Generalized Crosscorrelation Properties of Chu Sequences

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Abstract—In this paper, we present a theorem that shows
generalized properties for the cross-correlation function of Chu
sequences. The theorem gives the information on the maximum
magnitude and its position of cross-correlation function for all
kinds of Chu sequences with any length. By using this theorem,
we design the pilot sequences for multi-cell environment. It
mitigate effectively interference and we can confirm the superiority
of the proposed scheme to the conventional scheme.

Index Terms—Chu sequences, Maximum magnitude of cross-
correlation function, Pilot sequence, Multi-cell, Interference.

I. INTRODUCTION

Generally, in order to permit unambiguous message syn-
chronization, to minimize cochannel interference, and to sup-
port a large number of active user, large families of se-
quencies with good auto-correlation function and small cross-
correlation function values are required. A periodic sequences’
auto-correlation function which is 0 except for the period-
multiple-shift terms is called perfect auto-correlation function
[1]. Chu sequences [2] are class of sequences with perfect
auto-correlation. Furthermore, a lower bound of maximum
magnitude of the cross-correlation function is obtained in [3]
and it is proved that Chu sequences satisfy the lower bound
in specific cases [4]. However, we cannot always select Chu
sequences satisfying the lower bound, because the number of
Chu sequences which satisfy the lower bound is limited. In
section II, we present a theorem representing more generalized
cross-correlation properties of Chu sequences. The theorem
provides the information on the cross-correlation function of all
kinds of Chu sequences with any length. It is expected that
these properties give a useful guideline for the efficient miti-
gation of co-channel interference. In section III, we proposed
the channel estimation technique using the theorem for multi-
cell environment. It mitigate effectively multi-cell interference
due to the characteristic of the pilot sequence’s correlation.

II. CORRELATION PROPERTIES OF CHU SEQUENCES

A set of Chu sequences with length \( N \) is defined as \( \{a_r\} \) where \( a_r = \{a_r(0), a_r(1), ..., a_r(N-1)\} \) and

\[
a_r(k) = \begin{cases} 
\exp \left( i \pi \frac{r k^2}{N} \right), & \text{N even,} \\
\exp \left( i \pi \frac{r (k + 1)}{N} \right), & \text{N odd.}
\end{cases}
\]

(1)

Here, \( 0 \leq k, r < N \) and \( r \) is relatively prime with \( N \).

A. Periodic autocorrelation function

The autocorrelation function with lag \( j \) is defined as,

\[
x_r(j) = \sum_{k=0}^{N-j-1} a_r(k) a_r^*(k+j) + \sum_{k=N-j}^{N-1} a_r(k) a_r^*(k+j-N),
\]

(2)

where it was shown that Chu sequences have the following
properties [3].

\[
x(j) = \begin{cases} 
N, & j \mod N = 0 \\
0, & j \mod N \neq 0
\end{cases}
\]

(3)

B. Periodic cross-correlation function

Let \( a_r \) and \( a_s \) be any Chu sequences. Then, the cross-
correlation function \( y_{r,s}(j) \) of \( a_r \) and \( a_s \) with lag \( j \) is defined as

\[
y_{r,s}(j) = \sum_{k=0}^{N-j-1} a_r(k) a_s^*(k+j) + \sum_{k=N-j}^{N-1} a_r(k) a_s^*(k+j-N).
\]

(4)

When \( N \) is even, we can rewrite (4),

\[
y_{r,s}(j) = \sum_{k=0}^{N-j-1} \exp \left( i \pi \frac{r k^2}{N} \right) \exp \left( -i \pi \frac{s(k+j)}{N} \right)
\]

\[+ \sum_{k=N-j}^{N-1} \exp \left( i \pi \frac{r k^2}{N} \right) \exp \left( -i \pi \frac{s(k+j-N)^2}{N} \right).\]

(5)

It is proved in [2] that for an arbitrary integer \( d \),

\[
\exp \left( i \pi \frac{r(k+d)^2}{N} \right) = \exp \left( i \pi \frac{r(k+d)^2}{N} \right).
\]

(6)

is satisfied. Thus, (5) can be rewritten as

\[
y_{r,s}(j) = \sum_{k=0}^{N-1} \exp \left( i \pi \frac{r k^2}{N} \right) \exp \left( -i \pi \frac{s(k+j)^2}{N} \right),
\]

(7)

The following lemmas will be useful for proof of theorem 1.

Lemma 1: A primitive \( \ell \)th root of unity \( \xi \) is defined as

\[
\xi = \exp \left( i \frac{2u \pi}{h} \right),
\]

(8)

where \( u \) is any integer relatively prime to \( h \). It can be easily
for any integer \( v \), \( 0 < v \leq h - 1 \), the following relation is
Since \(\zeta \neq 1\) and \(\zeta^{\pm nk} = 0\), the following theorem holds.

\[
\sum_{k=0}^{h-1} \zeta^{nk} = 0, \quad \zeta \neq 1. \tag{9}
\]

**Lemma 2:** If \(l = k + e, e = 0, 1, \ldots, N\) and a function \(f(x)\) is periodic with period \(N\), then the following equation is satisfied.

\[
\sum_{l=0}^{N-1} f(l) = \sum_{e=0}^{N-1} f(e). \tag{10}
\]

Substituting \(k + e\) in \(\sum_{l=0}^{N-1} f(l)\), we obtain

\[
\sum_{l=0}^{N-1} f(l) = \sum_{e=0}^{N-1} f(e). \tag{11}
\]

Since \(f(x)\) is periodic with period \(N\), (10) is rewritten as

\[
\sum_{l=0}^{N-1} f(l) = \sum_{e=0}^{N-1} f(e) + \sum_{e=0}^{k-1} f(e) = \sum_{e=0}^{N-1} f(e). \tag{12}
\]

In this way, we proved the Lemma 2.

Then, the following theorem holds.

**Theorem 1:** Let \(G = \gcd(N, r-s)\), \(N = c_1 G\), \(r-s = c_2 G + d\), where \(c_1\) is prime with \(c_2\) and \(0 \leq c_3 \leq c_1\). Then, the squared absolute value of the cross-correlation function \(y_{r,s}(j), |y_{r,s}(j)|^2\), is given as

\[
|y_{r,s}(j)|^2 = \begin{cases} 
N G, & \text{if } N \text{ even, } c_1 c_2 \text{ even, } d = 0 \\
N G, & \text{if } N \text{ even, } c_1 c_2 \text{ odd, } d = \frac{G}{2} \\
N G, & \text{if } N \text{ odd, } d = 0 \\
0, & \text{otherwise}
\end{cases}
\]

**Proof:** We can rewrite (7) as

\[
y_{r,s}(j) = \exp\left( -i \pi \frac{s j^2}{c_1 G} \right) \cdot \sum_{k=0}^{c_1 G - 1} \exp\left( i 2 \pi \left( \frac{c_2 k^2}{2 c_1} - \frac{skc_2}{c_1} - \frac{skd}{c_1 G} \right) \right). \tag{13}
\]

Since \(|y_{r,s}(j)|^2 = y_{r,s}(j) y_{r,s}(j)^*\),

\[
|y_{r,s}(j)|^2 = \sum_{k=0}^{c_1 G - 1} \sum_{l=0}^{c_2 G - 1} \exp\left( i 2 \pi \left( \frac{c_2 k^2}{2 c_1} - \frac{sk(c_2 G + d)}{c_1 G} \right) \right) \cdot \exp\left( i 2 \pi \left( \frac{sl(c_2 G + d)}{c_1 G} - \frac{c_2 l^2}{2 c_1} \right) \right). \tag{14}
\]

We can know that the last term of (14) is periodic with period \(c_1 G\) from follows.

\[
\begin{align*}
\exp\left( i 2 \pi \left( \frac{sl(c_2 G + d)}{c_1 G} - \frac{c_2 l^2}{2 c_1} \right) \right) &= \exp\left( i 2 \pi \left( \frac{sl(c_2 G + d)}{c_1 G} - \frac{c_2 l^2}{2 c_1} \right) \right) \\
&= \exp\left( -i 2 \pi \left( \frac{c_2 k^2}{2 c_1} \right) \right) \exp\left( i 2 \pi (c_2 G + d + l G c_2) \right) \\
&= \exp\left( i 2 \pi \left( \frac{sl(c_2 G + d)}{c_1 G} - \frac{c_2 l^2}{2 c_1} \right) \right) .
\end{align*}
\]

(15)

If \(N\) and \(r-s\) are even, \(G = \gcd(r-s, N)\) is always even. Therefore, the last equality of (15) is valid. Consequently, we obtain \(|y_{r,s}(j)|^2\) from Lemma 2 and (15) as follows and divide in two parts for convenience.

\[
|y_{r,s}(j)|^2 = \sum_{e=0}^{c_1 G - 1} \exp\left( i 2 \pi \left( - \frac{c_2 e^2}{2 c_1} + \frac{se(c_2 G + d)}{c_1 G} \right) \right) \\
= A_{r,s}(j) + B_{r,s}(j)
\]

\[
A_{r,s}(j) = \sum_{e=c_1 m c_1}^{c_1 G - 1} \exp\left( i 2 \pi \left( - \frac{c_2 e^2}{2 c_1} + \frac{se(c_2 G + d)}{c_1 G} \right) \right) \\
B_{r,s}(j) = \sum_{e \neq mc_1}^{c_1 G - 1} \exp\left( i 2 \pi \left( - \frac{c_2 e^2}{2 c_1} + \frac{se(c_2 G + d)}{c_1 G} \right) \right) .
\]

(16)

Substituting \(e = mc_1\) in \(A_{r,s}(j)\),

\[
A_{r,s}(j) = \sum_{m=0}^{G-1} \exp\left( i 2 \pi \left( - \frac{c_1 c_2 m^2}{2} + smG + \frac{smd}{G} \right) \right) \\
= c_1 G \sum_{m=0}^{G-1} \exp\left( - \frac{c_1 c_2 m^2}{2} + i 2 \pi \frac{smd}{G} \right) .
\]

(17)

The last equality is derived, since \(c_1 G \sum_{k=0}^{G-1} \exp(-i 2 \pi c_2 km) = c_1 G\). Also, we assume that \(c_1 c_2\) is even, then

\[
A_{r,s}(j) = c_1 G \sum_{m=0}^{G-1} \exp\left( i 2 \pi \frac{smd}{G} \right) .
\]

(18)

If \(d = 0\), \(A_{r,s}(j) = c_1 G^2\). But, when \(d \neq 0\), since \(s\) is relatively with \(G\), we can know easily that \(A_{r,s}(j) \neq c_1 G^2 = 0\).
from Lemma 1. For \( e \neq m c_1 \), \( B_{r,s}(j) \) is satisfied from Lemma 1 that
\[
G \sum_{k=0}^{c_1-1} \exp \left( -i2\pi \frac{c_2ke}{c_1} \right) = 0. \tag{19}
\]
Therefore, we always obtain that \( B_{r,s}(j) = 0 \) regardless of \( j \).
Then, we can derive by (16) that
\[
|y_{r,s}(j = c_3G)|^2 = A_{r,s}(j = c_3G) = c_1G^2, \tag{20}
\]
\[
|y_{r,s}(j \neq c_3G)|^2 = 0. \tag{21}
\]
Now, we consider the case that \( c_1c_2 \) is odd in (17). For this case, if \( d = \frac{G}{2} \), then
\[
A_{r,s}(j = c_3G + \frac{G}{2}) = c_1G \sum_{m=0}^{G-1} \exp \left( i2\pi \left( -\frac{c_1c_2m^2 + sm}{2} \right) \right), \tag{22}
\]
In (22), \( s \) and \( c_1c_2 \) is odd. If \( m \) is odd, \( c_1c_2m \) is also odd and \( c_1c_2m + s \) is even since an odd number added to an odd number make an even number. Therefore, \( m(c_1c_2m + s) \) is even. Conversely, if \( m \) is even, \( m(c_1c_2m + s) \) is even too. In other word, \( m(c_1c_2m + s) \) is always even and then (22) is rewritten as
\[
A_{r,s}(j = c_3G + \frac{G}{2}) = c_1G^2. \tag{23}
\]
If \( d \neq \frac{G}{2} \), we obtain that
\[
A_{r,s}(j \neq c_3G + \frac{G}{2}) = c_1G \sum_{m=0}^{G-1} \exp \left( i2\pi \left( -\frac{m(c_1c_2m + s)}{2} + \frac{smc_4}{G} \right) \right), \tag{24}
\]
where \( 0 < c_4 < G \). Previously, since we know that \( m(c_1c_2m + s) \) is even, (24) is expressed as
\[
A_{r,s}(j \neq c_3G + \frac{G}{2}) = c_1G \sum_{m=0}^{G-1} \exp \left( i2\pi \frac{smc_4}{G} \right). \tag{25}
\]
From Lemma 1 and (25), we obtain that
\[
A_{r,s}(j \neq c_3G + \frac{G}{2}) = 0. \tag{26}
\]
When \( c_1c_2 \) is odd, \( B_{r,s}(j \neq c_3G + \frac{G}{2}) \) is same with the case that \( c_1c_2 \) is even. Therefore, by (16)
\[
|y_{r,s}(j = c_3G + \frac{G}{2})|^2 = A_{r,s}(j \neq c_3G + \frac{G}{2}) = c_1G^2, \tag{27}
\]
\[
|y_{r,s}(j \neq c_3G + \frac{G}{2})|^2 = 0. \tag{28}
\]
On that way we have proved the Theorem 1 when \( N \) is even. We can prove the case that \( N \) is odd in similar manner and will prove this in appendix A.

### III. APPLICATION TO PILOT PATTERNS FOR MULTI-CELL ENVIRONMENT

In this section, we design the pilot sequences using Chu sequences and receiver structure for multi-cell environment. By (3), we can know that Chu sequences have zero circular autocorrelation function. Therefore, when we transmit \( \alpha_r \) added CP, we can estimate the channel impulse response without multi-path interference. Also, if we use Chu sequences in pilots without multiuser interference, the \( k \)th user’s pilot sequence is \( kCP \) cyclic shifted version of \( \alpha_r \). However, if the active user increase or it is considered the user of the other cell like the case that \( K > (N/CP) \) where \( K \) is the number of the active user, it suffer severely from the interference. Therefore, we require new pilot pattern that effectively can mitigate the interference. The pilot sequences of the target cell and other cell is defined as
\[
p_r = [p_r,1, p_r,2, \cdots, p_r,K], p_s = [p_s,1, p_s,2, \cdots, p_s,K]
\]
where \( p_{r,k} \) is \( k \)th user’s pilot of target cell and \( p_{s,k} \) is \( k \)th user’s pilot of other cell.

The process is represented the proposed channel estimation technique at the receiver in Fig 1. After remove CP, in order to estimate \( k \)th user’s channel, we have the process which the received signal correlate with the \( k \)th user’s pilot sequences from zero cyclic shifted sequence to \( CP - 1 \) cyclic shifted sequence. Then the correlation value of received signal and \( m \) cyclic shifted \( k \)th user’s pilot is regard as \( h_{r,k}(m) \) where \( h_{r,k}(m) \) is \( m \) delayed channel’s estimated gain value of the target cell’s \( k \)th user. In this case, we can know that the pilot sequences’ correlation of the target cell and other cell influence the interference by theorem 1. Statistically, if \( \text{gcd}(r - s, N) \), the correlation’s maximum magnitude of the pilot sequences, is minimized, it is expected that the channel impulse response’s difference between the target cell and the other cell is largest. In this way, for each individual user, the channel gain value with largest power will be selected from the output of the correlator and is expressed as
\[
\hat{h}_{r,k}(m) = \begin{cases} 
\hat{h}_{r,k}(m), & \text{if } \left\{ \hat{h}_{r,k}(m) \right\} \subset \text{MAX}_M \\
0, & \text{else}
\end{cases} \tag{30}
\]
where \( \text{MAX}_M \) is the set of the taps with maximum power.
Through the technique above, we can mitigate effectively
and get the channel impulse response. To illustrate the performance of the channel estimator, we evaluate the MSE. The MSE performance is evaluated as a function of the average power from adjacent cells. For the simulation, we assume that the length of the pilot sequences is 1024, and pass through Rayleigh fading multi-path channels that are independently generated for base stations. Also, we assume that the number of the adjacent cells is 6 and the length of CP is 128. In order to minimize the maximum power of other cell, we select the pilot of the adjacent cells is 6 and the length of CP is 128. In order to apply the proposed scheme in section III. In this case, $\gcd(r - s, N) = 2$ regardless of $m$. For the conventional scheme, we select the pilot sequence of the target cell as $a_r = a_1$ and other cell’s pilot sequences randomly of the number that relatively prime with $N$.

In Fig. 2, the MSE of the proposed scheme and the conventional scheme is compared for signal to other cell interference ratio(SIR). We can confirm that the gap between the proposed and conventional scheme increase with the increase of other cell interference. In other words, we can conclude that the proposed scheme is superior to the conventional scheme in multi-cell environment.

IV. CONCLUSIONS

In this paper, we present the generalized cross-correlation properties for Chu sequences. The proved theorem can be applied to all kinds of Chu sequences with any sequence. Also, using this theorem, we propose the pilot pattern for multi-cell environment. Simulation result have confirmed the superiority of the proposed scheme to the conventional scheme.

APPENDIX A

PROOF OF THE THEOREM 1 WHEN $N$ IS ODD

Consider the case that $N$ is odd. From (1), (4), (6)

$$y_{r,s}(j) = \sum_{k=0}^{N-1} \exp \left( i\pi \frac{r k (k+1)}{N} \right) \cdot \exp \left( -i\pi \frac{s(k+j)(k+1+j)}{N} \right).$$

(31)

When $N$ is odd, let $G = \gcd(N, r - s), N = c_1 G, r - s = c_2 G, j = c_3 G + d$ in similar manner that $N$ is even. Substituting this variables in (31), we obtain

$$y_{r,s}(j) = \exp \left( -i\pi \frac{sj^2 + sj}{c_1 G} \right) \sum_{c_1 G - 1}^{c_1 G - 1} \exp \left( i2\pi \left( \frac{c_2 k^2}{2c_1} + \frac{c_2 k}{2c_1} - \frac{sk}{c_1} - \frac{skd}{c_1 G} \right) \right).$$

(32)

From (32), we can derive $|y_{r,s}(j)|^2$ from Lemma 1 easily in similar manner that $N$ is even. Then we obtain

$$|y_{r,s}(j)|^2 = \sum_{c_1 G - 1}^{c_1 G - 1} \exp \left( i2\pi \left( \frac{c_2 k^2}{2c_1} + \frac{c_2 k}{2c_1} - \frac{sk}{c_1} - \frac{skd}{c_1 G} \right) \right).$$

(33)

In (33), we divide in two parts for convenience like (16) and substitute $e = mc$ in (33),

$$A_{r,s}(j) = \sum_{c_1 G - 1}^{c_1 G - 1} \exp \left( i2\pi \left( -\frac{c_2 m^2 c_1 + m}{2} + \frac{sm(c_2 G + d)}{G} \right) \right) \sum_{k=0}^{G-1} \exp \left( -i2\pi c_2 km \right)$$

$$= c_1 G \sum_{m=0}^{G-1} \exp \left( i2\pi \left( -\frac{c_2 m(c_1 + 1)}{2} + \frac{sm(c_2 G + d)}{G} \right) \right).$$

(34)

An odd number is made up of odd number’s multiplication. Therefore, if $N$ is odd, $G$ and $c_1$ are odd. Then, $c_1 + 1$ is even number and (34) can be rewritten as

$$A_{r,s}(j) = c_1 G \sum_{m=0}^{G-1} i2\pi \frac{smd}{G}. $$

(35)

From (35), if $d = 0$, $A_{r,s}(j = c_3 G) = c_1 G^2$. Conversely, if $d \neq 0$ $A_{r,s}(j \neq c_3 G) = 0$ by Lemma 1. Furthermore, we can know that $B_{r,s}(j) = 0$ regardless of $d$ from (33) and Lemma 1. Therefore, we can derive as follows

$$|y_{r,s}(j = c_3 G)|^2 = A_{r,s}(j = c_3 G) = c_1 G^2,$$

(36)

$$|y_{r,s}(j \neq c_3 G)|^2 = 0.$$

(37)
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