Oscillation and Stability in Nonlinear Delay Differential Equations of Population Dynamics

I. KUBIACZYK AND S. H. SAKER
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Matejki 48/49,60-769 Poznan, Poland
<kuba><shaker>gamu.edu.pl
shaker@mm.unis.eg
(Received April 2001; accepted May 2001)

Abstract—In this paper, we shall study the oscillation of all positive solutions of the nonlinear delay differential equation

\[ x'(t) + \frac{\alpha V_m x(t) x^n(t - \tau)}{\theta x^{n+1}(t - \tau)} = \lambda, \]

(*)

and

\[ x'(t) + p x(t) - \frac{q x(t)}{r + x^n(t - \tau)} = 0 \]

(**)

about their equilibrium points. Also, we study the stability of these equilibrium points and prove that every nonoscillatory positive solution tends to the equilibrium point when \( t \) tends to infinity.

Where equation (*) proposed by Mackey and Glass [1] for a "dynamic disease" involving respiratory disorders and equation (**) is one of the models proposed by Nazarenko [2] to study a control of a single population of cells. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Oscillation, Nonlinear delay differential equations, Population dynamics.

1. INTRODUCTION

The literature on the applications of qualitative theory of nonlinear delay differential equations are growing rapidly day by day. This study is a relatively new field and is very interesting in applications in some mathematical models in ecology, biology, and spread of some infectious diseases in humans. For some contributions, we refer to the articles by Kulenovic and Ladas [3], Kulenovic, Ladas and Sficas [4], and Gopalsamy and Trofimchuk [5], which studied the oscillation and global attractivity in population dynamics of Wazewska-Czyzewska and Lasota [6] as a model for survival of red blood cells in animals. Gopalsamy, Kulenovic and Ladas [7] obtained some sufficient conditions for the existence of a globally attracting positive solution of the "food-limited" population model. Gopalsamy and Ladas [8] and Ladas and Qian [9] obtained some necessary and sufficient conditions for oscillation of all positive solution of delay logistic model about its
positive equilibrium point and obtained conditions under which the positive equilibrium is globally asymptotically stable. Berezansky and Braverman [10] studied oscillation and nonoscillation of a logistic model with several delays about its equilibrium point. Lalli and Zhang [11] studied the periodicity on delay population model with periodic coefficients. Recently, Olach [12] obtained some sufficient conditions for oscillation and nonoscillation of all positive solutions of the dynamics of a single species population model and Elabbasy, Saker and Saif [13] studied the oscillation of all positive solutions of host macroparasite model with delay time about its equilibrium point and proved that every nonoscillatory solution tends to this equilibrium point when $t$ tends to infinity. In particular, several authors have studied the oscillation caused by the deviating arguments in some mathematical models.

Our aim in this paper is to obtain some sufficient conditions for oscillation of all positive solutions of

$$
\dot{x}(t) + \frac{\alpha V_m x(t)x^n(t-\tau)}{\theta^n + x^n(t-\tau)} = \lambda, \quad t \geq 0,
$$

with

$$\alpha, V_m, \theta, \tau, \lambda \in (0, \infty), \quad n \in \mathbb{N}
$$

about the equilibrium point and give some sufficient conditions for locally asymptotically stable of this equilibrium point and prove that every nonoscillatory positive solution tends to the equilibrium point when $t$ tends to infinity.

Also, some necessary and sufficient conditions for oscillation of all positive solutions of

$$
\dot{x}(t) + px(t) - \frac{qx(t)}{r + x^n(t-\tau)} = 0, \quad t \geq 0,
$$

with

$$p, q, r, \tau \in (0, \infty), \quad n > 0, \quad \text{and} \quad \frac{q}{p} > r
$$

about the equilibrium point and give some sufficient conditions for locally asymptotically stable of this equilibrium point, and prove that every nonoscillatory positive solution tends to the equilibrium point when $t$ tends to infinity.

Mackey and Glass [1] have described some physiological control systems by three nonlinear delay differential equations. Two of them are proposed as models of hematopoiesis (blood cell production) these two equations are

$$\dot{p}(t) = \frac{\beta \theta^n}{\theta^n + p^n(t-\tau)} - \gamma p(t),
$$

$$\dot{p}(t) = \frac{\beta \theta^n p(t-\tau)}{\theta^n + p^n(t-\tau)} - \gamma p(t),
$$

where the time delay $\tau$ is the time between the production of immature cells in the bone marrow and their maturation for release in the circulating bloodstream, and $p(t)$ denotes the density of mature cells in blood circulation. Subsequently, many authors from various angels study these two equations. In particular, see [14–16].

Equation (1.1) proposed by Mackey and Glass [1] for a “dynamic disease” involving respiratory disorders, where $x(t)$ denotes the arterial CO$_2$ concentration of a mammal, $\lambda$ is the CO$_2$ production rate, $V_m$ denotes the maximum ventilation rate, and $\tau$ is the time between oxygenation of blood in lungs and stimulation of Chemoreceptors in the brainstem, and equation (1.3) is one of the models proposed by Nazarenko [2] to study a control of a single population of cells.

The paper is organized as follows, in Section 2, we study the oscillation of all solutions of equation (1.1) about its fixed point and study the stability of this point, and in Section 3, we study the oscillation of all positive solutions of equation (1.3) about its fixed point and study the stability of this point.
As usual a function \( z(t) \) is said to be oscillate about \( K \) if the function \((z(t) - K)\) has arbitrarily large zeros, we say that a solution oscillates if it oscillates about zero (otherwise is called nonoscillatory).

We only consider solutions of equation (1.1) and equation (1.3) with initial conditions of the form

\[
\begin{align*}
x(t) &= \phi(t), \quad \text{for } -\tau \leq t \leq 0, \\
\phi &\in C([-\tau, 0], [0, \infty]) \quad \text{and} \quad \phi(0) > 0.
\end{align*}
\]

By the method of steps one can prove that (1.1) and (1.7), and (1.3) and (1.7) have positive solutions for all \( t \geq 0 \).

2. OSCILLATION AND STABILITY OF EQUATION (1.1)

In this section, we establish some sufficient conditions for oscillation of all positive solutions of equation (1.1) about \( K^* = \frac{1}{k} \), where \( k \) is positive equilibrium point of the equation

\[
\dot{y}(t) = y(t) \left[ \frac{P_1}{q_1 + y^n(t-\tau)} - \gamma y(t) \right],
\]

with

\[
P_1 = \frac{a V_m}{\partial n} \quad \text{and} \quad q_1 = \frac{1}{\partial n}
\]

are positive constants.

Also, we investigate the sufficient condition of \( K^* \) to be locally asymptotically stable and prove that every nonoscillatory solution tends to \( K^* \) as \( t \to \infty \). Equation (2.1) given from equation (1.1) by setting \( x(t) = \frac{1}{y(t)} \).

It is clear that equation (1.1) oscillates about \( K^* \) iff \( y(t) \) oscillates about \( k \), which satisfy the equation

\[
\lambda k^2 - \frac{P_1 k}{q_1 + k^n} = 0
\]

or

\[
\lambda k^{n+1} + \lambda q_1 k - p_1 = 0.
\]

Set

\[
F(x) = \lambda x^{n+1} + \lambda q_1 x - p_1.
\]

Hence, \( F(0) = -p_1 \) and \( F(\infty) = \infty \). Thus, \( F(x) = 0 \) has a positive root \( k \in (0, \infty) \).

THEOREM 2.1. Assume that (1.2) and (2.2) hold. Then every positive solution of equation (1.1) oscillates about \( K^* \) if

\[
\frac{n p_1 k^n}{(q_1 + k^n)^2} e^{\lambda \tau \tau} > \frac{1}{e}.
\]

PROOF. We will consider equation (2.1). Set

\[
y(t) = k e^{z(t)}.
\]

Then \( y(t) \) oscillates about \( k \) iff \( z(t) \) oscillates about zero. And then \( x(t) \) oscillates about \( K^* \).

From (2.5), we have \( \dot{y}(t) = k \dot{z}(t) e^{z(t)} \), substitute in (2.1), we find that

\[
k \dot{z}(t) e^{z(t)} + \lambda k^2 e^{z(t)} - \frac{P_1 k e^{z(t)}}{q_1 + k^n e^{z(t-\tau)}} = 0
\]

or

\[
\dot{z}(t) + \lambda k e^{z(t)} - \frac{P_1}{q_1 + k^n e^{z(t-\tau)}} = 0,
\]

hence,

\[
\dot{z}(t) + \lambda k \left( e^{z(t)} - 1 \right) + \lambda k - \frac{P_1}{q_1 + k^n e^{z(t-\tau)}} = 0.
\]
But from (2.3), we have \( \lambda k = p_1/(q_1 + k^n) \), then

\[
\dot{z}(t) + \lambda k \left( e^{z(t)} - 1 \right) + \frac{p_1}{q_1 + k^n} = \frac{p_1}{q_1 + k^n e^{z(t-\tau)}} = 0
\]

or

\[
\dot{z}(t) + \lambda k \left( e^{z(t)} - 1 \right) + \frac{p_1 nk^n}{(q_1 + k^n)^2} \frac{(q_1 + k^n)(e^{z(t-\tau)} - 1)}{n(q_1 + k^n e^{z(t-\tau)})} = 0,
\]

then

\[
\dot{z}(t) + \lambda k f_1(z(t)) + \frac{p_1 nk^n}{(q_1 + k^n)^2} f_2(z(t-\tau)) = 0 
\]

(2.6)

with

\[
f_1(u) = e^u - 1 \quad \text{and} \quad f_2(u) = \frac{q_1 + k^n}{n} \frac{e^{nu} - 1}{q_1 + k^n e^{nu}}.
\]

Note that

\[
u f_1(u) > 0, \quad \text{for } u \neq 0, \quad \lim_{u \to 0} \frac{f_1(u)}{u} = 1 \tag{2.7}
\]

and

\[
u f_2(u) > 0, \quad \text{for } u \neq 0, \quad \lim_{u \to 0} \frac{f_2(u)}{u} = 1. \tag{2.8}
\]

The linearized equation associated with equation (2.6) is given by

\[
m(t) + \lambda km(t) + \frac{p_1 nk^n}{(q_1 + k^n)^2} m(t-\tau) = 0 \tag{2.9}
\]

and every solution of equation (2.9) oscillates if and only if (2.4) holds (see [15, Corollary 2.2.1]).

The proof is now elementary consequence of the linearized oscillation Theorem 4.1.1 in [15] according to which if (2.7) and (2.8) hold and every solution of equation (2.9) oscillates, then every solution of equation (2.6) also oscillates.

**Theorem 2.2.** Assume that (1.2) and (2.2) hold, and

\[
\frac{n p_1 k^n}{(q_1 + k^n)^2} e^{e^{k\lambda t}} = \frac{\pi}{2}, \tag{2.10}
\]

then \( K^* \) is locally asymptotically stable.

**Proof.** To prove that \( K^* \) is locally asymptotically stable, we prove that \( k \) is locally asymptotically stable for equation (2.1). The linearized equation of (2.1) about \( k \) is equation (2.9). Set \( u(t) = e^{-\lambda k t} m(t) \), then

\[
\dot{u}(t) + \frac{p_1 nk^n e^{\lambda k t}}{(q_1 + k^n)^2} u(t-\tau) = 0. \tag{2.11}
\]

Then if the trivial solution of (2.11) is locally asymptotically stable then the trivial solution of (2.9) is locally asymptotically stable. But (2.10) implies that the trivial solution of (2.11) is asymptotically stable. Therefore, by the linearized stability theory \( k \) is locally asymptotically stable. (see [15, Lemma 3]). Then \( K^* \) is locally asymptotically stable.

The following theorem deals with the asymptotic behavior of nonoscillatory solutions of equation (1.1)

**Theorem 2.3.** Assume that (1.2) and (2.2) holds. Let \( x(t) \) be a positive solution of equation (1.1), which does not oscillate about \( K^* \). Then \( \lim_{t \to \infty} x(t) = K^* \).

**Proof.** To prove this theorem, we may to prove that \( \lim_{t \to \infty} y(t) = k \).

Assume that \( y(t) > k \) for \( t \) sufficiently large (the proof when \( y(t) < k \) is similar and will be omitted). Set

\[
y(t) = ke^{x(t)}.
\]
Then $z(t) > 0$ for $t$ sufficiently large and

$$
\dot{z}(t) + \lambda k \left( e^{z(t)} - 1 \right) + \frac{p_1 k^n}{(q_1 + k^n)} \frac{(e^{nz(t-\tau)} - 1)}{(q_1 + k^n e^{nz(t-\tau)})} = 0,
$$

but

$$
0 < \left( e^{z(t)} - 1 \right).
$$

Then for $t$ sufficiently large, we have

$$
\dot{z}(t) + \frac{p_1 k^n}{(q_1 + k^n)} \frac{(e^{nz(t-\tau)} - 1)}{(q_1 + k^n e^{nz(t-\tau)})} \leq 0 \tag{2.12}
$$
or

$$
\dot{z}(t) \leq -\frac{p k^n}{(q + k^n)} \frac{(e^{nz(t-\tau)} - 1)}{(q + k^n e^{nz(t-\tau)})} < 0.
$$

Then $z(t)$ is decreasing and so $z(t - \tau) > z(t)$.

Hence,

$$
\lim_{t \to \infty} z(t) = \alpha \in [0, \infty),
$$

we prove that $\alpha = 0$, otherwise $\alpha > 0$. Then there exist $T > 0$ such that for $t \geq T$, $0 < \alpha - \varepsilon < z(t) < \alpha + \varepsilon$. As $z(t)$ is decreasing, we have

$$
\alpha - \varepsilon < z(t - \tau),
$$

then from (2.10), we have

$$
\dot{z}(t) + \frac{p_1 k^n}{(q_1 + k^n)} \frac{(e^{n(\alpha - \varepsilon)} - 1)}{(q_1 + k^n e^{n(\alpha - \varepsilon)})} < 0. \quad t > T. \tag{2.13}
$$

Integrating (2.13) from $T$ to $\infty$, we have a contradiction. Then $\alpha = 0$, and so $z(t)$ tends to zero as $t \to \infty$.

### 3. Oscillation and Stability of Equation (1.3)

In this section, we establish some necessary and sufficient conditions for oscillation of all positive solutions of equation (1.3) about its positive steady state $K$. Also, we investigate the sufficient condition of $K$ to be locally asymptotically stable, and prove that every nonoscillatory solution tends to $K$ as $t \to \infty$. Where equation (1.3) has a unique positive steady state

$$
K = \left[ \frac{q}{p} - r \right]^{1/n}.
$$

**Theorem 3.1.** Assume that (1.4) holds. Then every solution of equation (1.3) oscillates about $K$ iff

$$
\frac{n q K^n}{(r + K^n)^2} \tau > \frac{1}{e}. \tag{3.1}
$$

**Proof.**

$$
\dot{x}(t) = K e^{z(t)},
$$

which is invariant oscillation transformation, so we have

$$
\dot{z}(t) = K \dot{z}(t) e^{z(t)}.
$$
Substituting in (1.3), we get
\[ \dot{z}(t) + p - \frac{q}{r + K^n e^{nz(t-\tau)}} = 0. \]

But \( p = q/(r + K^n) \). Hence,
\[ \dot{z}(t) + \frac{q}{r + K^n} - \frac{q}{r + K^n e^{nz(t-\tau)}} = 0. \]

That is
\[ \dot{z}(t) + \frac{nqK^n}{(r + K^n)^2} \frac{(r + K^n)(e^{nz(t-\tau)} - 1)}{n(r + K^n e^{nz(t-\tau)})} = 0. \]

Hence,
\[ \dot{z}(t) + \frac{nqK^n}{(r + K^n)^2} f(z(t - \tau)) = 0, \]  

where
\[ f(u) = \frac{(r + K^n)(e^{nu} - 1)}{n(r + K^n e^{nu})}. \]

Note that
\[ uf(u) > 0, \quad \text{for } u \neq 0 \]  

and
\[ \lim_{u \to 0} \frac{f(u)}{u} = 1, \]  

also we claim that
\[ f(u) < u, \quad \text{for } u > 0. \]

The proof of (3.5) follows from the observation that \( f(0) = 0 \) and that
\[ \frac{d}{du} [f(u) - u] = \frac{(r + K^n)^2 e^{nu}}{(r + K^n e^{nu})^2} - 1 = - \left[ \frac{(r + K^n e^{nu})^2 - (r + K^n)^2 e^{nu}}{(r + K^n e^{nu})^2} \right] < 0, \quad \text{for } u > 0 \]

The linearized equation associated with equation (3.2) is
\[ \hat{y}(t) + \frac{nqK^n}{(r + K^n)^2} y(t - \tau) = 0 \]

and every solution of equation (3.6) oscillates iff (3.1) holds (see [7, Theorem 2.2.3]). The proof is now elementary consequence of the linearized oscillation Corollary 4.1.1 in [7] according (to which if (3.3)–(3.5) hold and every solution of equation (3.6) oscillates, then every solution of equation (3.2) also oscillates.

**THEOREM 3.2.** Assume that (1.4) holds, and
\[ \frac{nqK^n}{(r + K^n)^2} \tau < \frac{\pi}{2}, \]  

then \( K \) is locally asymptotically stable.

**Proof.** The linearized equation of (1.3) about \( K \) is (3.6). Then (3.7) implies that the trivial solution of (3.6) is asymptotically stable. Therefore, by the linearized stability theory \( K \) is locally asymptotically stable (see [7, Lemma 3]).
THEOREM 3.3. Let \( z(t) \) be a positive solutions of equation (1.3) which is nonoscillatory about \( K \). Then

\[
\lim_{t \to \infty} z(t) = K. \tag{3.8}
\]

PROOF. We shall assume that eventually \( z(t) \geq K \). The case where eventually \( z(t) \leq K \) is similar and will be omitted. Equation (1.3) can be written in the form

\[
x(t) + \frac{p(z(t))}{r + x^n(t-\tau)} = 0.
\]

That is

\[
\dot{x}(t) = -\left[ \frac{p(z(t))}{r + x^n(t-\tau)} \right]. \tag{3.9}
\]

From the hypothesis that \( x(t) \geq K \) and (3.9), we have

\[
x(t) \leq 0.
\]

Then \( x(t) \) is decreasing and so

\[
\lim_{t \to \infty} x(t) = L \in (0, \infty), \text{ exists,} \tag{3.10}
\]

we prove that \( L = K \). Otherwise, \( L < K \) and (3.9) yields

\[
\lim_{t \to \infty} \dot{x}(t) = -\left[ \frac{pL^{n+1} - pKL}{r + L^n} \right] < 0.
\]

Then \( \lim_{t \to \infty} x(t) = -\infty \), which contradicts (3.10).

REFERENCES