta}_{\lambda\nu}(r,t)\mathbb{D}^{\alpha\beta}_{\lambda\nu}(\varphi,\theta,\gamma). \tag{3.16}$$

Here the $\hat{\sigma}(x), \hat{\Sigma}(x)$ matrix acted on the comoving frame basis $|i>^{\pm}_{\nu}$, $|s>^{\pm}_{\lambda}$becomes the constant matrices in (3.9)(3.10) (c.f. (B8)-(B12)). Then, since the κ operator in (3.5) turns to be zero as shown after (3.20) , similar as (B13), this Dirac equation turns to be simply a "tangent vector equation along the left radial direction"

We note that in the second term of the eq. (3.14) the space spin S and the R spin T are coupled into

$$\begin{aligned}
&\epsilon_{3aj}(\sigma_j)_{\lambda'\lambda''} D^{\alpha S}_{\lambda\lambda'}\, |S\rangle^{\alpha}_{\lambda''} \underset{ST}{\otimes} (\sigma_a)_{\nu'\nu''} D^{\beta T}_{\nu\nu'}\, |I\rangle^{\beta}_{\nu''} \\
&= \epsilon_{3aj}(\hat{\sigma}_j)_{\lambda\lambda'}\, |s\rangle^{\alpha}_{\lambda'} \underset{ST}{\otimes} (\hat{\sigma}_a)_{\nu\nu'}\, |i\rangle^{\beta}_{\nu'} (-1)^{\lambda-\frac{1}{2}}(-\delta_{\alpha,\beta}\delta_{\lambda,\nu} + \delta_{\alpha,-\beta}\delta_{\lambda,-\nu}),
\end{aligned} \tag{3.17}$$

with nonvanishing (φ,θ) functions $\mathbb{D}$

$$\mathbb{D}^{\alpha\beta}_{\lambda\nu}(\varphi,\theta,\gamma) \equiv (-1)^{2\lambda+1}(-\delta_{\lambda,\nu}\delta_{\alpha,\beta} + \delta_{\lambda,-\nu}\delta_{\alpha,-\beta})\, |s\rangle^{\alpha}_{\lambda}\, |i\rangle^{\beta}_{\nu}\,. \tag{3.18}$$

by using

$$\frac{1}{4\pi}\sum_{\lambda',\nu'} C^{S,T,0}_{\lambda',\nu',0} D^{\alpha S}_{\lambda'\lambda}(\varphi,\theta,\gamma) D^{-\alpha T}_{\nu'\nu}(\varphi,\theta,\gamma) = \frac{1}{\sqrt{2\pi}} D^{0}_{0,\lambda+\nu}(\varphi,\theta,\gamma) C^{S,T,0}_{\lambda,\nu,0} = (-1)^{\lambda-\frac{1}{2}}\delta_{\lambda,-\nu}, \tag{3.19}$$

$$\frac{1}{4\pi}\sum_{\lambda',\nu'} C^{S,T,0}_{\lambda',\nu',0} D^{\alpha S}_{\lambda'\lambda}(\varphi,\theta,\gamma) D^{\alpha T}_{\nu'\nu}(\varphi,\theta,\gamma) = \frac{1}{\sqrt{2\pi}} D^{0}_{0,\lambda-\nu}(\varphi,\theta,\gamma) C^{S,T,0}_{\lambda,-\nu,0} = -(-1)^{\lambda-\frac{1}{2}}\delta_{\lambda,\nu}, \tag{3.20}$$

here the finite rotation matrices $D^{S}_{\mu'\mu}$ and $D^{T}_{\nu'\nu}$ from the fixed frames to the comoving frames is coupled to a singlet expressed by the well known Kroneker matrix $(-1)^{\mu-\frac{1}{2}}\delta_{\mu,\mp\nu}$, and we use ϵ^{3aj} as the $C^{\frac{1}{2}\frac{1}{2}0}_{\lambda\ \nu\ 0}$ to couple D^S and D^T to obtain $D^{J}_{m,\mu+\nu} s_{\mu} i_{\nu}$, this gives $J=0$, $\delta_{\mu,\pm\nu}$ and so $\kappa = 0$ in eq. (3.14) implicitly. Since the S T chirality have opposite helicity, so there are two terms in (3.17), with opposite sign.

We also note that for the last term of the eq. (3.14), S and T are coupled as

$$\eta_{r^-r^+}\sigma_{r^-} D^{\alpha S}\, |S\rangle \otimes \sigma_{r^+} D^{\beta T}\, |I\rangle = \eta^{r^-r^+}(\sigma^{r^-}|s>^{\alpha}_{\lambda} \underset{ST}{\otimes} \sigma^{r^+}\, |i\rangle^{\beta}_{\nu})(-1)^{\lambda-\frac{1}{2}}(-\delta_{\alpha,\beta}\delta_{\lambda,\nu} + \delta_{\alpha,-\beta}\delta_{\lambda,-\nu}), \tag{3.21}$$

Thus, the D funtion has been simply coupled into the Kronecker matrix.

From constraint for the zero mode sections in (3.13), the original $16 \otimes 16$ tensor product space $|s\rangle^{\alpha}_{\lambda}\, |i\rangle^{\beta}_{\nu}$ is reduced to 8 dimension, including $|s\rangle^{\alpha}_{\lambda}\, |i\rangle^{\alpha}_{\lambda}$ and $|s\rangle^{\alpha}_{\lambda}\, |i\rangle^{-\alpha}_{-\lambda}$ only. Then after factorize out the φ,θ dependent basis $|i\rangle$, $|s\rangle$, we can show as (B16)-(B18) that since for the zero mode section we have further constraint $f^{\alpha\beta}_{\lambda\nu} = -f^{\alpha\beta}_{-\lambda-\nu} \equiv f^{\alpha\beta}$. So only 4 independent component remain only.

Remind that, as has been proved by Hitchin [29] only the left imaginary null $\mathbf{K}_{T^-}$ and real null $\mathbf{K}_{r^-}$ contribute on α plane. The Dirac equation in the cup product frame turns to be

$$\left((\frac{\partial}{\partial r^+} + \frac{1}{r^+})I + \mathbf{K}_{T^-} + \mathbf{K}_{r^-} \right) |\xi\rangle \equiv \left((\frac{\partial}{\partial r^+} + \frac{1}{r^+})I + \mathbb{K}_+ \right) |\xi\rangle = 0, \tag{3.22}$$

$$|\xi\rangle \equiv \left| \begin{array}{c} f^{++} \\ -f^{+-} \\ f^{--} \\ -f^{-+} \end{array} \right\rangle,$$

where $\mathbf{K}_\mu$ are

$$\begin{aligned}
\mathbf{K}_{T^-} &= \sum_{i=1}^{4} \frac{F^i(\mathbb{r})}{\mathbb{r}} \mathbf{E}_{i,i+1}, \quad \mathbf{K}_{T^+} = -\sum_{i=1}^{4} \frac{F^i(\mathbb{r})}{\mathbb{r}} \mathbf{E}_{i+1,i} \\
\mathbf{K}_{r^-} &= \frac{1}{2} \sum_{i=1}^{4} G^i(\mathbb{r})(\mathbf{E}_{i,i} + \mathbf{E}_{i+1,i+1}), \\
F^i(\mathbb{r}) &= \omega^i F(\mathbb{r}), \quad G^i(\mathbb{r}) = \omega^i G(\mathbb{r}),
\end{aligned}$$

here we add an subscript $+$ to $\mathbb{K} = \mathbf{K}_{T^-} + \mathbf{K}_{r^-}$, to denote that it is the whole $\mathbb{K}$ along r^+. Further, we write $\mathbb{A}_+$ instead $\mathbf{K}_+$, since the $D_\mu^{(H)}$ becomes simply $\frac{\partial}{\partial r} + \frac{1}{r}$ in this gauge, by the vanishing of κ. The eq. (3.22) in each i-th $2 \otimes 2$ block generated by the $\mathbf{E}_{i+1,i}$, $\mathbf{E}_{i,i+1}$and $\mathbf{E}_{i,i}$, $\mathbf{E}_{i+1,i+1}$ is obtained in the same way as appendix B. Here we generalize the cup product $\underline{2} \otimes \underline{2} \rightarrow \underline{1} \otimes \underline{3}$ into $(2,1) \otimes ((2,1) \oplus (1,2)) \rightarrow (1,1) \oplus (3,1)$ and conjugate. That is the same decomposition of the quaternion product for the kernel of Dirac operator in SDYM. The outgoing Dirac equation is manipulated in the same way.

$$\langle\xi| \left((\frac{\partial}{\partial \bar{r}^-} + \frac{1}{\bar{r}^-}) I + \mathbb{K}_- \right) = 0 \qquad (3.23)$$

For the consistence of this equation with (3.22) at the intersection points of the outgoing and incoming waves, we should affinize the algebra by the inclusion of the **c** and the **d** term (see next section). The explicit form of $f^{\alpha\beta}(r^+, r^-)$ will be find in next section by solving the scattering equation of the incoming and outgoing waves.

4 Conformal affine massive models, affine ADHMN construction and ’t Hooft Hecke operator

In this section we will sketch how this zero mode solutions in the cup product form (section 3) in the background of the flat connection of the D bundles (section 2) are transformed to the conformal **affine** system by the affine Nahm construction, then turn to the ’t Hooft Hecke operator.

In previous section, by transform from the fixed frame to the tensor product form of the spherical comoving frame, the dependence of $\theta, \varphi, \gamma(\theta, \varphi)$ disappear by “Fourier transformation”, i.e. integrate with the Green function for these variable. Meanwhile we have choose the geodesics coordinate $\gamma = \pm\varphi$ in the north (south) patch of S^2 ($\sim \mathbb{C}P_1$). But the dependence on r^+ r^- remains, so we call it intermediate comoving frame. Now to get the final comoving frame without r^+ r^- dependence. we adopt the following step. We start from the cup product form in (3.22), then, firstly, we rotate the $|\xi\rangle$ in (3.22) by an angle ϕ such that $e^{\frac{1}{2}\phi} = coth(\mathbb{r})$ (c.f(4.2)) , so that D_{r^+} in (3.22) for each G^i, F^i term of the cup product becomes $\left(\frac{\partial}{\partial \mathbb{r}} + \frac{F^i \mathbb{r})}{\mathbb{r}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + G^i(\mathbb{r}) - \frac{1}{\mathbb{r}} \right)$. By noticing that$\left(\frac{d}{d\mathbb{r}}+\left(G^i(\mathbb{r})-\frac{1}{\mathbb{r}}\right)\right)\frac{F^i(\mathbb{r})}{\mathbb{r}}=0$, i.e. the diagonal function are the logarithm derivative of the off diagonal function. In these geodesic r,t fixed frame, we have $\mathbb{A}^i_\pm=\begin{pmatrix} G^i(\mathbb{r})-\frac{1}{\mathbb{r}} & \frac{F^i(\mathbb{r})}{\mathbb{r}} \\ \frac{F^i(\mathbb{r})}{\mathbb{r}} & G^i(\mathbb{r})-\frac{1}{\mathbb{r}} \end{pmatrix}\pm\zeta C$. Then for the outgoing and incoming waves scattering along r_+, r_- direction. The fixed frame $\mathbb{A}_\pm$ is given by integrate the light cone comoving frame $a_\pm$ along the spectral line.

$$\mathbb{A}_\pm=Pexp\int a_\pm(\mathbb{z};\mu),$$

where

$$a_\pm(\mathbb{z};\mu)=a_\pm(\mathbb{z})=\frac{\partial_{z_\pm}\left(G-\frac{1}{\mathbb{z}}\right)}{\left(G-\frac{1}{\mathbb{z}}\right)}\rho^{\mp2}+\left(\frac{\partial_{z_\pm}\left(\frac{F}{\mathbb{z}}\right)}{\left(\frac{F}{\mathbb{z}}\right)}\right)^{\pm1}\rho^{\mp1}E^{\pm1}+\zeta(\mathbb{z})c \tag{4.1}$$

here we have push down $\mathbb{r}=\mu r_++\mu^{-1}\bar{r}_-$ to $z_+=r_+,z_-=r_-$ on worldsheet. This $a_\pm(\mathbb{z})$turns to be the Lax connection $a_\pm(\mathbb{z})=\sum_{i=0}^{3}\partial_{z_\pm}\phi_i\omega^{\pm2i}E_{i,i}+e^{\pm\phi_i}(E^{\pm1})_{i,i\pm1}$, as we substitute in it the one soliton solution of the conformal affine Toda eq., by the following identification

$$\frac{\left(\frac{F(\mathbb{z})}{\mathbb{z}}\right)'}{\frac{F(\mathbb{z})}{\mathbb{z}}}=e^{|\phi_{soliton}|}=\frac{1+e^{2\mathbb{z}}}{1-e^{2\mathbb{z}}}=\coth\mathbb{z},\ \ \phi_i=|\phi|\,\rho_i,$$

$$\eta=\frac{\pi\mathrm{i}}{2},d\eta=0,\zeta=\ln(1-e^{2\mathbb{z}})\equiv\zeta_{sol}-\zeta_{vac}. \tag{4.2}$$

The Lax equation implies the existence of the transport matrix U, $\partial_\pm U=a_\pm(\mathbb{z})U$. Let the monodromy matrix $\mathcal{T}_\pm$, the loop operator of a along $r_\pm=0\to\infty$, $\mathcal{T}_+=\int_0^\infty a(\mathbb{z})dz$, $\mathcal{T}_-=\int_0^\infty a(\mathbb{z})d\bar{z}$, then $\mathcal{T}_\pm$ becomes independent of r,t and satisfy the **affine Nahm eq. or the affine Donaldson's imaginary eq.**

$$\lambda\frac{d\mathcal{T}^\alpha_+(\lambda)}{d\lambda}-[\mathcal{T}^h_+(\lambda),\mathcal{T}^\alpha_+(\lambda)]-\delta_+(s)=0; \tag{4.3}$$

$$-\lambda\frac{d\mathcal{T}^\alpha_-(\lambda)}{d\lambda}-[\mathcal{T}^h_-(\lambda),\mathcal{T}^\alpha_-(\lambda)]-\delta_-(s)=0; \tag{4.4}$$

real eq.

$$\lambda\frac{d\mathcal{T}^\mathfrak{h}_+(\lambda)}{d\lambda}-\frac{d\mathcal{T}^\mathfrak{h}_-(\lambda)}{d\lambda}+[\mathcal{T}^\mathfrak{h}_+(\lambda),\mathcal{T}^\mathfrak{h}_-(\lambda)]+[\mathcal{T}^\alpha_+(\lambda),\mathcal{T}^\alpha_-(\lambda)]-\delta(s)=\zeta(\mathbb{r}_0). \tag{4.5}$$

which depend on $\lambda=e^s$ only, here λ is the loop parameter of $\hat{g}$ and $\tilde{g}$, $\mathcal{T}_\pm(\lambda)=\mathcal{T}^\mathfrak{h}_-(\lambda)\rho+\mathcal{T}^\pm_\pm(\lambda)E^{\pm1}$

$$\delta_\pm(s)=\frac{1}{2}(\delta(s)\pm\frac{i}{\pi}\mathcal{P}\frac{1}{s}).$$

Meanwhile we change the r_+, r_- with respect to the scattering point (blow up point) $\mathbb{r}_0$ by $r_+\to r_+-r_{0+}$, $r_-\to r_--r_{0-}$, $\mathbb{r}_0\equiv\mu r_{0+}+\bar{\mu}r_{0-}$.

The residue of $\mathcal{T}$ around $\lambda=\mu,\bar{\lambda}=\bar{\mu}$ respectively are the E E^{-1} generators of the $A^{(1)}_3$. The parameter μ describes the central $U(1)^c$ of $U(4)$, i.e. the diffeomorphism, area preserving, dual twistor angle $U(1)$ symmetry, which we affinize, and it happens that this serves also as the common dilation, rapidity shift parameter in Rindler coordinate. This dilation operators also rotate the fixed frame $|\Lambda_{\max}\rangle$ to the moving frame $|\xi_{vac}\rangle$. After all this we have reach the affine twistor construction.

It is easy to check that the character of the blow up sheaf $\mathcal{M}_{n,r,k}$ (e.g. in [21]) is the pointygin class $c_2 = n = 1$, rank $r = 4$, 1st chern class $c_1 = k = 1$ case is the one soliton τ function of conformal affine Toda. Which can be factorize into $\left\langle \xi(\bar{\lambda}) \right| \xi(\lambda) \rangle$.

The distributions in fact are given by the trace twisted by the density matrix $\tilde{\rho}$ (4.7)

$$
\begin{aligned}
Trace\,|\xi\rangle\,\Gamma^{\pm}\,\langle\xi| &= \delta_{\pm}(\lambda),\\
Trace\,|\xi\rangle\,\Gamma\,\langle\xi| &= \delta(\lambda),\\
Trace M &= trace\tilde{\rho}M
\end{aligned}
$$

the occurrence of the density matrix, comes from that $|\xi\rangle$ as a state in massive integrable model are non-pure state.

To find the density matrix, besides the $\lambda\frac{\partial}{\partial\lambda} \equiv d$ and c, one should introduce the homotopy operator $\mathfrak{K}$. That is, we are dealing with a massive integrable field theory[3] as the Unrah effect, the bare vacuum should be replaced by introducing the density matrix ρ,

$$
< \xi_{vac}|T(\lambda_2)T(\lambda_1)|\xi_{vac} >= tr[\tilde{\rho}T(\lambda_1)T(\lambda_2)], \tag{4.6}
$$

$$
\tilde{\rho} = e^{2\pi i\mathfrak{K}}, \tag{4.7}
$$

how λ becomes the rapidity in the Rindler coordinates:

$$
\begin{aligned}
&x = r\cosh\alpha, t = r\sinh\alpha,\\
&-\infty < \alpha < +\infty, 0 < r < +\infty.
\end{aligned} \tag{4.8}
$$

We can introduce the rapidity shift operator $\mathfrak{K}$, such that,

$$
e^{\alpha\mathfrak{K}}T(\lambda)e^{-\alpha\mathfrak{K}} = T(\lambda - \alpha). \tag{4.9}
$$

where the shift operator, homotopy Bochner-Martinelli kernel operator $\mathfrak{K}$ is realized as the second terms in the right hand side of the following equation (4.10)

$$
\begin{aligned}
[\hat{X}, \hat{Y}] &= [\tilde{X}, \tilde{Y}]_{\sim} + \frac{1}{2}\oint \frac{d\lambda}{2i\pi} tr(\partial_\lambda \tilde{X}(\lambda)\cdot \tilde{Y}(\lambda))\, C\\
&= [\tilde{X}, \tilde{Y}]_{\sim} + \mathfrak{K}\tilde{X}(\tilde{Y})C,
\end{aligned} \tag{4.10}
$$

here $\hat{X}, \hat{Y} \in \hat{\mathfrak{g}}$, $\tilde{X}, \tilde{Y} \in \tilde{\mathfrak{g}}$.

Actually, the parameter λ is introduced by following Nahm [27], it is the Fourier transformation of time t originally restricted to the static time of BPS monopole in [27], but now generated to r_+, r_- for incoming and outgoing wave along r_+, r_-, which is further push down respectively to z_+, z_-, and further central extend upon scattering point $\mathbb{z}_0$. So this affine parameter is the rapidity, i.e. the Fourier transformation of the Rindler coordinates α. Together with the rotation matrix in the base space $D^S(\theta, \varphi)$ and in the target space $D^T(\theta, \varphi)$. The radial scattering function and the Rindler boost function

[3] We will following the discussion for the quantum massive integrable field by Lukyanov in [19].

expressed by the radial Bessel function gives the Fourier transformation which constitute the affine monopole function for an elementary blow up point.

Now as in [3], we rotate (dressing transform) the wavefunction $|\xi\rangle$ and$\langle\bar{\xi}|$ respectively from the $|\xi_{vac}\rangle$ to $|\xi_{sol}\rangle$, $\langle\bar{\xi}_{vac}|$ to $\langle\bar{\xi}_{sol}|$ at the same time changes a_{vac} to a_{sol} by conjugate by the vertex operator $V(z)$ ('t Hooft Hecke operator). The positive (negative) frequency part $V_-(z)$, $(V_+(z))$ is determined by the Riemann Hilbert transformation.

$$\tilde{V}_- = V_- e^{-\frac{1}{2}\zeta(\mathbb{r})}$$

$$\begin{aligned}\tilde{V}_-^{-1}(z) &= \frac{1}{2}\zeta^{-1}\frac{\partial}{\partial z_+}\zeta + \left(\frac{P_+}{\lambda-\mu} + \frac{P_-}{\lambda+\mu}\right) \\ &\equiv \left(\begin{array}{cc} \sqrt{\frac{e^{\mathbb{r}}+e^{-\mathbb{r}}}{e^{\mathbb{r}}-e^{-\mathbb{r}}}}I & 0 \\ 0 & \sqrt{\frac{e^{\mathbb{r}}-e^{-\mathbb{r}}}{e^{\mathbb{r}}+e^{-\mathbb{r}}}}I \end{array}\right) + \frac{2e^{2\mathbb{r}}}{\sqrt{1-e^{4\mathbb{r}}}}\left(\begin{array}{cc} \mu & \lambda \\ -\lambda & -\mu \end{array}\right)\frac{\mu}{\lambda^2-\mu^2} \qquad (4.11) \\ P_\pm &\equiv \frac{\mu e^{\mathbb{r}}}{\sqrt{1-e^{2\mathbb{r}}}}\left(\begin{array}{cc} \pm I & I \\ -I & \mp I \end{array}\right), \qquad (4.12)\end{aligned}$$

here all matrices are 4×4, the elements in each chiral blocks are 2×2 diagonal. $P_\pm$ is diagonal along the rotation axis for the operator V with rotation angle $\frac{1}{2}\varphi_{sol} = \coth(\mathbb{z})$.

Remark: The right hand side of this R. H. eq. is similar as the following operators in Hitchin's paper [18](p51)

$$i\frac{d}{d\lambda} + \left(\frac{1}{\lambda-1} + \frac{1}{\lambda+1}\right)T. \qquad (4.13)$$

The Hirota e.q. of τ, gives the background dependent holomorphic anomaly. After sum over various representation of $|\xi\rangle$ Nakajima [21] obtain the quantum τ. But the perturbative factor of the character (partition function), has been get by conjecture to fit the large N approximation. We have include the central extension with center **C** lies in the center $\mathbb{Z}$ of $U(4)$, to construct the universal bundle. This extension by homotopy is the way of nonabelian localization as calculated by Beasley and Witten for the Seifert manifold [25]

We may match our affine Nahm eq. of the 't Hooft Hecke operator with the Nahm eq. of the surface operator given by Gukov and Witten [26], since we have both 1st and 2nd chern class c_1 and c_2.

5 Discussion and Outlook

In fact eq. (4.10) is the integral kernel of the Bochner-Martinelli formula. From this, the reside theorem will give the Poincare-Hopf localization index, i.e. the analytic index. After doing the Thom isomorphism, the representative of this index can be written as the integral of the A-roof genus by using the Bott reside formula. This is just the topological index.

In fact, the analytic index is just the anti-holomorphic function on the sheaf $T(x)$ and $\psi(x)$ can be expressed using the $T(x)$ in [20]. The one corresponds to the real null vector, $T_0 \pm T_1$, is just B_1, and the one corresponds to the imaginary vector, $T_2 \pm iT_3$, is just B_2, and $\oint |\psi><\psi| d\lambda$ is just $i^*j + j^*i$. The character for the one affine monopole solution is just the one in [20] with $r = 2, n = 4, k = 1$.

If we use the Riemann-Hilbert transformation with N poles, we will get the N-soliton solution. The corresponding character is the character for the sheaf with the Young diagram that satisfies $|Y| = N$ in [20].

Now we turn to the problem of quantization. First we should use the Seiberg-Witten curve to determine the cut-off constant Λ, here the Seiberg-Witten curve is just the spectral curve of the Lax connections in the affine Toda system. There are four formalisms for quantization. The first one is the quantum group formalism, such as the massive integrable field theory (the one studied in [19]). The bosonic oscillator representation is given in [10] and [13] (here the relevant quantum group is $\hat{Sl}_4(p, q)$). In the second formalism, twistor and sheaf are used [20, 21]. The third formalism is the field theoretic formalism, where the hyperkahler quotient plays a important role. In the last one, the spectral curve in the moduli space form a Calabi-Yau 3-fold. Here, in the commutation relation, we should use $\phi_{quantum}$ to replace the ϕ_{vacuum} [22].

In the quantum group formalism, beside expanse the $\phi_{vaccuum}$ by the q-oscillator, we also obtain the exact explicit commutation relation of the vertex operator. We can obtain the exact result in the thermodynamics limit which is not an approximate result in the large N limit. Moreover, the two kinds of vertex operators in the quantum group formalism are corresponding to the Wilson loop and the 't Hooft loop in Yang-Mills theory, respectively. Modular transformation will exchange both these two kinds of operators and the corresponding states characterize respectively the order parameter and disorder parameter. This is also called the level-rank duality.

The area-preserving transformation of the dual twist angle φ ($\lambda = e^{i\varphi}$) is just the diffeomorphism transformation. So it will connect the phase angle φ of the center term of the topological field theory and the attractor parameter μ of the black hole entropy.

The quasi-Hopf quantum double of the Drinfeld's quantum group not only realizes the almost factorization in [23], but also give the center term of the holomorphic anomaly, which is obtained by solving the Hirota equation. In this situation, the polarization of the waving function can be used to explain the background independence.

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Appendix A. BPS monopole

The static spherical symmetric ansatz [4]

$$A_i^a(x) = \epsilon_{iaj}\frac{F(r)-1}{r}\hat{r}^j, \tag{A1}$$

$$A_0^a(x) = iG(r)\hat{r}^a(x), \tag{A2}$$

where $r = (x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}$, the unit radial vector $\hat{r}^j = \frac{r^j}{r}$, the space spin index $i, j = 1, 2, 3$, the "isospin" $SU(2)$ generator index $a = 1, 2, 3$. This ansatz is $U(1)$ symmetrical under the cooperative local rotation generated by both the space time spin $\hat{s}(x) = \hat{r}$ and the $SU(2)$ isospin $\mathbf{T}^r(x) = \sum\limits_a T^a\hat{r}^a$.

Decompose A as

$$A_\mu = H_\mu + K_\mu, H_i^a = -\varepsilon_{iaj}\frac{1}{r}\hat{r}^j, K_i^a = \epsilon_{iaj}\frac{F(r)}{r}\hat{r}^j, K_0^a = iG(r)\hat{r}^a, H_0^a = 0. \tag{A3}$$

Let[5] $D_\mu = \partial_\mu - iA_\mu = D_\mu^{(H)} - iK_\mu$,
herein

$$D_\mu^{(H)} = \partial_\mu - iH_\mu. \tag{A4}$$

Then we have that, $\mathbf{T}^r(x)$ is covariant constant under the $SU(2)$ "isospin" rotation; $D_i^{(H)}\mathbf{T}^r(x) = (\partial_i + iH_i^a T^{[a})(\sum_b T^{b]}\hat{r}^b) = \partial_i\hat{r}^a T^a - i\epsilon_{iaj}\hat{r}^j\hat{r}^b[T^a, T^b] = 0$; $D_0^{(H)}\mathbf{T}^r(x) = 0$; and meanwhile $\nabla_i\hat{r}^a = 0$ under the action of spin connection along the surface S^2 ($r = constant$).

Transform from the spacetime fixed frames in "spherical basis" [28] $e^M (M = 0, +1, -1)$:

$$e^0 = dx^3, e^{+1} \equiv -\frac{1}{\sqrt{2}}(dx^1 + idx^2), e^{-1} = \frac{1}{\sqrt{2}}(dx^1 - idx^2), \tag{A5}$$

to the local comoving frames on S^2 :

$$\mathfrak{e}^{T_+} = -\frac{1}{\sqrt{2}}re^{-i\gamma}\left(d\theta + \mathrm{i}\sin\theta d\varphi\right), \mathfrak{e}^{T_-} = \frac{1}{\sqrt{2}}re^{i\gamma}\left(d\theta - \mathrm{i}\sin\theta d\varphi\right), \tag{A6}$$

together with the normal vector $\mathfrak{e}^r = dr$ and time $\mathfrak{e}^t = dt$, $\gamma = \mp\varphi$ in the north (south) patch. In the spherical basis $\mathfrak{e}^m (m = 0, +1, -1)$, $\mathfrak{e}^0 \equiv \mathfrak{e}^r$, $\mathfrak{e}^{\pm 1} \equiv \mathfrak{e}^{T_\pm}$, we have

$$\mathfrak{e}^m = \sum_{M=0,\pm 1} D^1_{Mm}(\varphi, \theta, \gamma)e^M, \tag{A7}$$

change the basis of the fixed frame $SU(2)$ generator from the cartesian basis T^a ($a = 1, 2, 3$) to that in the spherical basis T^M, then gauge transform to the comoving frame $\mathbf{T}^m(x)$

$$\mathbf{T}^m(x) = \sum_{M=0,\pm 1} D^1_{Mm}(\varphi, \theta, \gamma)T^M \tag{A8}$$

[4] For the BPS monopole which is constituted by the Higgs Yang-Mills field or spontaneously breaking Yang-Mills field, the A_0^a should be replaced by the Higgs field $i\Phi^a$ and the ASD equation by the BPS equation.

Thus, we adopt the notation (A2) in view of further embedding into the SUSY affine monopole as the incoming and outgoing waves in the Sec. 2 and Sec. 3. There, in case of $d = 4$ SUYM, our ansatz A_μ turns to be the $\mathcal{A}_\mu$ of paper [2], $\mathcal{A}_\mu = A_\mu + i\phi_{\mu+4}$ with $A_0 = \phi_5 = \phi_6 = \phi_7 = 0$.

[5] In appendices, we adopt the physics convention with a Hermitian gauge field A_μ.

e.g.

$$\mathbf{T}^r(x) = \mathbf{T}^0(x) = \sum_{a=1,2,3} \hat{r}^a T^a,$$

underline $\mathbf{e}$ ($\mathbf{T}$) denotes the vector (generator) in comoving frame, here the letters from the beginning $(a, b, c = 1, 2, 3)$ and from the middle $(M, m = -1, 0, +1)$ of the alphabet denote the indices of orthogonal basis and spherical basis respectively. Capital letter $(M = -1, 0, +1)$ and Lowercase $(m = -1, 0, +1)$denote the indices of the fixed frame and the comoving frame respectively. Gauge transform $A_\mu(x)$ from the fixed gauge to the comoving gauge, The connection H_μ in (A3) gauge transform to the $U(1)$ Dirac potential (A9) while matrix tensor K_μ transform covariantly to (A10),

$$\mathbf{H}^0_{T_\pm} = \mathbf{A}^0_{T_\pm} = \frac{\mp\frac{\partial\gamma}{\partial\varphi} + \cos\theta}{r\sin\theta}, \tag{A9}$$

$$-\mathbf{K}^-_{T_+}(x) = -\mathbf{A}^-_{T_+}(x) = \mathbf{K}^+_{T_-}(x) = \mathbf{A}^+_{T_-}(x) = -\frac{i}{\sqrt{2}}\frac{F(r)}{r}, \mathbf{K}^0_t(x) = \mathbf{A}^0_t(x) = G(r), \tag{A10}$$

$$A(x) = \sum_{\substack{m=0,\pm1 \\ n=T_+,T_-,t}} \mathbf{A}^m_n(x)\mathbf{e}^n(x)\mathbf{T}^m(x). \tag{A11}$$

Separate $F_{\mu\nu}$ into the radial and the tangential electric and magnetic components

$$\begin{aligned} F^a_{i0} &= E_r(r)x_i x^a/r^2 + E_T(r)(\delta^a_i - x_i x^a/r^2), \\ \varepsilon_{ijk}F^a_{ij} &= M_r(r)x_k x^a/r^2 + M_T(r)(\delta^a_k - x_k x^a/r^2). \end{aligned} \tag{A12}$$

From eq. (A11) for $A(x)$, it is easy to see that the radial component $M_r(x)$ can be find by using the generalize Gauss equation (C9) for the reduced curvature;

$$M_r(x) = 2\mathbf{K}^-_+\mathbf{K}^+_- + F^{(H)}_{+-}.\mathbf{T}^0 = \frac{F^2(r)-1}{r^2}, \tag{A13}$$

where the $F^{(H)}_{+-} = \frac{1}{r^2}\mathbf{T}^0$ is the magnetic field of the Dirac monopole (A9) $\mathbf{T}^a.\mathbf{T}^b \equiv \frac{1}{2}Tr\mathbf{T}^a.\mathbf{T}^b$; the radial $E_r(x)$ and the tangential $E_T(x)$, $M_T(x)$ can be find by using the generalize Codazzi equation (C10)

$$E_r(x) = G'(r), \qquad E_T(x) = \frac{G(r)F(r)}{r}, \qquad H_T(x) = \frac{F'(r)}{r}. \tag{A14}$$

The normalizable solution of the anti-self-dual equation

$$F'(r) = G(r)F(r), \qquad G'(r) = \frac{F^2(r)-1}{r^2}, \tag{A15}$$

is

$$F(r) = \frac{\mu r}{\mathtt{sh}\,(\mu r)}, \qquad G(r) = \frac{1}{r} - \mu\mathtt{cth}(\mu r), \tag{A16}$$

which is holomorphic, and satisfies the condition $E_r = M_r \sim O(1) \Rightarrow F(r) \sim O(1)$, $G(r) \sim O(\frac{1}{r})$ as $r \to 0$, and asymptotically $E_r = M_r \sim \frac{1}{r^2}$, $E_T = M_T \sim O\left(\frac{1}{r^2}\right) \Rightarrow G \sim 1$, $F(r) \sim p(r)e^{-\mu r}$, here $p(r)$ is a polynomial.

Appendix B. The zero mode of fermion in static spherical symmetric BPS monopole

Dirac equation

$$\gamma_\mu D^\mu \Psi(x) = 0, \tag{B1}$$

where $D_\mu = \partial_\mu - i\sum_a T^a A^a_\mu$ with A_μ given by (A1) and (A2); $\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}$, $\gamma_0 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$,

$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ in the γ_5 diagonal representation.

Let $\Psi(x) = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}$ be static, then we get two decoupled 2-component equations with opposite chirality.

$$(\sigma_j D_j \mp G(r)\mathbf{T}^r(x))\chi^\pm = 0. \tag{B2}$$

Introduce the spin shift operator κ by couple the spin with the orbital momentum

$$\kappa = -\epsilon_{ijk}\sigma_i x_j D^{(H)}_k + 1, \tag{B3}$$

then

$$\sigma_i D^{(H)}_i = \sigma_r(\frac{\partial}{\partial r} + \frac{1}{r}) - \frac{1}{r}\sigma_r\kappa, \quad \sigma_r \equiv \sum_i \sigma_i \hat{r}_{i.} \tag{B4}$$

So in the Dirac operator besides the spin shift operator κ only the radial derivative remains.

$$[\sigma_r(\frac{\partial}{\partial r} + \frac{1}{r}) - \frac{1}{r}\sigma_r\kappa - i\epsilon_{ijk}\frac{F(r)}{r}\hat{r}_i\sigma_j T_k \mp G(r)\mathbf{T}^r]\chi^\pm = 0. \tag{B5}$$

In the spherical symmetric coordinate, let

$$\chi^\pm = \sum_J \sum_{\mu,\nu} f^{\pm J}_{\mu,\nu}(r)\mathbb{D}^J_{\mu,\nu}(\varphi,\theta,\gamma), \tag{B6}$$

$$\mathbb{D}^J_{\mu,\nu}(\varphi,\theta,\gamma) \equiv \sqrt{\frac{2J+1}{4\pi}} D^J_{M,\mu+\nu} s_\mu i_\nu, \quad M = -(2J+1), \cdots, 2J+1, \text{no summation for } \mu,\nu. \tag{B7}$$

$$s_\mu(\varphi,\theta,\gamma) = \sum_{\mu'} \sqrt{\frac{1}{2\pi}} D^S_{\mu',\mu}(\varphi,\theta,\gamma) S_{\mu'}, \quad \sigma_3 S_\mu = 2\mu S_\mu, S = \frac{1}{2} \tag{B8}$$

$$i_\nu(\varphi,\theta,\gamma) = \sum_{\nu'} \sqrt{\frac{1}{2\pi}} D^I_{\nu',\nu}(\varphi,\theta,\gamma) I_{\nu'}, \quad T_3 I_\nu = 2\nu I_\nu, I = \frac{1}{2} \tag{B9}$$

and

$$\sigma_r s_\mu = 2\mu s_\mu, \qquad T_r i_\nu = \nu i_\nu \tag{B10}$$

$\sigma_m(x) = \sum_{M=0,\pm1} D^1_{Mm}(\varphi,\theta,\gamma)\sigma^M$, $\sigma^r(x) = \sigma^0(x) = \sum_{a=1,2,3} \hat{r}^a\sigma^a$, or $(\sigma^a(x))_{\lambda\nu} = D^{\frac{1}{2}}_{\lambda\mu}(\varphi,\theta,\gamma)(\sigma^a)_{\mu\rho} D^{*\frac{1}{2}}_{\rho\nu}(\varphi,\theta,\gamma)$

here as in (A8) for the isospin generator $\mathbf{T}$, we introduce the spin generators $\sigma_1(x), \sigma_2(x), \sigma_3(x)$

in the comoving frames, later when we consider the spherical case, we always write σ_m simply, by omitting the argument x.

Then the $J=0,1$ component $\mathbb{D}^J_{\mu,\nu}(\varphi,\theta,\gamma)$ of the wave function χ satisfy

$$J^2\mathbb{D}^J_{\mu,\nu}=J(J+1)\mathbb{D}^J_{\mu,\nu},\qquad J=0,1$$

$$J_3\mathbb{D}^J_{\mu,\nu}=M\mathbb{D}^J_{\mu,\nu},\quad M=-(2J+1),...,2J+1$$

$$\sigma_r\mathbb{D}^J_{\mu,\nu}=2\mu\mathbb{D}^J_{\mu,\nu},\quad \mu,\nu=\pm\frac{1}{2}$$

$$T_r\mathbb{D}^J_{\mu,\nu}=\nu\mathbb{D}^J_{\mu,\nu},$$

$$\kappa\mathbb{D}^J_{\mu,\nu}=\kappa_\mu\mathbb{D}^J_{-\mu,\nu},$$

$$\varepsilon_{ijk}\hat{r}_i\sigma_j\mathbf{T}_k\mathbb{D}^J_{\mu,\nu}=-i2\mu\alpha^{\frac{1}{2}}_{\nu+\frac{1}{2}+\mu}\mathbb{D}^J_{-\mu,\nu+2\mu},$$

where

$$\alpha^I_\nu=(I+\nu)^{\frac{1}{2}}(I-\nu+1)^{\frac{1}{2}},\kappa_\nu=\alpha^J_{|\nu|+\frac{1}{2}}=\sqrt{(J+\frac{1}{2})^2-\nu^2}. \tag{B11}$$

So the radial components $f^{\pm J}(r)$ satisfy

$$(\frac{\partial}{\partial r}+\frac{1}{r})f^{\pm J}_{\mu,\nu}(r)-\frac{\kappa_\mu}{r}f^{\pm J}_{-\mu,\nu}(r)+\alpha^I_{\nu+\frac{1}{2}+\mu}\frac{F(r)}{r}f^{\pm J}_{-\mu,\nu+2\mu}(r)\mp 2\mu\nu G(r)f^{\pm J}_{\mu,\nu}(r)=0. \tag{B12}$$

The convergence at $r\to\infty$, requires $\mp 2\mu\nu G(r)>0$ asymptotically. Thus for the

$\gamma^5=1\quad(-1)$ solutions $f^{+J}_{\mu,\nu}\quad(f^{-J}_{\mu,\nu})$, $\mu\nu$ always >0 (<0). But then the $\frac{\kappa_\nu}{r}f^{\pm J}_{-\mu,\nu}$ term disobey this condition, so we should require $\kappa_\mu=0$. Thus J is restricted to be 0.The superscript J will be dropped later. This implies the well-known fact that for the zero mode, the total spin, which is contributed by the space spin and the isospin induced by the field, cancels. Based on the same reason in section 3, we simply use the $J=0$ component in the tensor product of S and T. From $F(r)f^{\pm}_{-\mu,\nu+2\mu}$ term , $-\mu(\nu+2\mu)$ should be >0 (<0) for $f^+(f^-)$. Hence $\nu=-\mu$ and $f^+=0$, only f^- is convergent, it satisfies

$$(\frac{\partial}{\partial r}+\frac{1}{r})f^-_{\mu,-\mu}(r)-\frac{F(r)}{r}f^-_{-\mu,\mu}(r)-\frac{1}{2}G(r)f^-_{\mu,-\mu}(r)=0$$

i.e.

$$\begin{aligned}(\frac{\partial}{\partial r}+\frac{1}{r})f^-_{\frac{1}{2},-\frac{1}{2}}(r)-\frac{F(r)}{r}f^-_{-\frac{1}{2},\frac{1}{2}}(r)-\frac{1}{2}G(r)f^-_{\frac{1}{2},-\frac{1}{2}}(r)=0,\\ (\frac{\partial}{\partial r}+\frac{1}{r})f^-_{-\frac{1}{2},\frac{1}{2}}(r)-\frac{F(r)}{r}f^-_{\frac{1}{2},-\frac{1}{2}}(r)-\frac{1}{2}G(r)f^-_{-\frac{1}{2},\frac{1}{2}}(r)=0.\end{aligned} \tag{B13}$$

The unique convergent solution is

$$f^-(r)=f^-_{\frac{1}{2},-\frac{1}{2}}(r)=-f^-_{-\frac{1}{2},\frac{1}{2}}(r)\simeq r^{-\frac{1}{2}}(sh\frac{\beta r}{2})^{\frac{1}{2}}(ch\frac{\beta r}{2})^{-\frac{3}{2}}.$$

which satisfy

$$(\frac{\partial}{\partial r}+\frac{1}{r})f^-(r)+\frac{F(r)}{r}f^-(r)-\frac{1}{2}G(r)f^-(r)=0. \tag{B14}$$

In fact, we have a local (comoving) homotopy isomorphism between the spin s_μ and isospin i_ν in the reduced cup product $\underline{2}\,\dot{\underset{ST}{\otimes}}\,\underline{2} \overset{cup\ product}{\Longrightarrow} \underline{1}$, in the tensor product $s_\mu \otimes i_\nu$.

$$\underline{2}\underset{ST}{\otimes}\underline{2}=\underline{1}\oplus\underline{3}, \tag{B15}$$

the nondegenerate **zero mode** lies in $\underline{1}$ only. So we simply have

$$\chi^-=\sum_{\mu\nu} f^-_{\mu\nu}s^-_\mu i_\nu(-1)^{\mu-\frac{1}{2}}\delta_{\mu,-\nu} \tag{B16}$$

by using

$$\frac{1}{4\pi}\sum_{\mu',\nu'}C^{S,T,J}_{\mu',\nu',M}D^S_{\mu'\mu}(\varphi,\theta,\gamma)D^T_{\nu'\nu}(\varphi,\theta,\gamma)=\frac{1}{\sqrt{2\pi}}D^J_{M,\mu+\nu}(\varphi,\theta,\gamma)C^{S,T,J}_{\mu,\nu,0}=(-1)^{\mu-\frac{1}{2}}\delta_{\mu,-\nu}, \tag{B17}$$

$$S=T=\frac{1}{2}, J=0.$$

Here, the finite rotation matrices $D^S_{\mu'\mu}$ and $D^T_{\nu'\nu}$ from the fixed frames to the comoving frames is coupled to a singlet expressed by the well known Kroneker matrix $(-1)^{\mu-\frac{1}{2}}\delta_{\mu,-\nu}$.

The fixed frame Dirac eq (B5) turns to be the eq.(B12) in the **tensor product comoving frame**, the last two terms comes from (B12)

$$\gamma_i K_i=\gamma_i K^a_i T^a=\gamma_i\epsilon_{ija}\frac{F(r)}{r}\hat{r}_j T_a,$$
$$\gamma_0 K_0=\gamma_0\delta^{ab}K^a_0 T^b=\gamma_0\delta^{ab}iG(r)\hat{r}^a T^b.$$

And further turn to be that in the **cup product form**, that is

$$\chi^-\text{ part: }(i\epsilon_{3jk}\sigma_j\mathbf{T}_k\frac{F(r)}{r})\chi^-=i(\sigma_1\mathbf{T}_2-\sigma_2\mathbf{T}_1)\frac{F(r)}{r}\chi^- \overset{cup\ product}{\Longrightarrow}\frac{i}{2}(\mathbf{E}^2_1-\mathbf{E}^1_2)\frac{F(r)}{r}|f^-\rangle,$$
$$G(r)\hat{r}^aT^a\chi^- \overset{cup\ product}{\Longrightarrow}\frac{1}{2}G(r)(\mathbf{E}^1_1-\mathbf{E}^2_2)|f^-\rangle,$$

where the cup product basis matrix $(\mathbf{E}^j_i)_{\alpha\beta}=\delta_{i\alpha}\delta_{j\beta}$. Notice that the radial gradient term has a σ_r, which turns to $\frac{1}{2}(\mathbf{E}^1_1-\mathbf{E}^2_2)$ also, so we have at last the zero mode equation in cup product form (B13)

$$\left((\frac{\partial}{\partial r}+\frac{1}{r})I-\frac{F(r)}{2r}\begin{pmatrix}0&1\\1&0\end{pmatrix}-\frac{1}{2}G(r)I\right)|f^-\rangle=0,$$
$$|f^-\rangle=\frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}f^-_{\frac{1}{2},-\frac{1}{2}}(r)\\f^-_{-\frac{1}{2},\frac{1}{2}}(r)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}f^-_{\frac{1}{2},-\frac{1}{2}}(r)-f^-_{-\frac{1}{2},\frac{1}{2}}(r)\\-f^-_{\frac{1}{2},-\frac{1}{2}}(r)+f^-_{-\frac{1}{2},\frac{1}{2}}(r)\end{pmatrix} \tag{B18}$$

Here we have factorize out the spherical dependence of the wave function in the reduced cup product frame

$$s^-_-i_+ \quad - \quad s^-_+i_-,$$

(the superscript− of s denote the chirality).

By(A10) we have

$$\frac{i}{2}(\mathbf{E}_1^2-\mathbf{E}_2^1)\frac{F(r)}{r}=\frac{1}{2}(\mathbf{E}_+^- -\mathbf{E}_-^+)\frac{F(r)}{r}\Rightarrow\frac{1}{2}(\mathbf{K}_+^- -\mathbf{K}_-^+)=\mathbf{K}_+^-.$$

so at last (B18) turns to be

$$((\frac{\partial}{\partial r}+\frac{1}{r})+\mathbf{K}_{T^+}-\frac{1}{2}\mathbf{K}_t)|f^-\rangle=0$$

i.e. the eq. (B14).

Remark: Following Nahm [27] by Fourier transform to the momentum space, but different from Nahm, we adopt the light cone spherical frame coordinates, then the $|f^-\rangle$ becomes the holomorphic sheaf, i.e. the mini-twistor [26 p584]. Here, the geodesic flow along the spectral line, left real null line, $\nabla_r-\frac{1}{2}\mathbf{K}_t\sim\nabla_U-i\Phi$ [29], and the $\mathbf{K}_{T^+}\sim\nabla_x+i\nabla_y=\bar{\partial}$ [29], spans the left null plane (α plane), as the natural flat connection on it.

Appendix C. The generalized Gauss Codazzi equation

1. Condition of reducibility.

The necessary and sufficient condition for the gauge field with group G defined on space-time manifold M to be reducible into gauge field with subgroup H, may be formulated as follows: A connection on the principal bundle $P(M,G)$ can be reduced into the connection of subbundle $Q(M,H)$, when and only when the associated bundle $E(M,G/H,G)$ have a section $\mathbf{n}(x):M\to G/H$, which is invariant under parallel displacement. In order to employ this condition, we must find out proper expression for G/H, consequently we decompose the left invariant algebra g of group G canonically into $g=\mathfrak{h}+m$, where $\mathfrak{h}$ is the subalgebra corresponding to the stationary subgroup of the element on G/H. Observing the natural correspondence between G/H and the subspace spanned by $\mathfrak{h}$ in the left invariant Lie algebra, we may perform the reduction as follows.

2. Gauge field with group $G=SU(2)$ which may be abelianized into $H=U(1)$. in this case, $\mathfrak{h}$ is one dimensional everywhere, and its normalized base is taken as $\hat{\mathbf{n}}$, $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\equiv-2tr(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})=1$. Then the set of $\hat{\mathbf{n}}$ makes a unit sphere $S^2\sim SU(2)/U(1)$ in the space of adjoint representation. The section $\hat{\mathbf{n}}(x)$ is the mapping of space-time M(except the singular point) onto S^2. This unit isospin field is invariant under the parallel displacement by gauge potential $\mathbf{A}_\mu(x)$,

$$\tilde{\nabla}\hat{\mathbf{n}}(x)\equiv\partial_\mu\hat{\mathbf{n}}(x)+e[\mathbf{A}_\mu(x),\hat{\mathbf{n}}(x)]=0$$

$$\mathbf{A}_\mu(x)\in g,\quad \mu=0,1,2,3 \qquad (C1)$$

where, under infinitesimal gauge transformation

$$\mathbf{A}'_\mu(x)=\mathbf{A}_\mu(x)+e[\mathbf{A}_\mu(x),\alpha(x)]+e\partial_\mu\alpha(x)$$

$$\alpha(x)\in g \qquad (C2)$$

Using identity $\mathbf{V} = (\mathbf{V}\cdot\mathbf{N})\mathbf{N} + [\mathbf{N},[\mathbf{V},\mathbf{N}]]$, we can easily see that the necessary and sufficient condition of (C1) is

$$\mathbf{A}_\mu = (\mathbf{A}_\mu \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \frac{1}{e}[\hat{\mathbf{n}}, \partial_\mu \hat{\mathbf{n}}] \qquad (C3)$$

Here, the potential is $SU(2)$ formally, but in reality it may be transformed at least locally into $U(1)$ potential with a constant $\hat{\mathbf{n}}(x)$ as the generator, i.e. the gauge could be chosen to turn $\hat{\mathbf{n}}(x)$ into the same direction in some region of x, $\partial_\mu\hat{\mathbf{n}}(x) = 0$. Then $\mathbf{A}'_\mu(x)$ becomes explicit Abelian, i.e., it equals $(\mathbf{A}'\cdot\mathbf{n})\mathbf{n}$ in this region. If we fix the direction of $\hat{\mathbf{n}}(x)$, but rotate a gauge angle $\Gamma(x)$ around $\hat{\mathbf{n}}(x)$, then we obtain the $U(1)$ transform generated by $e\hat{\mathbf{n}} : \mathbf{A}'_\mu = \mathbf{A}_\mu + e\hat{\mathbf{n}}\partial_\mu\Gamma$.

Substituting (C3) into

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + e[\mathbf{A}_\mu, \mathbf{A}_\nu] \qquad (C4)$$

we get

$$\begin{aligned}\mathbf{F}_{\mu\nu} &= [\partial_\mu(\mathbf{A}_\nu\cdot\hat{\mathbf{n}}) - \partial_\nu(\mathbf{A}_\mu\cdot\hat{\mathbf{n}})]\hat{\mathbf{n}} - \frac{1}{e}[\partial_\mu\hat{\mathbf{n}}, \partial_\nu\hat{\mathbf{n}}] \\ &= (\mathbf{F}_{\mu\nu}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} \end{aligned} \qquad (C5)$$

In the region where $\hat{\mathbf{n}}(x)$ is well defined, $\mathbf{F}_{\mu\nu}\cdot\hat{\mathbf{n}}$ satisfies

$$\partial^\mu({}^*\mathbf{F}_{\mu\nu}\cdot\hat{\mathbf{n}}) = \tilde{\nabla}^\mu({}^*\mathbf{F}_{\mu\nu}\cdot\hat{\mathbf{n}}) = (\tilde{\nabla}^{\mu*}\mathbf{F}_{\mu\nu})\cdot\hat{\mathbf{n}} +^* \mathbf{F}_{\mu\nu}\tilde{\nabla}^\mu\hat{\mathbf{n}} = 0 \qquad (C6)$$

where $*\mathbf{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\mathbf{F}^{\lambda\rho}$.Locally $\mathbf{F}_{\mu\nu}\cdot\hat{\mathbf{n}}$ is the same as the ordinary electromagnetic field without magnetic charge, and in explicit Abelian gauge it may be expressed by the $U(1)$ potential $\mathbf{A}_\mu\cdot\hat{\mathbf{n}}$; $\mathbf{F}_{\mu\nu}\cdot\hat{\mathbf{n}} = \partial_\mu(\mathbf{A}_\nu\cdot\hat{\mathbf{n}}) - \partial_\nu(\mathbf{A}_\mu\cdot\hat{\mathbf{n}})$. But, globally its magnetic flux through some two dimensional space like close surface M' may be non-zero,

$$\frac{1}{2}\iint_{M'} \mathbf{F}_{\mu\nu}\cdot\hat{\mathbf{n}}dx^\mu \wedge dx^\nu = \frac{l}{e}\iint_{S^2}\hat{\mathbf{n}}\cdot[d\hat{\mathbf{n}}, \delta\hat{\mathbf{n}}] = -\frac{4\pi l}{e} \qquad (C7)$$

Here integer l is the times by which the surface M' covers the isospin sphere S^2 through the mapping $\hat{\mathbf{n}}(x)$. Physically it is the quantum number of the magnetic charge surrounded by the surface M'. If $l \neq 0$, it is impossible to turn $\hat{\mathbf{n}}(x)$ into one and the same direction globally by non-singular single-valued gauge transformation. Then there must be either singularity or overlapping regions with transition function, the corresponding Abelian potential being the Dirac-Schwinger potential with string or the Wu-Yang global potential. Above all, the characteristic $\pi_1(S^1)$ of bundle $Q(M,H)$ of $U(1)$ gauge field corresponds one to one to $\pi_2(S^2)$ of the section $\mathbf{n}(x)$ on the associated coset bundle of $SU(2)$, $\pi_1(S^1) \sim \pi_2(S^2)$. Their common characteristic number is determined physically by the dual charge.Mathematically $\hat{\mathbf{n}}(x)$ is the generator of the holonomy group of $P(M,G)$ under given connection;physically e$\hat{\mathbf{n}}(x)$ is the charge operator;abelianizable $\mathbf{A}_\mu$ is the potential of pure electromagnetic field $\mathbf{F}_{\mu\nu}$; and meantime, $\hat{\mathbf{n}}$ is the common isodirection of the six space-time components of $\mathbf{F}_{\mu\nu}$.

3.Non-abelianizable $SU(2)$ potential $\mathbf{A}_\mu(x)$ and field strength $\mathbf{F}_{\mu\nu}(x)$. Now, the holonomy group are whole $SU(2)$, thus it is impossible to choose from its generators some $\hat{\mathbf{n}}(x)$ which remains invariant under parallel displacement. The noninvariant section $\hat{\mathbf{n}}(x)$ must be given otherwise. Physically,as the charge operator, $\hat{\mathbf{n}}(x)$ is the phase axis of wave functions of charged particles, or the isodirection of its current vector,or $\hat{\mathbf{n}}(x) = \phi(x)/|\phi(x)|$, where $\phi(x)$ is the Higgs particle. In pure gauge field without other particles, $\hat{\mathbf{n}}(x)$ may be the privileged direction determined by the intrinsic symmetry of the field, e.g. the generator of the stationary subgroup H for G invariant connection. (In case of synchronous of space spin and isospin spherical symmetry field as the BPS monopole, the privileged direction is synchronous with the vector radius $\hat{\mathbf{r}}$.)

Once a section $\hat{\mathbf{n}}(x)$ is given, it determines a corresponding subbundle $Q(M,H)$, whose characteristic class is decided by the homotopic property of $\hat{\mathbf{n}}(x)$.Physically, as soon as the charge operator $\hat{\mathbf{n}}(x)$ is given, one can separate the $U(1)$ Dirac component $\mathbf{H}_\mu(x)$ from the $SU(2)$ potential $\mathbf{A}_\mu(x)$ as follows: Here $\mathbf{H}_\mu(x)$ is the $U(1)$ gauge potential with $\hat{\mathbf{n}}(x)$ as the generator. Now from $\widetilde{\nabla}_\mu\hat{\mathbf{n}} \equiv \partial_\mu\mathbf{n} + e[\mathbf{A}_\mu, \hat{\mathbf{n}}]$ which is not vanishing now, we get

$$\mathbf{A}_\mu = (\mathbf{A}_\mu \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \frac{1}{e}[\hat{\mathbf{n}}, \partial_\mu\hat{\mathbf{n}}] + \frac{1}{e}[\hat{\mathbf{n}}, \nabla_\mu\hat{\mathbf{n}}] \equiv \mathbf{H}_\mu + \mathbf{K}_\mu \qquad (C8)$$

Here we have set$\mathbf{H}_\mu \equiv (\mathbf{A}_\mu \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \frac{1}{e}[\hat{\mathbf{n}}, \partial_\mu\hat{\mathbf{n}}]$.It is easy to prove that $\mathbf{H}_\mu$ satisfies(C1)-(C3),hence it is the U(1) part in $\mathbf{A}_\mu$, the remainder $\frac{1}{e}[\hat{\mathbf{n}}, \nabla_\mu\hat{\mathbf{n}}] \equiv \mathbf{K}_\mu$ is gauge covariant and represents the charged vector particles. Geometrically $e\mathbf{K}_\mu$corresponds to the second fundamental form, e.g.$\nabla_\mu\hat{\mathbf{n}} = [e\mathbf{K}_\mu, \hat{\mathbf{n}}]$ is the generalized Weingarten formula (since $|\hat{\mathbf{n}}| = 1$, the "normal" component is absent). Substituting (C8) into $\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + e[\mathbf{A}_\mu, \mathbf{A}_\nu]$ and making comparison with (C5). We get

$$\begin{aligned}\mathbf{F}_{\mu\nu} \cdot \hat{\mathbf{n}}\ \hat{\mathbf{n}} - [\mathbf{K}_\mu, \mathbf{K}_\nu] &= F_{\mu\nu} \cdot \hat{\mathbf{n}}\ \hat{\mathbf{n}} - [\nabla_\mu\hat{\mathbf{n}}, \nabla_\nu\hat{\mathbf{n}}] \\ &= \partial_\mu(\mathbf{H}_\nu \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \partial_\nu(\mathbf{H}_\mu \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \frac{1}{e}[\partial_\mu\hat{\mathbf{n}}, \partial_\nu\hat{\mathbf{n}}] \\ &= \mathbf{F}^{(H)}_{\mu\nu} \qquad (C9)\end{aligned}$$

This is just the 't Hooft expression. Here $\mathbf{F}^{(H)}_{\mu\nu}$ is the $U(1)$ field part in $\mathbf{F}_{\mu\nu}$contributed by $\mathbf{H}_\mu$ (The Higgs particle does not contribute this $U(1)$ field, only its isodirection coincides with that of charge operator.) Geometrically (C9) is the generalized Gauss equation. $\mathbf{F}^{(H)}_{\mu\nu} \cdot \hat{\mathbf{n}}$, as the $U(1)$ "subcurvature" of the total curvature $\mathbf{F}_{\mu\nu}$, satisfies the Bianchi identity on subbundle. At the same time we get the generalized Codazzi equation,

$$[\hat{\mathbf{n}}, [\mathbf{F}_{\mu\nu}, \hat{\mathbf{n}}]] = \widetilde{\nabla}_\mu\mathbf{K}_\nu - \widetilde{\nabla}_\nu\mathbf{K}_\mu \qquad (C10)$$

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