A Neurodynamic Optimization Approach to Robust Pole Assignment for Synthesizing Linear State Feedback Control Systems

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Abstract—This paper presents a neurodynamic optimization approach to robust pole assignment for synthesizing linear control systems via state feedback. A pseudoconvex objective function is minimized as a robustness measure. A neurodynamic model is applied whose global convergence was theoretically proved for constrained pseudoconvex optimization. Compared with existing approaches on benchmark problems, the convergence of proposed neurodynamic approach to global optimal solutions can be guaranteed. Simulation results of the proposed neurodynamic approach is reported to demonstrate its superiority.

I. INTRODUCTION

Pole assignment is a basic design approach for linear control systems. Various closed-loop design specifications can be achieved via proper assignment of closed-loop poles or eigenvalues [1]. When all of the state variables of a system are completely controllable and measurable, appropriate gains for state feedback control can be synthesized to achieve desired objectives [2], [3].

As practical control systems can hardly be precisely modeled and parameter uncertainties are always existed in designed model, a more desirable property of the control system is proposed as the insensitivity of the poles to parameter perturbation in the coefficient matrices of the system equations. As a result, robust pole assignment approach is more desirable for synthesis of linear feedback control system. Given a linear system and the desired closed-loop system equations. As a result, robust pole assignment approach is more desirable for synthesis of linear feedback control system. Given a linear system and the desired closed-loop system equations. As a result, robust pole assignment approach is more desirable for synthesis of linear feedback control system.

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II. PRELIMINARIES & PROBLEM FORMULATION

Consider a linear time-invariant control system as follows:

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \]  

where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the control vector, \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) are known coefficient matrices associated with \( x(t) \) and \( u(t) \), respectively. As usual, the linear system described in (1) is assumed to be completely state controllable. If state variables are measurable, then a linear state feedback control law:

\[ u(t) = r(t) + K x(t) \]  

can be applied to control the state of the system. The closed-loop system is in the following form:

\[ \dot{x}(t) = (A + BK)x(t) + Br(t), \quad x(0) = x_0, \]  

where \( r \in \mathbb{R}^m \) is a reference input vector, and \( K \in \mathbb{R}^{m \times 2} \) is a state feedback gain matrix. Feedback gain matrix \( K \) must be chosen by using different design strategies, depending on the design requirements. In this paper, we will focus on the pole assignment approach.

Based on the controllability assumption of pair \( (A, B) \), there exists at least one feedback matrix \( K \) for a particular
choice of almost any set of desired poles \(\lambda_1, \lambda_2, \ldots, \lambda_n\) of the closed loop system. The objective of pole assignment for synthesizing the control system is to find such a \(K\). For complex conjugate eigenvalue pairs \(\sigma_j \pm \omega_j\), a real pseudo-diagonal \((2 \times 2)\) blocks are defined as follows:

\[
\begin{bmatrix}
\sigma_i & \omega_i \\
-\omega_i & \sigma_i
\end{bmatrix},
\]

where \(j = \sqrt{-1}\). Denote \(\text{spec}(\Lambda)\) as the set of eigenvalues of the matrix \(\Lambda\). Given a real pseudo-diagonal matrix \(\Lambda \in \mathbb{R}^{n \times n}\) with \(\text{spec}(\Lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n)\), find real matrix \(K\) and nonsingular matrix \(Z\) satisfying

\[
(A + BK)Z = Z\Lambda. \quad (4)
\]

Let \(G\) denote \(KZ\). The pole assignment problem is then equivalent to the problem of finding a matrix pair \(\Lambda\) and \(G\), such that the following Sylvester equation holds [11]:

\[
AZ + BG = Z(\Lambda + Z^{-1} \Delta Z).
\]

The perturbed system is stable if \((\Lambda + Z^{-1} \Delta Z)\) is stable. It is known in [3] that the robust stability can be guaranteed if \(\kappa_2\) is known in [3] that the robust stability can be guaranteed if \(\kappa_2\) is known in [3] that the robust stability can be guaranteed if

\[
\|Z\|_2^2 \|Z^{-1}\|_F = \sqrt{\lambda_{\max}(Z^T Z)/\lambda_{\min}(Z^T Z)}
\]

is the spectral condition number of the assigned eigensystem. \(\lambda_{\max}\) and \(\lambda_{\min}\) are the largest and smallest eigenvalues, respectively. From (6) we can see that a smaller \(\kappa_2\) will lead to a bigger bound of \(\Delta\). Hence, minimizing the spectral condition number \(\kappa_2\) becomes a focal point.

In view that \(\kappa_2(Z)\) is nonconvex and \(1 \leq \kappa_2(Z) \leq \|Z\|_F\|Z^{-1}\|_F\), an alternative robustness measure is \(\|Z\|_F\|Z^{-1}\|_F\), where \(\|\cdot\|_F\) is the Frobenius norm. It is shown in [6] that any local (global) optimal solution is also a local (global) optimal solution to another alternative robust minimum measure: \(\|Z\|_F^2 + \|Z^{-1}\|_F^2\). As both objective functions above are nonconvex, generally it is very difficult to obtain the global optimal solutions. In [12], a matrix variable \(Y\) is introduced as \(Z^{-1}\) with additional concave constraints, global optimal solution is also not guaranteed.

Here we propose a new approach to minimize the condition number directly. In view that the optimal solutions of \(\kappa_2\) and \(\kappa_2^2\) is the same, \(\kappa_2^2\) is to be used instead of \(\kappa_2\). By using Kronecker product and vectorization techniques, the equation \(AZ + ZA + BG = 0\) can be written as

\[
Mz = 0, \quad (7)
\]

where \(z\) denotes a vector obtained by stacking all column vectors of \(Z\) and \(G\) together,

\[
M = [I_n \otimes A - \Lambda^T \otimes I_n] [I_n \otimes B].
\]

Hence, the robust pole assignment problem for state feedback formulate as the following optimization problems:

minimize \(\kappa_2^2(Z)\)

s.t. \(Mz = 0\). \quad (8)

As \(\lambda_{\max}(Z^T Z) = \|Z\|_2^2\) is positive and convex, and \(\lambda_{\min}(Z^T Z) = 1/\|Z^{-1}\|_2^2\) is positive and concave, the ratio objective function \(\kappa_2^2(Z) = \lambda_1(Z^T Z)/\lambda_n(Z^T Z)\) is pseudoconvex according to Theorem 3.2.10 in [37]. Thus, some pseudoconvexity optimization approach can be applied for solving (8).

### III. NEURODYNAMIC APPROACHES

#### A. Condition Number Minimization

Neural networks as computational model are widely used in control systems, however, most existing neurodynamic optimization approaches are suitable for convex optimization only. It was until recent years that several recurrent neural networks were developed for solving pseudoconvex optimization problems [27]-[31]. In particular, a one-layer recurrent neural network [31] is suitable for solving linearly constrained nonsmooth and pseudoconvex optimization problems such as the problems in (8) formulated in the preceding section:

\[
e_1 \frac{dz}{dt} \in -\partial f(z) - \sigma M^T g(Mz),
\]

where \(g(y)\) is a vector valued discontinuous activation function with each component is defined as

\[
g_i(y) = \begin{cases}
1, & y > 0 \\
0, & y = 0 \\
-1, & y < 0
\end{cases}
\]

\(e_1\) is a positive scaling constant, \(\partial f(z)\) is the generalized gradient of a given objective function \(f(z)\), \(M \in \mathbb{R}^{n \times (n^2 + mn)}\) satisfies the linear equality constraints \(Mz = 0\), and \(\sigma\) is nonnegative constant. It is proved in [31] that the dynamic inclusion (9) is capable of solving pseudoconvex optimization problems with linear equality . From random initial points, the state variables will converge to the unique optimal solution. The convergence and optimality conditions of proposed neural networks have been already investigated by using the Lyapunov method and differential inclusion theory according to [31]. Assume there exist \(\hat{z} \in \mathbb{R}^n\) and \(r > 0\), \(B(\hat{z}, r) = \{z \in \mathbb{R}^n : \|z - \hat{z}\|_2 < r\}\) is the \(r\) neighborhood of \(\hat{z}\). The objective function \(f = \kappa_2^2(Z)\) is pseudoconvex, regular, and Lipschitz bounded on \(B(\hat{z}, r)\). Denote \(l_f\) as an upper bound of Lipschitz constant of \(f(x)\) on \(B(\hat{z}, r)\), and set \(\sigma > l_f/\sqrt{\lambda_{\min}(M^TM)}\), the state vector of neural network is convergent to an optimal solution thereafter following for any \(z_0 \in B(\hat{z}, r)\).

The objective function \(\kappa_2^2(Z)\) can be calculated as follows:

\[
\frac{\partial \kappa_2^2(Z)}{\partial Z} = \frac{\partial \kappa_2^2(Z)}{\partial (Z^T Z)} \frac{\partial (Z^T Z)}{\partial Z} = 2Z \frac{\partial \kappa_2^2(Z)}{\partial (Z^T Z)},
\]
where $Z \in \mathbb{R}^{n \times n}$ is nonsingular. Note that $Z^T Z$ is symmetric and positive definite, denote $\tilde{Z} = Z^T Z$. In [38], the gradient of the symmetric and positive definite matrix $\kappa(Z)$ can be calculated as follows:

$$\partial \kappa(Z) = \frac{\partial(\lambda_{\text{max}}(Z)/\lambda_{\text{min}}(Z))}{\partial(Z)}$$

$$= \frac{\lambda_{\text{min}}(Z)\partial\lambda_{\text{max}}(Z) - \lambda_{\text{max}}(Z)\partial\lambda_{\text{min}}(Z)}{\lambda_{\text{min}}^2(Z)},$$

where $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ are the largest and smallest eigenvalues of $Z$, respectively. According to [38], we know

$$\partial\lambda_{\text{max}} = \{v_{\text{max}} v_{\text{max}}^T | v_{\text{max}} \in \mathbb{R}^n, \|v_{\text{max}}\| = 1, \tilde{Z} v_{\text{max}} = \lambda_{\text{max}} v_{\text{max}}\},$$

$$\partial\lambda_{\text{min}} = \{v_{\text{min}} v_{\text{min}}^T | v_{\text{min}} \in \mathbb{R}^n, \|v_{\text{min}}\| = 1, \tilde{Z} v_{\text{min}} = \lambda_{\text{min}} v_{\text{min}}\}.$$

Then the gradient of the objective function $\partial \kappa(Z)$ can be derived as follows:

$$\partial \kappa(Z) = 2Z \lambda_{\text{min}} v_{\text{max}} v_{\text{max}}^T - \lambda_{\text{max}} v_{\text{min}} v_{\text{min}}^T,$$

$$= 2\|Z\|_2 \left(\frac{\lambda_{\text{min}} - \lambda_{\text{max}}}{\lambda_{\text{min}}^2}\right) \leq 2 \lambda_{\text{max}}^2 (Z_0^T Z_0),$$

where $Z_0$ is the initial values of $Z$. By properly setting $\sigma$ in the following way:

$$\sigma = \frac{2 \lambda_{\text{max}}^2 (Z_0^T Z_0)}{\lambda_{\text{min}}^2 (Z_0^T Z_0)} \sqrt{\lambda_{\text{min}}(M M^T)},$$

the neurodynamic model (9) is capable of solving pseudo-convex optimization problem (8), and its finite-time convergence to a feasible solution can be guaranteed.

**B. Eigenvector Computation**

In (10), $\partial \kappa(Z)$ contains the eigenvalues and eigenvectors of $Z$, which cannot be expressed explicitly. A neurodynamic model is shown in [39] to be capable for computing the largest eigenvalue $\lambda_{\text{max}}$ and corresponding eigenvector $v_{\text{max}}$ of $Z$. For symmetric and positive definite matrices (note that the eigenvector $v_{\text{min}}$ corresponding to the smallest eigenvalue is equivalent to the eigenvector corresponding to the largest eigenvalue of $-\tilde{Z} + \lambda_{\text{max}} I$):

$$\epsilon_2 \frac{du_{\text{max}}}{dt} = \tilde{Z} u_{\text{max}} - u_{\text{max}}^T \tilde{Z} u_{\text{max}},$$

$$\epsilon_2 \frac{du_{\text{min}}}{dt} = (\tilde{Z} + \lambda_{\text{max}} I) u_{\text{min}} - u_{\text{min}}^T (\tilde{Z} + \lambda_{\text{max}} I) u_{\text{min}},$$

$$\lambda_{\text{max}} = u_{\text{max}}^T \tilde{Z} u_{\text{max}}, \quad v_{\text{max}} = u_{\text{max}},$$

$$\lambda_{\text{min}} = u_{\text{min}}^T \tilde{Z} u_{\text{min}}, \quad v_{\text{min}} = u_{\text{min}},$$

where $\epsilon_2$ is another positive scaling constant and $u_{\text{max}}, u_{\text{min}} \in \mathbb{R}^n$. Denote equilibrium vector of $u_{\text{max}}$ as $\tilde{u}_{\text{max}}$ and $u_{\text{min}}$ as $\tilde{u}_{\text{min}}$. Then the largest eigenvalue $\lambda_{\text{max}}$, smallest eigenvalue $\lambda_{\text{min}}$ and the corresponding nominal eigenvectors $v_{\text{max}}$ and $v_{\text{min}}$ can be computed. According to [39], the convergence of the recurrent neural network can be guaranteed with any nonzero $v_{\text{max}}(0)$ and $v_{\text{min}}(0)$.

**C. Neurodynamic Robust Pole Assignment**

The robust pole assignment processes for synthesizing state feedback control system via robust pole assignment.

![Fig. 1. Block diagram of the neurodynamics-based state feedback control system via robust pole assignment.](image)

In this section, we discuss the simulation results of two illustrative examples in detail to demonstrate the effectiveness and characteristics of the proposed method.

**Example 1:** Consider the multiple-input multiple-output system discussed in 5, 16.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

The desired poles are $\Lambda = \{-1, -2, -3\}$, $M$ and $l_f$ can be obtained with known $A$, $B$, $\Lambda$ and initial variables. If

$$Z_0 = \begin{bmatrix} 0.5455 & -1.0516 & 0.3975 \\ -0.7519 & 1.5163 & -0.0326 \\ 1.6360 & -0.4251 & 0.5894 \end{bmatrix},$$

IV. SIMULATION RESULTS

In this section, we discuss the simulation results of two illustrative examples in detail to demonstrate the effectiveness and characteristics of the proposed method.

**Example 1:** Consider the multiple-input multiple-output system discussed in 5, 16.
then $\sigma = 4412$ according to (11). Let $\epsilon_1 = 10^{-3}$, and $\epsilon_2 = 10^{-6}$, the optimal condition number for feedback is 10.7738. The assigned poles are: $-1.0000$, $-2.0001$, $-3.0003$ for feedback system. The convergent values of $Z$, $G$ and corresponding $K$ are presented as follows:

$$
\tilde{Z} = \begin{bmatrix}
-0.2810 & -0.0345 & -0.2403 \\
0.0348 & 0.0343 & 0.0974 \\
0.3098 & -0.0339 & 0.3318
\end{bmatrix},
$$

$$
\tilde{G} = \begin{bmatrix}
0.5907 & 0.0693 & 1.2476 \\
-0.3446 & -0.0346 & -0.6238
\end{bmatrix},
$$

$$
\tilde{K} = \begin{bmatrix}
2.4644 & 7.7385 & 3.2732 \\
-0.8581 & -3.3700 & -1.5122
\end{bmatrix}.
$$

Figs. 2 and 3 depict the transient behaviors of the state variables $z$ of RNN$_1$ and corresponding feedback gain matrix $K$, respectively. Figs. 4 and 5 depict transient behaviors of the resulting real and imaginary parts of the closed-loop poles during robust pole assignment. Fig. 6 depicts the transient behaviors of the spectral condition number and the norm of constraints $||AZ - Z\Lambda + BG||_2$ during robust pole assignment. It indicates that the exact pole assignment can be made in about two nanoseconds and the spectral condition number $\kappa_2(Z)$ reaches the minimum in about forty nanoseconds and stays thereafter. The results substantiated the global convergence properties of the neurodynamic approach.

**Example 2:** Consider the distillation column model discussed in [3], [5], [6], [10] and [12]. The objective is to synthesize a robust state feedback controller such that the
closed-loop system poles are \(-0.2, -0.5, -1\) and \(-1 \pm j\).

![Fig. 6](image6.png)  
**Fig. 6.** Transient behaviors of \(\kappa_2(Z)\) and \(\|AZ - Z \Lambda + BG\|_2\) of the feedback control system in Example 1.

\[
A = \begin{bmatrix}
-0.1094 & 0.0628 & 0 & 0 & 0 \\
1.3062 & -2.132 & 0.9807 & 0 & 0 \\
0 & 1.595 & -3.149 & 1.547 & 0 \\
0 & 0.0355 & 2.632 & -4.257 & 1.855 \\
0 & 0.00227 & 0 & 0.1636 & -0.1625 \\
\end{bmatrix},
\]

\[
B^T = \begin{bmatrix}
0 & 0.0638 & 0.0838 & 0.1004 & 0.0063 \\
0 & -0.1396 & -2.060 & -0.0128 \\
-0.2 & 0 & 0 & 0 & 0 \\
0 & -0.5 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 \\
\end{bmatrix},
\]

\[
\Lambda = \begin{bmatrix}
-0.5 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

Let \(\epsilon_1 = 10^{-3}, \epsilon_2 = 10^{-6}\), and \(\sigma = 10^4\), the optimal condition number for feedback system is 31.48. The assigned poles are: \(-0.2000, -1.0000, -0.5000, -1.0000 \pm 1.0001 j\) for feedback system. The convergent value of \(Z, G\) and corresponding \(K\) are presented as follows:

\[
\bar{Z} = \begin{bmatrix}
-0.285 & 0.393 & 0.062 & 0.105 & -0.017 \\
0.411 & -2.447 & -0.880 & -1.212 & 1.907 \\
0.544 & 0.832 & 0.211 & -0.961 & 2.413 \\
-0.142 & 1.522 & 0.590 & -0.607 & 1.684 \\
-1.892 & -2.083 & -0.249 & 0.202 & 0.027 \\
\end{bmatrix},
\]

\[
\bar{G} = \begin{bmatrix}
-80.555 & 90.692 & -193.534 & 161.404 & -41.083 \\
\end{bmatrix},
\]

\[
\bar{K} = \begin{bmatrix}
-0.832 & -0.211 & 0.590 & -0.607 & 1.684 \\
-1.892 & -2.083 & -0.249 & 0.202 & 0.027 \\
-80.555 & 90.692 & -193.534 & 161.404 & -41.083 \\
\end{bmatrix}.
\]

Fig. 7 depicts the transient behaviors of eigenvector variables \(u_{\text{max}}\) and \(u_{\text{min}}\), which substantiates that the recurrent neural networks RNN2 converges within ten picoseconds. Fig. 8 depicts the convergent behavior of \(\kappa_2(Z)\) in a few microseconds. Compared with Figs. 7 and 8, multiple time-scales can be identified. A comparison with the other published results is summarized in Table I that shows that the neurodynamic optimization approach is able to result in good performance in robust of pole assignment.
TABLE I

<table>
<thead>
<tr>
<th>Method</th>
<th>( \kappa_2 )</th>
</tr>
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<tbody>
<tr>
<td>Kautsky-Nichols-Dooren</td>
<td>39.4</td>
</tr>
<tr>
<td>method [3]</td>
<td></td>
</tr>
<tr>
<td>Byrne-Nash method [5]</td>
<td>33.1</td>
</tr>
<tr>
<td>Yang-Tits [7]</td>
<td>39.46</td>
</tr>
<tr>
<td>Lam-Yan method [6]</td>
<td>33.6</td>
</tr>
<tr>
<td>Lam-Yan method [9]</td>
<td>31.6</td>
</tr>
<tr>
<td>Ho-Lam-Xu-Tâm method [10]</td>
<td>32.1</td>
</tr>
<tr>
<td>Rami-Faiz-Benzaouia-Taddeo method [16]</td>
<td>33.069</td>
</tr>
<tr>
<td>Method herein</td>
<td>31.48</td>
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V. CONCLUSIONS

In this paper, a novel neurodynamic optimization approach is proposed for the synthesis of linear state feedback control systems via robust pole assignment. By minimizing the spectral condition number of the eigensystem in real time, the proposed approach is shown to be capable of making robust pole assignment. The performance and characteristics of the neurodynamic optimization approach are demonstrated in details by using two benchmark problems.

REFERENCES


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TABLE I

<table>
<thead>
<tr>
<th>Comparison of results in Example 1</th>
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<tbody>
<tr>
<td>Method</td>
</tr>
<tr>
<td>Kautsky-Nichols-Dooren method</td>
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<tr>
<td>Byrne-Nash method</td>
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<tr>
<td>Method herein</td>
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