Blow-up phenomena for a class of quasilinear parabolic problems under Robin boundary condition

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Abstract. This paper deals with a class of heat emission processes in a medium with a nonegative source, a nonlinear decreasing thermal conductivity and a linear radiation (Robin) boundary condition. For such heat emission problems, using a differential inequality technique, we establish conditions on the data sufficient to guarantee that the blow-up of the solutions does occur or does not occur. In addition, the same technique is used to determine a lower bound for blow-up time blow-up occurs.

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1 Introduction

In a recent paper, L.E. Payne and P.W. Schaefer [5] have investigated the blow-up phenomena to the following semilinear heat equation
\[ \Delta u - u_t = -f(u), \]
in a bounded domain \( \Omega \subset \mathbb{R}^N \), with a radiation (Robin) boundary condition and a prescribed non-negative initial condition. By means of first-order differential inequality technique, they established conditions on the data sufficient to guarantee the blow-up of solution at some finite time or conditions for the global boundedness of solutions. Moreover, a lower bound on blow-up time was obtained when blow-up occurs.

The mathematical investigation of the blow-up phenomena of solutions to nonlinear parabolic equations and systems received a great deal of attention during the last decades (we refer the reader especially to the books of Quittner-Souplet [13] and Samarskii [14], to the survey papers of Levine [4] and Galaktionov [3] or the references therein). Nowadays, a variety of methods are known for the study of blow-up phenomena in parabolic problems. However, in a serie of recent papers, a new interesting differential inequality method was used by L.E. Payne and his colaborators to study various aspects of the blow-up phenomena (see XXXX).

In this short note, we show that the results of [5] may be extended to the following class of quasilinear initial-boundary value problems
\[
\begin{align*}
(g(u)u_x)_x - u_t = -f(u), & \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial n} + \gamma u(x, t) = 0, & \quad x \in \partial\Omega, \ t > 0, \\
u(x, 0) = h(x), & \quad x \in \Omega,
\end{align*}
\]
where \( \gamma \) is a nonegative constant (so the Neumann boundary condition is covered as well), \( g \) is a positive non-increasing function and \( f, h \) are nonnegative functions assumed to satisfy \( f(0) = 0 \) and \( h(x) = 0 \) on \( \partial\Omega \). We notice that a similar class of parabolic equations, but under Dirichlet boundary condition, was treated in C. Enache [2].
It is known that the solution of problem (1.1) may not exists for all time and the only way that the solution can fail to exist is by becoming unbounded at some finite time $t^*$. In Section 2 of this paper we establish conditions on the data of problem (1.1) forcing the solution $u(x,t)$ to blow-up at some finite time $t^*$ and, under these conditions we determine an upper bound for the blow-up time $t^*$. In Section 3 we determine a lower bound on the blow-up time for a similar problem when blow-up of the solution does occur. In the final section, we determine conditions on the data sufficient to guarantee global boundedness of solution. In each section, our approach will follow naturally the first-order differential technique used in C. Enache [2] (in the case of the Dirichlet boundary condition) and L.E Payne-P.Schaefer (in the case $g \equiv 1$).

## 2 Criterion for blow-up of solution $u(x,t)$ in problem (1.1)

Let us introduce the following auxiliary functions

$$
F(s) := \int_0^s f(y)g(y)dy, \quad G(s) := 2\int_0^s yg(y)dy, \quad H(s) := \gamma \int_0^s g^2(y)dy, \quad s > 0
$$

$$
A(t) := \int_\Omega G(u(x,t))dx, \quad B(t) := 4\left\{\int_\Omega \left[F(u(x,t)) - \frac{1}{2}g^2|\nabla u|^2\right]dx - \int_{\partial\Omega} H(u(x,t))ds\right\}, \quad t \geq 0,
$$

where $u(x,t)$ is the solution of problem (1.1). The main result of this section is formulated in the following theorem:

**Theorem 2.1.** Let $u(x,t)$ be the classical solution of the parabolic problem (1.1). Assume that the data of problem (1.1) satisfy the following conditions

$$
sf(s)g(s) \geq 2(1 + \alpha)F(s), \quad s > 0,
$$

where $\alpha$ is a positive parameter, and $B(0) \geq 0$, i.e.

$$
\int_\Omega \left[F(h) - \frac{1}{2}g^2(h)|\nabla h|^2\right]dx - \int_{\partial\Omega} H(h)ds \geq 0.
$$

We then conclude that $u(x,t)$ blows up at some finite time $t^* < T$, with

$$
T := \frac{1}{\alpha(\alpha + 1)}A(0)B^{-1}(0) \leq \infty.
$$

**Proof.** We compute

$$
A'(t) = 2\int_\Omega fgudx + 2\int_\Omega gu_i'u_i'dx
$$

$$
= 2\int_\Omega fgudx - 2\int_{\partial\Omega} g^2u^2dx - 2\int_\Omega gu_i'u_i'u + gu_i'dx
$$

$$
\geq 4(1 + \alpha)\int_\Omega F(u)dx - 2(1 + \alpha)\int_\Omega g^2|\nabla u|^2dx + 2\int_{\partial\Omega} H(u)ds
$$

$$
= (1 + \alpha)B(t),
$$

where we have used successively the differential equation (1.1), the divergence theorem, the fact that $g' \leq 0$, the assumption (2.2) and the definition of $B(t)$.

On the other hand,

$$
B'(t) = -4\left\{\gamma \int_{\partial\Omega} g^2u_i'u_i'ds + \int_\Omega \left[gg'u_i'u_i'\nabla u|^2 + g^2\nabla u\nabla u_i' - fg'u_i'dx\right]\right\}
$$

$$
\geq 4\int_\Omega gu_i'dx
$$

where we have used again the divergence theorem. Therefore, $B(t)$ is a nondecreasing function in $t$ and, in view of (2.3) we have

$$
B(t) \geq B(0) \geq 0.
$$

Next, we make use of Schwartz inequality, of (2.5)-(2.6) and of the fact that $g' \leq 0$ to obtain the following chain of inequalities

$$
(1 + \alpha)A'B(t) \leq A'(t)^2 \leq (2\int_\Omega gu_i'dx)^2 \leq 4\int_\Omega gu_i'dx\int_\Omega gu^2dx
$$

$$
\leq 8\int_\Omega \left(\int_0^s sg(s)ds\right)dx \int_\Omega gu_i'dx
$$

$$
\leq AB(t)
$$

so that we have

$$
\frac{d}{dt}BA^{-(1+\alpha)} \geq 0,
$$

$$
\frac{d}{dt}(BA^{-(1+\alpha)}) \geq 0,
$$

$$
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$$

2
Integrating (2.9) from 0 to $t$ we obtain

$$B(t)/B(0) \geq [A(t)/A(0)]^{1+\alpha}.$$  \hspace{1cm} (2.10)

Therefore, combining (2.5) and (2.10) we have

$$A'(t) \geq MA(t)^{1+\alpha}, \quad \text{where } M := (1 + \alpha)B(0)A(0)^{-(1+\alpha)} \geq 0.$$  \hspace{1cm} (2.11)

Integrating now (2.11) from 0 to $t$, we obtain the inequality

$$A(t)^{-\alpha} \leq A(0)^{-\alpha} - \alpha Mt.$$  \hspace{1cm} (2.12)

But this inequality cannot hold for

$$t \geq T = \frac{1}{\alpha(\alpha + 1)}A(0)B^{-1}(0).$$  \hspace{1cm} (2.13)

In conclusion, the solution $u(x,t)$ of problem (1.1) fails to exist by blowing up at some finite time $t^* < T$, with $T > 0$ given in (2.4). □

3 A lower bound for global time

In this section we seek a lower bound on blow-up time for a non-negative solution of (1.1) when blow-up occurs at some finite time $t^*$. We assume that $\Omega$ is a bounded smooth convex domain in $\mathbb{R}^3$ and that the nonlinear functions $f$ and $g$ satisfy

$$0 < f(s) \leq kg(s) \left( \int_0^s \frac{1}{g(y)} dy \right)^{p+1} \text{ for all } s > 0,$$  \hspace{1cm} (3.1)

We have the following result:

**Theorem 3.1.** Assume that the data of problem (1.1) satisfy. Let $u(x,t)$ be a classical solution of the parabolic problem (1.1) in a convex domain $\Omega \subset \mathbb{R}^3$, which become unbounded at some finite time $t^*$. Then, the blow-up time $t^*$ is explicitly bounded below as it follows:

$$t^* \geq \int_{\eta_0}^{\infty} \frac{d\eta}{K_1 \eta^{3/2} + K_2 \eta^3}, \quad \text{where } \eta_0 := \int_\Omega \left( \int_0^{\eta(x)} \frac{1}{g(s)} ds \right) dx.$$  \hspace{1cm} (3.2)

and

$$K_1 = 3^{3/4}pk\rho^{-3/2}, \quad K_2 = \frac{p^7k^4}{2^5(2p-1)^3g_m^2} \left( \frac{d}{\rho} + 1 \right)^6, \quad \rho := \min_{\partial \Omega} \sum_{i=1}^3 x_in_i, \quad d^2 := \max_{\partial \Omega} |x|^2,$$  \hspace{1cm} (3.3)

$n_i$ being the $i$-th component of the unit outer normal vector to $\partial \Omega$ and summation from 1 to 3 is understood on repeated indices.

**Proof.** We define the auxiliary function

$$\Phi(t) := \int_\Omega v(u(x,t))^{2p} dx, \quad \text{with } v(s) := \int_0^s \frac{1}{g(y)} dy,$$  \hspace{1cm} (3.4)

and compute

$$\Phi'(t) = 2p \int_\Omega v^{2p-1} \left( g\partial_uu + f(u) \right) dx$$

$$= -2p \int_\Omega v^{2p-1} u dx - 2p \int_\Omega \left( (2p-1) v^{2p-2} \frac{|\nabla u|^2}{g} - v^{2p-1} \frac{g'}{g} |\nabla u|^2 - v^{2p-1} \frac{f'}{g} \right) dx$$

$$\leq 2pk \int_\Omega v^{2p} dx - \frac{2(2p-1)g_m}{p} \int_\Omega |\nabla v|^2 dx.$$  \hspace{1cm} (3.5)

since $g' \leq 0$ and

$$|\nabla v|^2 = p^2 v^{2p-2} |\nabla v|^2.$$  \hspace{1cm} (3.6)

We now remind an integral inequality due to L.E. Payne and P.W. Schaefer (see (2.16) in [6])

$$\int_\Omega v^{\beta} dx \leq \frac{1}{3^{\beta/2}} \left\{ \frac{3}{2p} \int_\Omega v^{2p} dx + \left( \frac{d}{\rho} + 1 \right) \left( \int_\Omega v^{2p} dx \right)^{1/2} \left( \int_\Omega |\nabla v|^2 dx \right)^{1/2} \right\}^{\beta/2}.$$  \hspace{1cm} (3.7)
Using in (3.7) the following inequalities
\[(a + b)^{3/2} \leq 2^{3/2} \left(a^{3/2} + b^{3/2}\right), \quad a^{1/4}b^{3/4} \leq \frac{1}{4}a\beta^{-3} + \frac{3}{4}b\beta \text{ for all } a, b, \beta \in \mathbb{R}_+^*,\] (3.8)
we obtain
\[
\int_{\Omega} v^3 dx \leq \frac{2^{1/2}}{3^{3/4}} \left( \left( \frac{3}{2p}\right)^{3/2} \left( \int_{\Omega} v^2 dx \right)^{3/2} + \left( \frac{d}{\rho} + 1 \right)^{3/2} \left[ \frac{1}{4}b^{-3} \left( \int_{\Omega} v^2 dx \right)^3 + \frac{3}{4}b \int_{\Omega} |\nabla v|^2 dx \right] \right). \] (3.9)
We now choose \(\beta\) as
\[
\beta := \frac{2^{1/2} (2p - 1) g_m (d/\rho + 1)^{-3/2}}{3^{1/4} p^2 k}, \tag{3.10}
\]
and make use of (3.9) in (3.5), to obtain
\[
\Phi'(t) \leq K_1 \Phi^{3/2} + K_2 \Phi^3, \tag{3.8}
\]
Integrating (3.8) from \(\Phi(0)\) to \(\infty\), we deduce the desired lower bound (3.2) for the blow-up time, assuming that the solution of (1.1) blows up at finite time \(t^*\). \(\square\)

4 A criterion for non blow-up

We note by \(\mu\) the first eigenvalue in the elastic membrane problem
\[
\Delta u + \mu u = 0, \quad x \in \Omega, \quad \frac{\partial u}{\partial n} + \sigma u = 0, \quad x \in \partial \Omega, \tag{4.1}
\]
with \(\sigma := \frac{4p^2\gamma}{4p - 1}\).

We have the following result.

**Theorem 4.1.** If \(u(x, t)\) is a nonnegative classical solution of (1.1) in a convex domain \(\Omega \subset \mathbb{R}^3\) where data \(f\) and \(g\) satisfy (3.1) with \(p \geq 1/4\) and \(h\) satisfies
\[
\int_{\Omega} h^4 dx \leq \left( \frac{4(p - 1)}{pM} \mu \right)^4, \quad \text{where} \quad M := \frac{2^{5/4}pk}{3^{3/8}} \left[ \left( \frac{\mu}{\sigma} \right)^{3/4} + \sigma^{3/4} \right], \tag{4.3}
\]
then, the solution \(u(x, t)\) exists for all time.

**Proof.** With \(v\) given in (3.1), we define
\[
\Psi(t) := \int_{\Omega} vu(x, t)^4 dx \tag{4.4}
\]
and compute
\[
\Psi'(t) = 4p \int_{\Omega} v^{4p - 1} \frac{1}{q} \left[ (gu, i) + f(u) \right] \]
\[
= 4p \int_{\Omega} v^{4p - 1} \frac{L}{q} dx - 4p \int_{\Omega} \frac{1}{q} u^{4p - 1} dx - 4p \int_{\Omega} \left[ (4p - 1) v^{4p - 2} u^{4p - 1} - u^{4p - 1} \frac{q}{q - 1} u, i \right] gu, dx \tag{4.5}
\]
\[
\leq 4pk \int_{\Omega} v^{5p} dx - 4p \gamma g_m \int_{\Omega} v^p C ds - \frac{4p - 1}{p} g_m \int_{\Omega} |\nabla v^2|^2 dx
\]
since \(g' \leq 0\) and (3.6)

Now, in order to bound the third integral in (4.5), we make use of the following integral inequality which was derived by L.E. Payne and P.W. Schaefer (see (4.8) in [5])
\[
\int_{\Omega} v^{5p} dx \leq \frac{1}{3^{1/8}} \left( \frac{d}{\rho} + 1 \right)^{3/2} \left[ \sigma \int_{\Omega} v^p ds + \int_{\Omega} |\nabla v^2|^2 dx \right]^{3/4} + \sigma^{3/2} \left( \int_{\Omega} v^p \right)^{3/4} \int_{\Omega} v^4 dx \tag{4.6}
\]
Moreover, from (4.1) it follows that
\[
Q := \sigma \int_{\Omega} v^p ds + \int_{\Omega} |\nabla v^{2p}|^2 dx \geq \mu_1 \int_{\Omega} v^p dx. \tag{4.7}
\]
Using now (4.6) and (4.7) in (4.5), we have
\[
\Psi'(t) \leq Q^{3/4}\left\{-\frac{4p-1}{p}\mu^{1/4}\Psi^{1/4} + \frac{4pk}{3^{1/8}(2\sigma)^{3/4}}\Psi^{1/2}\right\} + \frac{2^{5/4}pk}{3^{1/8}3^{3/4}\psi^{5/4}}.8 \tag{1}
\]

\[
\leq Q^{3/4}\{\}
\tag{2}
\]

As long as the brace in (4.8) is negative, we can use again (4.7) to obtain
\[
\Psi'(t) \leq -\Psi\left[\frac{4p-1}{p}\mu - M\Psi^{1/4}\right]. \tag{3}
\]

Hence, from (4.3) follows that the solution $u(x,t)$ decays in norm and will exist for all time.

References