Classes of structures with universe a subset of $\omega_1$

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Abstract

We continue recent work on computable structure theory in the setting of $\omega_1$. We prove the analogue of a result from Fokina et al. (2012 J. Symbolic Logic, 77, 122–132) saying that isomorphism of computable structures lies ‘on top’ among $\Sigma^1_1$ equivalence relations on $\omega$. Our equivalence relations are on $\omega_1$. In the standard setting, $\Sigma^1_1$ sets are characterized in terms of paths through trees. In the setting of $\omega_1$, we use a new characterization of $\Sigma^1_1$ sets that involves clubs in $\omega_1$. Finally, we present some new results about $\omega_1$-computable categoricity for fields.

Keywords: $\omega_1$-computability, computable uncountable structure, turing computable embeddings, computable categoricity, complete equivalence relation.

1 Introduction

Recent work on computable structure theory has come to include the setting of $\omega_1$-computability, in addition to the standard setting of $\omega$-computability. In [13], there is a sample of results, including some that transfer immediately from the standard setting, some that transfer in modified form, and some that do not transfer at all. The main motivation for this work is that there are familiar uncountable structures, such as the field of real numbers and the field of complex numbers, which feel computable. With suitable assumptions, these structures actually are computable in $\omega_1$. In [5], there is a definition of the arithmetical hierarchy in the setting of $\omega_1$, and of ‘computable $\Sigma^\alpha_\alpha$’ and ‘computable $\Pi^\alpha_\alpha$’ formulas, for countable ordinals $\alpha$. (These are formulas of $L_{\omega_2, \omega_1}$, as opposed to $L_{\omega_1^{<\omega}}$.) The main result of [5] is that for countable ordinals $\alpha$, a relation $R$ on a computable structure $A$ is ‘relatively intrinsically $\Sigma^\alpha_\alpha$’ iff it is definable by a computable $\Sigma^\alpha_\alpha$ formula.

In [14] and [12], there are results on computable categoricity in the setting of $\omega_1$. In [14], it is shown that the ‘Zil’ber field’ of size $\aleph_1$ is not computably categorical, while the ‘Zil’ber cover’ of size $\aleph_1$ is computably categorical. The Zil’ber fields are structures that resemble the field of complex numbers, with complex exponentiation. The Zil’ber covers are a related class of structures. There is
2 \( \omega_1 \)-Computable structure theory

a general condition on ‘quasi-minimal excellent’ classes, saying exactly when the member of size \( \aleph_1 \) will be computably categorical. In [12], there is a characterization of the linear orderings that are computably categorical in the setting of \( \omega_1 \).

In the remainder of Section 1, we give a very brief summary of the basic definitions and elementary results on computability and computable structures in the setting of \( \omega_1 \). For more details, we refer the reader to [13]. In Section 2 we define an analogue of the \( \Sigma^1_1 \) sets, and imitate a result of Kleene on \( \Sigma^1_1 \)-completeness in order to give an analogue of a theorem from [7] concerning \( \Sigma^1_1 \) equivalence relations. Our result involves clubs (closed and unbounded sets) instead of paths through trees. In Section 3, we define the notion of a computable embedding of one class of structures (of size \( \aleph_1 \)) into another, giving several examples. In particular, we give an analogue of a result of H. Friedman and Stanley [8], concerning universality of linear orderings. Finally, in Section 4, we study computable categoricity of fields in the context of \( \omega_1 \).

1.1 Basic computability in the setting of \( \omega_1 \)

We assume at least that all subsets of \( \omega \) are constructible, and in some places, we assume that all subsets of \( \omega_1 \) are constructible. The basic definitions come from ‘\( \alpha \)-recursion’ theory, where \( \alpha = \omega_1 \) (see [23]).

Definition 1.1

- A set or relation on \( \omega_1 \) is computably enumerable, or c.e., if it is defined in \( (L_{\omega_1}, \in) \) by a \( \Sigma_1 \)-formula \( \varphi(\bar{c}, x) \), with finitely many parameters—a \( \Sigma_1 \) formula is finitary, with only existential and bounded quantifiers.
- A set or relation is computable if it and its complement are both computably enumerable.
- A (partial) function is computable if its graph is c.e.

Results of Gödel give us a \( 1 \rightarrow 1 \) function \( g \) from \( \omega_1 \) onto \( L_{\omega_1} \) such that the relation \( g(\alpha) \in g(\beta) \) is computable. The function \( g \) gives us ordinal codes for sets, so that computing on \( \omega_1 \) is really the same as computing on \( L_{\omega_1} \). There is also a computable function \( \ell \) taking \( \alpha \) to the code for \( L_\alpha \). Using the fact that \( L_{\omega_1} \) is closed under \( \alpha \)-sequences for any countable ordinal \( \alpha \), we may allow relations and functions of arity \( \alpha \), where \( \alpha \) is any countable ordinal.

As in the standard setting, we have indices for c.e. sets. There is a c.e. set \( C \) of codes for pairs \( (\varphi, \vec{\tau}) \), representing \( \Sigma_1 \) definitions, where \( \varphi(\bar{\tau}, x) \) is a \( \Sigma_1 \)-formula and \( \vec{\tau} \) is a tuple of parameters appropriate for \( \bar{\tau} \). There is a computable function \( h \) mapping \( \omega_1 \) onto \( C \). A set \( X \) has index \( \alpha \) if \( h(\alpha) \) is the code for a pair \( (\varphi, \vec{\tau}) \) representing a \( \Sigma_1 \) formula \( \varphi(\vec{\tau}, x) \), and \( X \) is the set defined in \( (L_{\omega_1}, \in) \) by this formula. We write \( W_\alpha \) for the c.e. set with index \( \alpha \). Suppose \( W_\alpha \) is determined by the pair \( (\varphi, \vec{\tau}) \); i.e., \( \varphi(\vec{\tau}, x) \) is a \( \Sigma_1 \) definition. We say that \( x \in W_\alpha \) at stage \( \beta \), and we write \( x \in W_\alpha, \beta \), if \( L_\beta \) contains \( x \), the parameters \( \vec{\tau} \), and witnesses making the formula \( \varphi(\vec{\tau}, x) \) true. The relation \( x \in W_\alpha, \beta \) is computable. Let \( U \subseteq (\omega_1)^2 \) consist of the pairs \( (\alpha, \beta) \) such that \( \beta \in W_\alpha \). Then \( U \) is \( m \)-complete c.e. It is not computable, since the ‘halting set’ \( K = \{ \alpha : \alpha \in W_\alpha \} \) is c.e. and not computable.

In the setting of \( \omega_1 \), we have a good notion of relative computability.

Definition 1.2

- A relation is c.e. relative to \( X \) if it is \( \Sigma_1 \)-definable in \( (L_{\omega_1}, \in, X) \).
- A relation is computable relative to \( X \) if it and its complement are both c.e. relative to \( X \).
- A (partial or total) function is computable relative to \( X \) if the graph is c.e. relative to \( X \).
A c.e. index for $R$ relative to $X$ is an ordinal $\alpha$ such that $g(h(\alpha)) = (\varphi, \tau)$, where $\varphi$ is a $\Sigma_1$ formula (in the language with $\in$ and a predicate symbol for $X$), and $\varphi(\tau, x)$ defines $R$ in $(L_{\omega_1}, \in, X)$. We write $W^X_\alpha$ for the c.e. set with index $\alpha$ relative to $X$.

As in the standard setting, we have a universal c.e. set of partial computations using oracle information. Let $U$ consist of the codes for triples $(\sigma, \alpha, \beta)$ such that $\sigma \in 2^\rho$ (for some countable ordinal $\rho$), and for $X$ with characteristic function extending $\sigma$, $\beta \in W^X_\alpha$. Then $U$ is c.e.

**Definition 1.3**

The jump of $X$ is the set $X' = \{(\alpha, x) : x \in W^X_\alpha\}$.

We can iterate the jump function through countable levels. We let $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$, and for limit $\alpha$, $X^{(\alpha)}$ is the set of codes for pairs $(\beta, x)$ such that $\beta < \alpha$ and $x \in X^{(\beta)}$. As $L_{\omega_1}$ is closed under countable sequences, it follows that for countable limit $\lambda$, $X^{(\lambda)}$ is the least upper bound of the $X^{(\alpha)}$ for $\alpha < \lambda$, in the ordering of relative computability.

### 1.2 A little computable structure theory

We consider structures with universe a subset of $\omega_1$. As in the standard setting, we usually identify a structure with its atomic diagram. A structure is computable if the atomic diagram is computable. A structure is decidable if the complete diagram is computable. We mention some simple examples, taken from [13]. The ordered field of reals has a computable copy with universe $\omega_1$. If we think of the reals as a subset of $L_{\omega_1}$, where each number is represented by a rational cut, this is a computable structure. The field of complex numbers has a computable copy. We may even add exponential functions such as exp, noting that any analytic function is determined by the countable sequence of coefficients of a power series.

In the standard setting, Morley [20] and Millar [19] showed that for any countable complete decidable elementary first-order theory $T$, there is a decidable saturated model if and only if there is a computable enumeration of the complete types consistent with $T$. In the setting of $\omega_1$, we have the following (see [13]).

**Proposition 1.4**

For any countable complete elementary first-order theory $T$ (with infinite models), $T$ has a decidable saturated model with universe $\omega_1$.

In the standard setting, the first non-computable ordinal, $\omega^{CK}_1$, is the next admissible ordinal after $\omega$. In the setting of $\omega_1$, the first non-computable ordinal comes much before the next admissible after $\omega_1$. This was well-known in the 1970s. The simple proof is found in [9] and [13]. In the standard setting, the Harrison ordering is a computable ordering of type $\omega^{CK}_1 \cdot (1 + \eta)$. This ordering has initial segments isomorphic to all computable well orderings. In the setting of $\omega_1$, we have the following (see [13]).

**Theorem 1.5 (Greenberg-Knight-Shore)**

There is a computable ordering $H$ with initial segments isomorphic to all computable ordinals.

**Sketch of Proof.** We take a uniformly computable list of linear orderings, representing all computable isomorphism types, and carry out a finite-injury priority construction to produce $H$ with

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1Here is the argument: let $\alpha$ be the least admissible after $\omega_1$. Then the set of computable well orderings of $\omega_1$ is an element of $L_\alpha$ and the function $f$ that takes such a well ordering to its length is $\Sigma_1$ definable over $L_\alpha$; it follows that the range of $f$ is bounded in $\alpha$. 
4 \( \omega_1 \)-Computable structure theory

an initial segment that is a sum of intervals representing the well-ordered \( A_\alpha \), in order, followed by various other intervals that are not well ordered.

In the standard setting, the property of being well ordered is complete \( \Pi^1_1 \). We have seen that in the setting of \( \omega_1 \) being well ordered is relatively simple. The following result holds in the standard setting [7].

**Theorem 1.6 (Fokina-S.Friedman-Harizanov-Knight-McCoy-Montalbán)**
For every \( \Sigma^1_1 \) equivalence relations \( E \) on \( \omega_1 \), there is a uniformly computable sequence of trees \((T_n)_{n \in \omega} \) (subtrees of \( \omega^{<\omega_1} \)) such that

\[
mEn \iff T_m \cong T_n.
\]

In the proof of Theorem 1.6, in the standard setting, the isomorphism type of each tree \( T_n \) is determined by an \( \omega \)-sequence of computable ordinals, or \( \infty \).

In [7], the result for trees is used to show that isomorphism on computable members of certain other classes lies on top in the same way: notably, torsion-free Abelian groups and Abelian \( p \)-groups.

In the next section, we shall lift Theorem 1.6 to the setting of \( \omega_1 \). Since there are so few computable ordinals, we shall need some fresh ideas.

**Theorem 1.7**
Assume \( V = L \). For any \( \Sigma^1_1 \) equivalence relation \( E \) on \( \omega_1 \), there is a uniformly computable sequence of structures \( M^*(\alpha)_{\alpha<\omega_1} \) (with universe \( \omega_1 \)) such that \( \alpha \mathrel{E} \beta \iff M^*(\alpha) \cong M^*(\beta) \).

2 \( \Sigma^1_1 \) sets

Recall that in the standard setting, a set \( S \subseteq \omega \) is \( \Sigma^1_1 \) if there is a computable relation \( R(x,u) \) such that

\[
n \in S \iff (\exists f \in \omega^\omega)(\forall s \in \omega) R(n,f \upharpoonright s).
\]

Kleene showed the following.

**Theorem 2.1 (Kleene)**
If \( S \) is \( \Sigma^1_1 \), then there is a uniformly computable sequence of subtrees \((T_x)_{x \in \omega} \) of \( \omega^{<\omega} \) such that \( x \in S \) iff \( T_x \) has a path.

In the standard setting, a computable tree with no path has a tree rank that is a computable ordinal. The ordinal tree ranks were crucial to the proof of Theorem 1.6. In our setting, we do not have enough computable ordinals, so we will need a new idea. We take the following as our definition of \( \Sigma^1_1 \) subset of \( \omega_1 \).

**Definition 2.2**
A set \( S \subseteq \omega_1 \) is \( \Sigma^1_1 \) if there is a computable relation \( R \), on ordinals and functions in \( \omega_1^{<\omega_1} \), such that \( x \in S \) iff \( \langle \exists f \in \omega_1^{<\omega_1} \rangle (\forall \beta \in \omega_1) R(x,f \upharpoonright \beta) \).

Thus \( \Sigma^1_1 \) sets are projections of ‘co-c.e.’ subsets of \( \omega_1^{<\omega_1} \), defined using \( \Pi^1_1 \) formulas and parameters from \( L_{\omega_1} \). As in the standard setting, we can replace \( \Pi^1_1 \) by \( \Pi^1_n \) here for any finite \( n \), using Skolem functions to replace alternating quantification over \( L_{\omega_1} \) with existential quantification over functions in \( \omega_1^{<\omega_1} \) (for the details see Chapter 16 of [22]).
A subtree $T$ of $\omega_1^{<\omega_1}$ is a subset of $\omega_1^{<\omega_1}$ which is closed under initial segments (i.e. if $\sigma$ belongs to $T$ then so does $\sigma \upharpoonright \beta$ for all $\beta$). Two such trees are isomorphic if there is a bijection between them which preserves the initial segment relation $\sigma \subseteq \tau$. The same definitions apply to subtrees of $A^{<\omega_1}$ for any set $A$.

**Lemma 2.3**
For any $\Sigma^1_1$ set $S \subseteq \omega_1$, there is a uniformly computable sequence $(T_x)_{x<\omega_1}$ of subtrees of $\omega_1^{<\omega_1}$ such that $x \in S$ iff $T_x$ has an $\omega_1$-branch.

**Proof.** We do just what Kleene did. Let $T_x$ consist of those $\sigma \in \omega_1^{<\omega_1}$ such that $\forall \beta < \text{length}(\sigma) R(x, \sigma \upharpoonright \beta)$.

The structures that we produce for our main result (Theorem 1.7) are not members of any familiar class. The structures in the range of our embedding will each code a sequence of sets $(X_\beta)_{\beta<\omega_1}$, up to an equivalence relation $\sim$, which is defined as follows.

**Definition 2.4**
For $X, Y \subseteq \omega_1$, $X \sim Y$ iff $X \Delta Y$ is not stationary, where $X \Delta Y$ denotes the symmetric difference of $X$ and $Y$.

**Lemma 2.5**
For any $\Sigma^1_1$ set $X \subseteq \omega_1$, there is a uniformly computable sequence $(S_x)_{x<\omega_1}$ of subsets of $\omega_1$ such that $x \in X$ iff $S_x$ contains a club.

**Proof.** Choose a uniformly computable sequence of trees $(T_\alpha)_{\alpha<\omega_1}$ as in Lemma 2.3. Thus $\alpha \in S$ iff $T_\alpha$ has an $\omega_1$-branch. Choose a parameter $p \in L_{\omega_1}$ and a $\Sigma_1$ formula $\varphi$ so that $\sigma \in T_\alpha$ iff $L_{\omega_1} \models \varphi(p, \alpha, \sigma)$. This is possible as the uniformly computable sequence of $T_\alpha$’s is $\Sigma_1$ definable with some parameter over $L_{\omega_1}$.

For ordinals $\alpha < \beta \leq \omega_1$ such that $p$ belongs to $L_\beta$ we let $T_\alpha^\beta$ be the interpretation of the tree $T_\alpha$ in $L_\beta$, i.e. $\{ \sigma \mid L_\beta \models \varphi(p, \alpha, \sigma) \}$. This may not be a tree for all such pairs $\alpha < \beta$. By definition we have that $T_\alpha^\alpha = T_\alpha$.

Now let $S_\alpha$ be the set of countable ordinals $\beta > \alpha$ such that for some countable $\gamma > \beta$,

1. $L_\gamma \models ZF^-$ (ZF minus Power Set),
2. $\omega_1^\gamma = \beta$,
3. $T_\alpha^\beta$ is a tree which has a branch of length $\beta$ in $L_\gamma$.

First, suppose that $T_\alpha$ has an $\omega_1$-branch $b$. We show that $S_\alpha$ contains a club.

Suppose that $M$ is a countable elementary substructure of $L_{\omega_1}$ such that $b \in M$. Then the transitive collapse, denoted by $\overline{M}$, has the form $L_\gamma$. Let $\beta = \omega_1^{\overline{M}}$. Since $b$ is an $\omega_1$-branch through the tree $T_\alpha = T_\alpha^{\omega_1}$, $b \upharpoonright \beta$ is a $\beta$-branch through the tree $T_\alpha^{\beta}$ that belongs to $L_\gamma$ and therefore $\gamma$ witnesses that $\beta$ belongs to $S_\alpha$.

Now form a continuous chain $(M_i)_{i<\omega_1}$ of countable elementary substructures of $L_{\omega_1}$. Let $\overline{M_i}$ be the transitive collapse of $M_i$. Then $\overline{M_i} = L_{\gamma_i}$, for some countable ordinal $\gamma_i$. Let $\beta_i = \omega_1^{\overline{M_i}}$. Then the sequence $(\beta_i)_{i<\omega_1}$ enumerates a club $C$ in $\omega_1$. For each $i$, the image of $b$ under the transitive collapse of $M_i$, $\pi_i(b)$, is a $\beta_i$-branch through $T_\alpha^{\beta_i}$ belonging to $L_{\gamma_i}$, witnessing that $\beta_i$ belongs to $S_\alpha$. Thus $C$ is the required club.

Conversely, we show that if $T_\alpha$ has no $\omega_1$-branch, then $S_\alpha$ does not contain a club.
ω₁-Computable structure theory

Suppose that \( C \) is a club and we will show that some element of \( C \) does not belong to \( S_\alpha \). Let \( M \) be the least elementary substructure of \( L_{\omega_2} \) such that \( C, \alpha, \omega_1 \in \text{M}. \) In \( L_{\omega_2}, T_\alpha = T_\alpha^{\omega_1} \) has no \( \omega_1 \)-branch, so the same holds in \( M \). Again, we take the transitive collapse \( \pi(M) = \overline{M} \sim L_\gamma \). We have \( \beta = \omega_1^{\overline{M}} \in C \) and \( T_\alpha^\beta \) has no \( \beta \)-branch in \( L_\gamma \). We claim that \( \beta \) does not belong to \( S_\alpha \). Indeed, suppose otherwise and that the ordinal \( \gamma \) witnesses this. Then \( \gamma \) must be greater than \( \gamma \), as \( T_\alpha^\beta \) has no \( \beta \)-branch in \( L_\gamma \). But if \( \gamma \) is greater than \( \gamma \), then \( \beta \) is countable in \( L_\gamma \); as \( M \) was chosen to be the least elementary substructure of \( L_{\omega_2} \) containing the parameters \( C, \alpha, \omega_1 \), it follows that \( \overline{M} \) is the least elementary substructure of \( \overline{M} \) containing the parameters \( \pi(C), \pi(\alpha), \pi(\omega_1) \) and therefore \( \overline{M} \), as well as \( \beta \in \overline{M} \), is countable in \( L_{\gamma + 2} \). We have reached the desired contradiction.

Let \( E \) be a \( \Sigma_1^1 \) equivalence relation on \( \omega_1 \). We identify pairs of ordinals with single ordinals and let \( S \) be as above, so that \( \alpha E \beta \) iff \( S_{\alpha, \beta} \) contains a club. For any \( X \subseteq \omega_1 \), let \( L(X) \) be the \( \aleph_1 \)-like linear order formed by stacking \( \omega_1 \) many copies of the rational order, and at limit stage \( \alpha \) putting in a supremum iff \( \alpha \in X \). More precisely: let \( Q \) denote the rational order and \( Q_0 \) the rational order together with a least element; then \( L(X) \) is obtained from the ordering \( (\omega_1, <) \) by replacing \( \alpha \) with a copy of \( Q_0 \) if \( \alpha \) is a limit ordinal in \( X \) and otherwise replacing \( \alpha \) by a copy of \( Q \).

**Lemma 2.6**
For \( X, Y \subseteq \omega_1 \), \( L(X) \equiv L(Y) \) iff \( X \sim Y \) (as in Definition 2.4).

**Proof.** Suppose \( L(X) \) is isomorphic to \( L(Y) \) via the isomorphism \( \pi \). For countable \( \alpha \) let \( L(X)|\alpha \) be the initial segment of \( L(X) \) obtained from the order \( (\alpha, <) \) by replacing \( i < \alpha \) by \( Q_0 \) if \( i \) is a limit ordinal in \( X \) and by \( Q \) otherwise. Then for club-many \( \alpha \), the restriction of \( \pi \) to \( L(X)|\alpha \) is an isomorphism from \( L(X)|\alpha \) onto \( L(Y)|\alpha \). For such \( \alpha, \alpha \) belongs to \( X \) iff \( Y \) agrees with \( \alpha \), as otherwise the restriction of \( \pi \) to \( L(X)|\alpha \) would not be extendible to an isomorphism from \( L(X) \) onto \( L(Y) \). Thus \( X, Y \) agree on a club and \( X \sim Y \).

Conversely, suppose that \( X \sim Y \) and choose a club \( C \) on which \( X, Y \) agree. By induction on \( \alpha \) in \( C \), build an isomorphism between \( L(X)|\alpha \) and \( L(Y)|\alpha \). The base case is easy, as there is a unique countable dense linear order without endpoints. The limit cases are trivial, as the limit of isomorphisms is an isomorphism. For the case where \( \alpha \) is the \( C \)-successor to \( \beta \in C \), use the fact that \( X, Y \) agree at \( \beta \) to conclude that the ordinal \( \beta \) is replaced by the same ordering in \( L(X)|\alpha \) as it is in \( L(Y)|\alpha \).

We use ideas from [7]. For any finite chain \( c = (\alpha, \gamma_1, \gamma_2, \ldots, \gamma_n, \beta) \), let

\[
S(c) = S_{\alpha, \gamma_1} \cap S_{\gamma_1, \gamma_2} \cap \cdots \cap S_{\gamma_n, \beta}
\]

If \( \alpha' E \alpha \) then \( S_{\alpha', \alpha} \) contains a club. Therefore, for each finite chain \( c \) from \( \alpha \) to \( \beta \), \( S_{\alpha', \alpha} \cap S(c) \sim S(c) \). It follows that if we define \( S^*(\alpha, \beta) \) to be the set of the \( S(c) \), where \( c \) is a chain starting with \( \alpha \) and ending with \( \beta \), and \( \alpha E \alpha' \), then \( S^*(\alpha, \beta) \) agrees with \( S^*(\alpha', \beta) \), in the sense that they have the same elements modulo the ideal of nonstationary sets. Let \( M(\alpha, \beta) \) be the structure that is the ‘free union’ of \( \omega_1 \)-many copies of the linear orders \( L(X) \) for \( X \in S^*(\alpha, \beta) \). One way to make this precise is to let \( M(\alpha, \beta) \) consist of two disjoint sets \( A, B \) of size \( \omega_1 \), with a relation \( R(a, b_0, b_1) \) for \( a \) in \( A \) and \( b_0, b_1 \) in \( B \) so that for each fixed \( a \), \( R(a, -, -) \) defines a linear order of \( B \) isomorphic to one of the \( L(X) \), for \( X \in S^*(\alpha, \beta) \), and each such order occurs for exactly \( \omega_1 \)-many such \( a \) in \( A \).

Alternatively, we may let \( M(\alpha, \beta) \) have an equivalence relation with an ordering on each equivalence class, so that for each set \( X \in S^*(\alpha, \beta) \), the ordering \( L(X) \) is copied in uncountably many equivalence classes, and for each equivalence class, the ordering on the equivalence class is isomorphic to \( L(X) \) for some \( X \in S^*(\alpha, \beta) \). We note that in either case, the language of the structures \( M(\alpha, \beta) \) is finite.
LEMMA 2.7
(1) If \( \alpha E \alpha' \), then for all \( \beta \), \( M(\alpha, \beta) \equiv M(\alpha', \beta) \).
(2) If it is not the case that \( \alpha E \alpha' \), then \( M(\alpha, \alpha) \not\equiv M(\alpha', \alpha) \).

PROOF. (1) is clear. For (2), we note that if it is not the case that \( \alpha E \alpha' \), then there is no set \( X \in S^*(\alpha, \alpha') \) that contains a club, but there is such a set in \( S^*(\alpha, \alpha) \). From this it follows that \( M(\alpha, \alpha) \) is not isomorphic to \( M(\alpha', \alpha) \). \( \blacksquare \)

The structures \( M(\alpha, \beta) \) have a finite language. Finally, let \( M^*(\alpha) \) be the sequence (not the free union) of the structures \( M(\alpha, \beta) \), for \( \beta < \omega_1 \). To make this precise, we could add to the language a disjoint family of unary predicates \( (U_\beta)_{\beta<\omega_1} \), where \( U_\beta \) is the universe of a copy of \( M(\alpha, \beta) \); but as we would like to keep the language finite, we instead let \( M^*(\alpha) \) be a structure that includes a copy of \( \omega_1 \) with the usual ordering, and has a predicate associating to each \( \beta < \omega_1 \) one of a family of sets, disjoint from \( \omega_1 \) and disjoint from each other. We put a copy of \( M(\alpha, \beta) \) on the set associated with \( \beta \).

LEMMA 2.8
For all \( \alpha, \alpha' \), \( \alpha E \alpha' \) iff \( M^*(\alpha) \equiv M^*(\alpha') \).

PROOF. If \( \alpha E \alpha' \), then for all \( \beta \), \( M(\alpha, \beta) \equiv M(\alpha', \beta) \). Then \( M^*(\alpha) \equiv M^*(\alpha') \). If it is not the case that \( \alpha E \alpha' \), then \( M(\alpha, \alpha) \not\equiv M(\alpha', \alpha) \). Then we have \( M^*(\alpha) \not\equiv M^*(\alpha') \). \( \blacksquare \)

This completes the proof of Theorem 1.7.

Our structures \( M^*(\alpha) \) are in a finite relational language, and we may use standard coding tricks to transform them into undirected graphs. We represent each element by a point attached to a triangle. For an \( n \)-place relation symbol \( R \), we represent each \( n \)-tuple of elements by a special point, attached by chains of length 1, 2, ..., \( n \) to the points representing the elements. Therefore, the structures \( M^*(\alpha) \) of Theorem 1.7 can be chosen to be undirected graphs.

3 Turing computable embeddings

H. Friedman and Stanley [8] introduced the notion of Borel embedding for comparing classification problems for classes of countable structures. The notion of Turing computable embedding [2] allows some finer distinctions. Here we define the analogue of Turing computable embedding for structures with universe a subset of \( \omega_1 \). We have said that in the setting of \( \omega_1 \), \( A \leq_f B \) if there is some \( \alpha \) such that \( \psi^B_\alpha = \chi_B \). We begin by saying something about indices for partial computable functions, and for partial computable functions relative to an oracle. We write \( \psi_\alpha \) for the partial computable function \( f \) derived in the following way from the c.e. set \( W_\alpha \). We order the elements according to their Gödel codes. We let \( f(x) = y \) if for the first \( y \) such that there is a pair \( (x, u) \in W_\alpha \), \( y \) is the first such \( u \)—if there is no pair \( (x, u) \in W_\alpha \), then \( f(x) \) is undefined. Similarly, \( \psi^X_\alpha \) is the partial function \( f \) derived from \( W^X_\alpha \) such that \( f(x) = y \) if for the first \( y \) such that there is some pair \( (x, u) \in W^X_\alpha \), \( y \) is the first such \( u \). If we let \( X \) vary, then we obtain an operator \( \Phi = \psi_\alpha \) taking each set \( X \) to the partial function \( \psi^X_\alpha \).

DEFINITION 3.1
(1) Let \( K \) and \( K' \) be classes of structures with universe a subset of \( L_{\omega_1} \). A computable transformation from \( K \) to \( K' \) is a computable operator \( \Phi = \psi_\alpha \) such that for each \( A \in K \), there is some \( B \in K' \) such that \( \psi^B_\alpha = \chi_B \). We write \( \Phi(A) = B \).
(2) We write \( K \leq_c K' \) if there is a computable transformation \( \Phi = \psi_\alpha \) from \( K \) to \( K' \) such that \( A, A' \in K \), \( A \equiv A' \) iff \( \Phi(A) \equiv \Phi(A') \).
In the standard setting, the class of undirected graphs lies on top among classes of countable structures under \( \leq_{\aleph_1} \). The same is true in our setting. Let \( L \) be a computable relational language—\( L \) may be uncountable, and it may include symbols of arity \( \alpha \) for computable ordinals \( \alpha \). When we say that the language is computable, we mean that the set of relation symbols is computable, and we have a computable function assigning a countable ordinal arity to each symbol. Let \( \text{Mod}(L) \) be the class of \( L \)-structures with universe a subset of \( \omega_1 \). Let \( UG \) be the class of undirected graphs.

**Proposition 3.2**

\( \text{Mod}(L) \leq_{\aleph_1} UG \)

**Proof.** We first give a transformation that replaces the language \( L \) by one with just finitely many relations of finite arity. We have a predicate \( U \), for elements of the structure \( M \). We have a predicate \( O \) with an ordering of type \( \omega_1 \). We have another predicate \( S \) for special points, representing triples \( (R, \alpha, \sigma) \), where \( R \) is a predicate symbol \( R \) in \( L \) of arity \( \alpha \) and \( \sigma \) is an \( \alpha \)-tuple from \( M \). There is a relation \( Q \) that holds of \( x \in O \), \( p \in S \) and \( a \in U \) if \( p \) is the special point corresponding to a triple \( (R, \alpha, \sigma) \) such that \( x \) is the \( \beta^{th} \) element of \( O \), and \( \sigma(\beta) = a \). We let \( T \) be the set of special points \( p \in S \) representing atomic facts that are true in \( M \); i.e., \( p \) is the special point corresponding to \( (R, \alpha, \sigma) \), where \( R \) is a relation symbol of arity \( \alpha \), \( \alpha \in M^{\omega_1} \), and \( M \models R(\sigma) \). The unary predicates \( U, O, \) and \( S \) are disjoint, and the universe of our structure \( M^* \) is the union. Beyond these, we have a binary relation—the ordering on \( O \), the ternary relation \( Q \), and the set \( T \subseteq S \). So, the language is finite. It is not difficult to see that \( M_1 \cong M_2 \) iff \( M_1^* \cong M_2^* \).

Marker included in his basic model theory text [16] a simple way of coding an arbitrary structure (for a finite relational language \( L \)) in an undirected graph. (There are further coding methods that accomplish the same thing, described by Lavrov, Nies, and others.) In Marker’s transformation, for each element \( b \) of the input structure, the graph has an element \( g_b \), which we mark by making it one vertex of a triangle. For each relation symbol \( R \) in the finite language \( L \), \( j = 0, \ldots \), we designate a pair of shapes, a \( (2j+4) \)-gon, and a \( (2j+5) \)-gon, which we use to indicate that the relation holds, or does not hold. For each ordered \( n \)-tuple of elements \( b_1, \ldots, b_n \), we introduce a special point \( t_{R_j, b_1, \ldots, b_n} \). This point is connected to \( g_b \) by a chain of length \( i \). The special point \( t_{R_j, b_1, \ldots, b_n} \) is one vertex of a \( (2j+4) \)-gon if \( R(b_1, \ldots, b_n) \) holds, and a \( (2j+5) \)-gon, otherwise. The \( 1 \)-gons have no points in common, aside from the special points. We can easily make this transformation into a computable embedding in the setting of \( \omega_1 \).

We start with a large \( \omega_1 \)-computable graph \( G^* \), such that there are \( \aleph_1 \) elements that are the special vertex of a triangle, for each \( n \)-ary relation symbol \( R \) in \( L \) and each \( n \)-tuple of special vertices of triangles \( v_1, \ldots, v_n \), there is a special point \( g_{R_j, b_1, \ldots, b_n} \), connected to \( b_1 \) by a chain of length \( i \). This point \( g_{R_j, b_1, \ldots, b_n} \) is one vertex of both a \( (2j+4) \)-gon, and a \( (2j+5) \)-gon. We note that the elementary first-order theory of the desired \( G^* \) is totally categorical. It follows that there is an \( \omega_1 \)-computable model \( G^* \) of size \( \aleph_1 \)—see [13], or [14]. The set \( V \) of special vertex elements is computable. To see this, note that there is a \( \Sigma_1 \) definition of \( V \)—saying that the element is part of a triangle and is connected to further elements. There is also a \( \Sigma_1 \) formula defining the elements not in \( V \)—a disjunction of formulas indicating the position in an \( n \)-gon, or on a chain from a vertex of a triangle to a vertex of some other \( n \)-gon. For each \( n \)-ary relation symbol \( R_j \), there is a computable function taking \( n \)-tuples \( b_1, \ldots, b_n \) of points in \( V \) to the special point \( g_{R_j, b_1, \ldots, b_n} \). There are obvious \( \Sigma_1 \) definitions of the graph. We have a computable function \( f \) taking \( \alpha \) to the \( \alpha^{th} \) element of \( V \), which we call \( b_\alpha \). The functions are defined by \( \Sigma_1 \) recursion. We have a \( \Sigma_1 \) formula saying how \( f(\alpha) \) is obtained from \( f/\alpha \).

Now, for an input \( L \)-structure \( A \), with universe that we can computably identify with a subset of \( \omega_1 \), \( \Phi(A) \) is the subgraph of \( G^* \) consisting of the following elements: \( b_\alpha \), for \( \alpha \in A \), the special points...
This embedding is effective. We can use the same idea, with some modifications, to give an embedding in \([13]\). We describe the embedding below. We state the result just for characteristic 0, although it is the modification from \([4]\), both transfer directly to the uncountable setting, with no fresh ideas. The coding idea was used already in the uncountable setting for a result of Hirschfeldt that is included in \([13]\). We describe the embedding below. We state the result just for characteristic 0, although it is true also for any finite characteristic.

**Proposition 3.3**

If \(K\) is the class of undirected graphs, and \(K'\) is the class of fields of characteristic 0, then \(K \leq_{tc} K'\). (The same is true for finite characteristic.)

**Proof.** Let \(F^*\) be a large algebraically closed field of characteristic, 0 with independent transcendentalts \(b_\alpha\), for \(\alpha < \omega_1\). Since the theory is \(\aleph_0\)-categorical, we get an \(\omega_1\)-computable model \(F^*\) of size \(\omega_1\)—see [13]. We may suppose that the universe is \(\omega_1\). Checking independence of a countable tuple is computable. Let \(f(\alpha)\) be the first element not algebraic over \([f(\beta): \beta < \alpha]\). Then \(ran(f)\) is a transcendence basis for \(F^*\). This function is computable. We write \(b_\alpha\) for \(f(\alpha)\). For an input graph \(G\), with universe a subset of \(\omega_1\), we let \(\Phi(G)\) be the subfield of \(F^*\) generated by the elements \(b_\alpha\), for \(\alpha \in G\), elements algebraic over a single \(b_\alpha\), and elements \(\sqrt{b_\alpha + b_\beta}\), if in \(G\) there is an edge between \(\alpha\) and \(\beta\). It may seem clear that for graphs \(G\) and \(G'\), \(G \cong G'\) iff \(\Phi(G) \cong \Phi(G')\). For characteristic \(p \neq 2\), the construction is the same, except that \(F^*\) has characteristic \(p\). For characteristic 2, it is necessary to use a different coding to indicate the presence of an edge—we may use the cube root of \(b_\alpha + b_\beta\) instead of the square root. The hardest part of Friedman and Stanley’s proof is showing \(\sqrt{b_\alpha + b_\beta}\) do not get in by accident. The proof in [8] works for characteristic 0. There is a reference in [10] to work of Shapiro [24] (from algebraic number theory) that completes the proof. For characteristic 0, there is a geometric proof, due to Dwyer, which was included in Calvert’s thesis [6].

In [8], H. Friedman and Stanley give a Borel embedding of undirected graphs into linear orderings. This embedding is effective. We can use the same idea, with some modifications, to give an embedding in our uncountable setting.

**Proposition 3.4**

If \(K\) is the class of undirected graphs and \(K'\) is the class of linear orderings, then \(K \leq_{tc} K'\).

**Proof.** By Proposition 1.4, a countable complete theory has a computable saturated model \(Q\) with universe \(\omega_1\). We could apply this result to the theory of dense linear orderings without endpoints. We want a theory of dense linear orderings without endpoints, partitioned into infinitely many disjoint dense subsets. Let \(T\) be the theory of a structure whose universe is the disjoint union of predicates \(U\) and \(V\), where on \(V\), there is a dense linear ordering without endpoints, \(U\) is infinite, and there is a function \(f\) from \(V\) onto \(U\) such that for each \(u \in U\), \(f^{-1}(u)\) is dense in \(V\). We consider a computable saturated model \(Q^*\) of this theory, with universe \(\omega_1\). We have a type \(\omega_1\) ordering on the elements of \(U^{Q^*}\), inherited from the ordering on \(\omega_1\). This is not part of the structure \(Q^*\).

We identify the elements of \(U^{Q^*}\) with the countable ordinals. We write \(Q\) for \(V^{Q^*}\). Let \(Q_\alpha\) be \(f^{-1}(\alpha)\), where \(\alpha\) is the \(\alpha^{th}\) element of \(U\). Now, \(Q\), with the ordering from \(Q^*\), is a saturated dense linear ordering without endpoints. The sets \((Q_\alpha)_{\alpha < \omega_1}\) partition \(Q\) into dense subsets. The elements of
The ordering corresponding to a given input graph $G$ will be a sub-ordering $\Phi(G)$ of this ordering, consisting of the sequences $\sigma$ of length $2\beta + 2$ such that for some $\beta$-tuple $\overline{\sigma}$ from $G$, satisfying the atomic type $t_\alpha$, we have

1. for $\gamma < \beta$, $\sigma(2\gamma) \in Q_0$, and $\sigma(2\gamma + 1) \in Q_{2+\alpha}$, 
2. $\sigma(2\beta) \in Q_1$, 
3. $\sigma(2\beta + 1)$ is an element of $U$ identified with an ordinal less than $\alpha$.

This is the construction. In [8], $Q$ is a countable dense linear ordering without endpoints, partitioned into disjoint dense sets $(Q_n)_{n \in \omega}$.

Let $\sigma \in \Phi(G)$. We say that $\sigma$ represents the tuple $\overline{\sigma}$ if $\sigma$ is related to $\overline{\sigma}$ in the way described. The ordering $\Phi(G)$ is made up of intervals having the order type $\alpha$, for the various atomic types $t_\alpha$ realized in $G$. It takes effort to show that $G_1 \equiv G_2$ if $\Phi(G_1) \equiv \Phi(G_2)$. In [8], the details are omitted. We say a little here. First, suppose $G_1 \equiv G_2$ via $f$. To show that $\Phi(G_1) \equiv \Phi(G_2)$, we consider the set $\mathcal{F}$ of countable partial isomorphisms $p$, where for some countable family of $\sigma_i \in \Phi(G_1)$,

1. if $p(\sigma_i) = \tau_i$, where $\sigma_i$ represents the tuple $\overline{\sigma_i}$ (of countable arity), then $\tau_i$ represents the corresponding tuple $f(\overline{\sigma_i})$,
2. if $\sigma_i$ represents a tuple $\overline{\sigma_i}$ realizing type $t_\alpha$, then $p$ maps the full interval of type $\alpha$ containing $\sigma_i$ to the corresponding interval of type $\alpha$ containing $\tau_i$.

Clearly, $\mathcal{F}$ is closed under unions of countable chains. The fact that $Q^*$ is saturated allows us to show that it has the back-and-forth property. Say $p \in \mathcal{F}$ maps $\sigma_i$ representing $\overline{\sigma_i}$ to $\tau_i$ representing $\overline{\tau_i}$. We indicate how to go forth, with a further element $\sigma$ of $\Phi(G_1)$, representing $\overline{\tau_i}$. We need an image $\tau$ representing $\overline{\tau} = f(\overline{\tau})$. If $\sigma$ has length $2\beta + 2$, we choose $\tau$, of the same length, representing $f(\overline{\tau})$. For $\gamma < \beta$, $\tau(2\gamma) \in Q_{1(\alpha)}$, located to the left, or right, of $\sigma(2\gamma + 1)$ is located to the left, or right, of $\sigma(2\gamma + 1)$. Similarly, for $\gamma < \beta$, $\tau(2\gamma) \in Q_0$, $\tau(2\beta) \in Q_1$, located in the proper relation to $\tau(2\gamma)$, or $\tau(2\beta)$. Finally, $\tau(2\beta + 1)$ matches $\sigma(2\beta + 1)$. Going back, with a further element of $\Phi(G_2)$ is similar.

Next, suppose $\Phi(G_1) \equiv \Phi(G_2)$ via $f$. We define an isomorphism $g$ from $G_1$ onto $G_2$. The universes of $G_1$ and $G_2$ are subsets of $\sigma_0$, so we have lists of elements $(a_\alpha)_{\alpha < \omega_1}$ and $(b_\alpha)_{\alpha < \omega_1}$. At step $\alpha$, we have a countable partial isomorphism $g_\alpha$. At stage $\alpha + 1$, we add an element to the domain or the range. Take the first $\beta$ such that $a_\beta$ is not in the domain or $b_\beta$ is not in the range. If $a_\beta$ is not in the domain, then we add $a_\beta$ to the domain, and otherwise, we add $b_\beta$ to the range.

The partial isomorphisms will form a continuous chain. Of course $g_0 = \varnothing$. For $g_1$, we must add $a_0$ to the domain. Take $\sigma \in \Phi(G_1)$ representing the sequence $(a_0)$, of atomic type $t_0$. Then $\sigma$ has the form $q_0, q_0, q_0$, where $q_0 \in Q_0$, $q_0 \in Q_{2+\alpha}$, $r_1 \in Q_1$, and $\beta < \delta$. Note that $\sigma$ is part of a maximal well-ordered interval of type $\delta$, and $t_0$ is the atomic type of the sequence of length 1. Now, $f(\sigma)$ must be in the same position in a maximal well-ordered interval of type $\delta$. Therefore, $f(\sigma)$ also represents a 1-tuple—if it is a sequence of the form $r_0', q_0', r_1'$, $\beta$, where $r_0' \in Q_0$, $q_0' \in Q_d$, for $d \in G_2$, and $r_1' \in Q_1$. We get $g_1(a_0) = d$. Assuming that $b_0 \neq d$, we define $g_2$ with $b_0$ in the range. For this, we take $\tau \in \Phi(G_2)$ of length 6, representing $(d, b_0)$, such that $\tau$ extends $r_0'q_0'r_1'$. Let $t_{\alpha'}$ be the atomic type of $d, b_0$. Then $\tau$ has the form $r_0', q_0', q_1', r_2', \delta'$, where $r_1' \in Q_1$, $q_1' \in Q_{b_0}$, $r_1' \in Q_1$, and $\delta' < \alpha'$. Note that in the interval between $f(\sigma)$ and $\tau$, every element represents an extension of $(d)$—there is no sequence representing
the type of the empty sequence. Since \( \tau \) lies in a maximal well-ordered interval of type \( \alpha' \), the same
is true of \( f^{-1}(\tau) \). Between \( f(\sigma) \) and \( \tau \), there is no interval of type \( \gamma \), where \( \gamma \) is the atomic type of
the empty set, so all elements of this interval represent extensions of \((q_0)\). We can see that \( f^{-1}(\tau) \)
will have the form \((r_0, q_0, r_1', q_1, r_2, \delta')\), where \( q_1 \in Q_c \), for some \( c \in G_1 \), and \( r_2 \in Q_1 \), and \( \delta'' < \alpha' \).
We let \( g_1(\cdot) \mapsto b_0 \).

We continue in this way. At each stage \( \alpha \), we have \( g_\alpha \) mapping a countable
sequence \((c_\beta)_{\beta < \alpha} \) from \( G_1 \) to a countable sequence \((d_\beta)_{\beta < \alpha} \) from \( G_2 \). We also have sequences \((\sigma_\beta)_{\beta < \alpha} \) and \((\tau_\beta)_{\beta < \alpha} \) in \( \Phi(G_1) \) and \( \Phi(G_2) \), such that \( \sigma_\beta \), of length \( 2 \beta + 2 \), represents the sequence \((c_\gamma)_\gamma < \beta \) and \( \tau_\beta \), also of
length \( 2 \beta + 2 \), represents the sequence \((d_\gamma)_\gamma < \beta \). We arrange that between \( \sigma_\beta \) and \( \tau_\beta \) in \( \Phi(G_1) \), all
elements represent extensions of the sequence represented by \( \sigma_\beta \) and between \( \tau_\beta \) and \( \tau_{\beta+1} \) in \( \Phi(G_1) \), all
elements represent extensions of the sequence represented by \( \tau_\beta \). (We choose one side, and the
isomorphism \( f \) takes care of the other side.) For limit \( \alpha \), we let \( g_\alpha = \bigcup_{\beta < \alpha} g_\beta \), as required. We let \( \sigma_\alpha \)
be the sequence of length \( 2 \alpha + 2 \) such that for \( \beta < \alpha \), \( \sigma_\alpha \) agrees with \( \sigma_\beta \) on the first \( 2 \beta \) terms. Let \( t_\rho \)
be the atomic type of the domain, and range, of \( g_\alpha \). At the end of \( \sigma_\alpha \), we put two last terms \( r_{2\alpha} \in Q_1 \) and \( \delta < \rho \). We let \( \tau_\alpha = f(\sigma_\alpha) \). This extends the initial part of \( \tau_\beta \) of length \( 2 \beta \).

4 Results on fields

Here we consider arbitrary \( \omega_1 \)-computable fields of characteristic 0. The domain of each field is
either \( \omega_1 \) or possibly just \( \omega \), and the field operations are all \( \omega_1 \)-computable. We believe that our first
results carry over equally well to fields of positive characteristic, and so we denote the prime subfield
(either \( Q \) or \( F_p \)) of a field \( F \) by \( Q \).

**Lemma 4.1**

Every \( \omega_1 \)-computable field \( F \) has a computable transcendence basis over its prime subfield \( Q \). (\( Q \)
itself is \( \omega_1 \)-computable, being countable.)

**Proof.** This follows from the proof of [13, Lemma 4.6], although it is stated there only for \( \omega_1 \)-
computable presentations of \( C \). For each \( \alpha \in F \) we define \( \alpha \in B \) iff

\[
(\forall(\beta_1, \ldots, \beta_n) \in \alpha^{-\omega})(\forall p \in Q[X_1, \ldots, X_n, Y])
\]

\[
[p(\beta_1, \ldots, \beta_n, \alpha) = 0 \Rightarrow p(\beta_1, \ldots, \beta_n, Y) = 0].
\]

This statement quantifies only over countable sets which we can enumerate uniformly and know when
we have finished enumerating each one. It says that \( \alpha \) lies in \( B \) iff \( \alpha \) satisfies no non-zero polynomial
over the subfield \( Q(\beta : \beta < \alpha) \) generated by all elements \( \beta < \alpha \). Clearly this \( B \) is a transcendence basis
for \( F \).

**Corollary 4.2**
The field \( \mathbb{C} \) of complex numbers is relatively \( \omega_1 \)-computably categorical.

**Proof.** Given any two \( \omega_1 \)-computable fields \( E \cong F \cong \mathbb{C} \), use the lemma to find computable
transcendence bases \( B \) for \( E \) and \( C \) for \( F \). Let \( f \) be any computable bijection from \( B \) onto \( C \) (for
instance, let \( f(\alpha) \) be the least element of \( C \) which is \( > f(\beta) \) for every \( \beta < \alpha \)). We now extend this \( f \)
effectively to an isomorphism from \( E \) onto \( F \). For each element \( \xi \in E \) in order, find any polynomial
\( q \in Q(B)[X] \) satisfied by \( \xi \) and take the finite subset \( B_0 \subseteq B \) of those elements of \( B \) actually used in
the coefficients of \( q \). Let \( E_0 \) be the countable subfield of \( E \) generated by \( B_0 \cup \xi \) (with \( \xi \) now denoting
the set of those ordinals \( < \xi \), so that \( E_0 \) lies within the domain on which \( f \) has already been defined),
and find the minimal polynomial \( p(X) \) of the field element \( \xi \) over \( E_0 \). (This can be done by brute force, just by checking all of the countably many polynomials in \( E_0[X] \).) Every coefficient in \( p(X) \) lies in \( E_0 \), hence already has an image in \( F \) under \( f \), and we choose \( f(\xi) \) to be the least root in \( F \) of the image of this polynomial \( p(X) \) in \( F[X] \) under \( f \), which must exist, \( F \) being algebraically closed.

Thus we recursively build a field embedding \( f : E \rightarrow F \). But since \( f \) maps \( B \) onto the transcendence basis \( C \) for \( F \), \( f \) must map \( E \) onto all of \( F \): every \( \eta \in F \) has a minimal polynomial \( p(X) \in Q(C)[X] \) of some degree \( d \), and the roots \( \xi_1, \ldots, \xi_d \) of its preimage in \( E[X] \) must map one-to-one to the \( d \)-many roots of \( p(X) \) in \( F \), forcing \( \eta \in \text{ran}(f) \).

The foregoing proof relativizes to the degree of any field \( E \cong \mathbb{C} \), yielding relative \( \omega_1 \)-computable categoricity.

For our next results, it is useful to have an \( \omega_1 \)-computable bijection \( f : \omega_1 \rightarrow (\omega_1)^2 \). Say that the pair \( (\alpha, \beta) \) lies on the diagonal \( D_\gamma \) in \( (\omega_1)^2 \) if \( \alpha + \beta = \gamma \). (Geometrically, this is a misnomer: for instance, the ‘diagonal’ \( D_\omega \) contains the pairs \( (0, \omega), (1, \omega), \ldots, \) and \( (\omega, 0) \).) Let \( f(0) = (0, 0) \), and to define \( f(\delta) \) recursively, find the least \( \gamma \) for which \( D_\gamma \) is not a subset of \( \text{ran}(f \upharpoonright \delta) \), and the least \( \alpha \leq \gamma \) for which the unique pair \( (\alpha, \beta) \) in \( D_\gamma \) is not in \( \text{ran}(f \upharpoonright \delta) \), and set \( f(\delta) = (\alpha, \beta) \). If \( \delta = \theta + 1 \) is a successor and \( f(\theta) = (\alpha, \beta) \), this defines

\[
f(\theta + 1) = \begin{cases} 
(0, \alpha + \beta + 1), & \text{if } \beta = 0; \\
(\alpha + 1, \beta - 1), & \text{if } 0 < \beta < \omega; \\
(\alpha + 1, \beta), & \text{otherwise.}
\end{cases}
\]

One checks that this \( f \) really is a bijection, and then uses angle brackets to write \( (\alpha, \beta) = f^{-1}(f((\alpha, \beta)) \in \omega_1 \), borrowing the notation from \( \omega \)-computability. Moreover, \( f \) is clearly computable, hence allows us to partition \( \omega_1 \) effectively into \( \omega_1 \)-many uniformly computable disjoint subsets of size \( \omega_1 \):

\[\omega_1^{[\omega]} = \{ (\alpha, \beta) : \beta \in \omega_1 \} .\]

One guesses that every countably generated extension of the field of complex numbers should be computably categorical. First, we show that there is a computable copy.

**Proposition 4.3**

Let \( F \) be countably generated extension of the field \( \mathbb{C} \) of complex numbers; i.e., \( F = \mathbb{C}(a_0, a_1, \ldots) \), for some countable sequence \( a_0, a_1, \ldots \). Then \( F \) has an \( \omega_1 \)-computable copy.

**Proof.** First, let \( C_0 \) be a countable algebraically closed subfield of \( \mathbb{C} \) such that if \( a_i \) is algebraic over \( \mathbb{C}(a_1, a_2, \ldots) \), then it is algebraic over \( C_0(a_1, a_2, \ldots) \); i.e., the minimal polynomial for \( a_i \) over \( C_0(a_1, a_2, \ldots) \) has coefficients in \( C_0(a_1, a_2, \ldots) \). Define \( A_0 = C_0(a_0, a_1, \ldots) \) using the same minimal polynomials as in \( F \); this is all countable, hence can be done \( \omega_1 \)-effectively. If \( C_1 \) is an algebraically closed extension of \( C_0 \) of transcendence degree 1 over \( C_0 \), we set \( A_1 = C_1(a_1, a_2, \ldots) \). Thus, if \( a_i \) is algebraic over \( C_1(a_1, a_2, \ldots) \), its minimal polynomial has coefficients in \( C_0(a_1, a_2, \ldots) \). We continue, building \( C_{\alpha+1} \) and \( A_{\alpha+1} \) from \( C_{\alpha} \) and \( A_{\alpha} \) for all countable ordinals \( \alpha \), with \( C_0 \subseteq C_\alpha \subseteq C_{\omega} \subseteq \mathbb{C} \), where each \( C_\alpha \) is the algebraic closure of the first \( \alpha \)-many elements in a computable transcendence basis for \( \mathbb{C} \) over \( C_0 \). We obtain \( C_{\omega} \) and \( A_{\omega} \) from \( C_\omega \) and \( A_\omega \) in the same way we obtained \( C_1 \) and \( A_1 \) from \( C_0 \) and \( A_0 \). For limit \( \alpha \), let \( C_\alpha = \bigcup_{\beta < \alpha} C_\beta \), and \( A_\alpha = \bigcup_{\beta < \alpha} A_\beta \). Finally, let \( C = \bigcup_{\alpha < \omega} C_\alpha \) and \( A = \bigcup_{\alpha < \omega} A_\alpha \). Clearly, \( C \) is isomorphic to \( \mathbb{C} \), via an isomorphism extending the (identity) map from the subfield \( C_0 \) of the copy of \( \mathbb{C} \) within \( F \) to the copy of \( C_0 \) within \( A_0 \). Moreover, if \( a_i \) is algebraic over \( C((a_j)_{j < i}) \), then its minimal polynomial has coefficients in \( C_0((a_j)_{j < i}) \). Therefore, \( A \cong F \).
We can prove computable categoricity, modulo a conjecture, for every field $F$ such as described in Proposition 4.3. We believe that the conjecture holds, but the proof seems delicate. Notice that within each such field $F$, although there are many subfields isomorphic to $\mathbb{C}$, one such subfield must contain all the others, and indeed this largest one is the only copy of $\mathbb{C}$ over which $F$ is countably generated. (If $\mathbb{C}_1 \not\subseteq \mathbb{C}_2$ are copies of $\mathbb{C}$ within a larger field, then $\mathbb{C}_1$ cannot be countably generated over $\mathbb{C}_2$; each $z_1 \in \mathbb{C}_1 \setminus \mathbb{C}_2$ must be transcendental over $\mathbb{C}_2$, so for each $z_2$ in a transcendence basis for $\mathbb{C}_2$, a separate generator is required to produce $\sqrt{z_1 + z_2}$.)

**Conjecture 4.4**

Suppose $F$ is a countably generated extension of a subfield $C$ isomorphic to $\mathbb{C}$, and let $C_0$ and $A_0$ be as in the previous proof. Then $C$ is definable in $F$ by a computable $\Sigma_1$-formula with a countable tuple of parameters from $A_0$, and hence is relatively intrinsically $\omega_1$-c.e.

We believe that it is possible to fix an element $z_0 \in C \setminus C_0$ such that, for $x \in F$, $x \in C$ if and only if the algebraic closure of the countable set $\{x\} \cup C_0(\{z_0\})$ is contained within $F$. This containment is a $\Sigma_1$ statement, as the algebraic closure would have to be contained within some countable initial segment $\sigma$ of dom($f$) = $\omega_1$, and so this would prove Conjecture 4.4. The details remain elusive, and here we leave this claim as a conjecture. Once shown to be true, it will imply the following theorem.

**Theorem 4.5** (modulo Conjecture 4.4)

If $F$ is a countably generated extension of $\mathbb{C}$, then $F$ is relatively $\omega_1$-computably categorical.

**Proof using Conjecture 4.4.** For simplicity, we identify $F$ with one of its computable copies, and let $F'$ be another computable copy. Let $C_0$ and $A_0$ be the countable parts of $F$ from the proof of Proposition 4.3. There is a non-computable isomorphism $\rho$ from $F$ onto $F'$, and we write $C_0'$ and $A_0'$ for the images of $C_0$ and $A_0$ under $\rho$. Let $C$ and $C' = \rho(C)$ be the largest subfields isomorphic to $\mathbb{C}$ within $F$ and $F'$, respectively. Note that $A_0 = C_0(a_0, a_1, \ldots)$, and if $a'_i = \rho(a_i)$, then $A_0' = C_0'(a'_0, a'_1, \ldots)$.

We build a computable isomorphism $f$ from $F$ onto $F'$ recursively. We define, by $\Sigma_1$ recursion a chain of countable subfields $A_\alpha \subseteq F$ and $A'_\alpha \subseteq F'$ with a chain of functions such that $f_\alpha$ is an isomorphism from $A_\alpha$ onto $A'_\alpha$. We will have $F = \cup_\alpha A_\alpha$ and $F' = \cup_\alpha A'_\alpha$. Then $f = \cup_\alpha f_\alpha$ will be a computable isomorphism from $F$ onto $F'$. Inside $A_\alpha$ and $A'_\alpha$, we will have algebraically closed subfields $C_\alpha$ and $C'_\alpha$ such that $C_\alpha \subseteq C$ and $C'_\alpha \subseteq C'$. For each $\alpha$, $A_\alpha$ will be the field generated by the elements of $C_\alpha$ and the elements $a_i$. Similarly, $A'_\alpha$ will be the field generated by the elements of $C'_\alpha$ and the elements $a'_i$. Once we have $f_\alpha$, taking $C_\alpha$ isomorphically onto $C'_\alpha$, and knowing that $f_\alpha(a_i) = a'_i$, the rest of $f_\alpha$ is determined.

To start off, we have $A_0$, $C_0$, $A'_0$, and $C'_0$. We let $f_0$ be the restriction of $\rho$ to $A_0$. Given $A_\alpha$, $C_\alpha$, $A'_\alpha$, $C'_\alpha$, with the isomorphism $f_\alpha$ taking $C_\alpha$ to $C'_\alpha$, and taking $a_i$ to $a'_i$, we extend as follows. Applying Conjecture 4.4, we let $c$ be the first element that we find in $C' \setminus A'_\alpha$. Since $C_\alpha$ is algebraically closed, $c$ is not algebraic over $C_\alpha$. It is also not algebraic over $A_\alpha = C_\alpha(a_0, a_1, \ldots)$. Similarly, let $c'$ be the first element that we find in $C' \setminus A'_\alpha$. This is not algebraic over $A_\alpha$. Let $C_{\alpha + 1}$ be the algebraic closure of $C_\alpha(c)$ in $C$, and let $C'_{\alpha + 1}$ be the algebraic closure of $C'_\alpha(c')$ in $C'$. We let $f_{\alpha + 1}$ be the extension of $f_\alpha$ taking $c$ to $c'$, and taking $C_{\alpha + 1}$ isomorphically onto $C'_{\alpha + 1}$. Extend in the obvious way to an isomorphism from $A_{\alpha + 1}$ onto $A'_{\alpha + 1}$. For limit ordinals $\alpha$, $C_\alpha$, $A_\alpha$, $C'_\alpha$, $A'_\alpha$, and $f_\alpha$ are all defined by taking limits.

It is clear that no element of $C$ can be left out of the domain of $f$, and no element of $C'$ can be left out of the range. The $a_i$ are all in the domain, and the $a'_i$ are all in the range. Therefore, the domain includes all of $F$ and the range includes all of $F'$, so $f$ is the desired computable isomorphism from $F$ onto $F'$.  


Relative $\omega_1$-computable categoricity again follows just by relativizing the argument to the degree of any $F \equiv E$. ■

It is natural to ask whether Theorem 4.5 would hold for fields of countable transcendence degree over $\mathbb{C}$. Such fields need not be countably generated over $\mathbb{C}$, so the theorem does not apply to them directly, and indeed such a field need not be computably categorical.

**Theorem 4.6**

There exists an $\omega_1$-computable field $F$, with transcendence degree 1 over a computable subfield isomorphic to $\mathbb{C}$, such that $F$ is not $\omega_1$-computably categorical.

**Proof.** We use a computable listing $\{\varphi_\alpha: \alpha \in \omega_1\}$ of all partial $\omega_1$-computable functions from $\omega_1$ into $\omega_1$. With this listing, we build the following two computable fields $E$ and $F$, with $E \cong F$ and each with $\mathbb{C}$ as a computable subfield, but diagonalizing to satisfy requirements $R_\alpha$:

$$R_\alpha: \varphi_\alpha \text{ is not an isomorphism from } E \text{ onto } F.$$ 

Set $E_{-2} = F_{-2} = \mathbb{C}$ to be a computable copy of $\mathbb{C}$ with a computable transcendence basis $\{z_\alpha: \alpha \in \omega_1\}$, and let $E_{-1} = F_{-1} = \mathbb{C}(x)$ with $x$ purely transcendental over $\mathbb{C}$. We then build $E_0 = F_0$ by adjoining every $\sqrt{x + z_\alpha}$, for every $\alpha$, so that $E_0 = F_0 = E_{-1}(\sqrt{x + z_\alpha: \alpha \in \omega_1}) = F_0$. (Of course, adjoining a square root also adjoins its conjugate. When we write $\sqrt{x + z_\alpha}$ below, we will always mean the root actually adjoined to $E$ here, which is taken to be a lesser element than its conjugate in the domain $\omega_1$ of $E$.) With these fields, we are ready to begin diagonalizing. It is important to note that $E_0$ can be viewed as a subfield of the real numbers $\mathbb{R}$ (by considering the elements inside the square roots to be positive), and that henceforward every $E_\alpha$ will also embed into $\mathbb{R}$, by the same trick. Hence $E$ will not contain any square root of $-1$.

At stage 0, we *initialize* every requirement $R_\alpha$, by declaring it unsatisfied and setting $y_{\alpha,0} = 0$. We also set every $\tau_\alpha = 0$, and define $f_0:E_0 \rightarrow F_0$ to be the identity map.

At stage $\sigma + 1$, we have $E_\sigma \equiv E_\sigma$ via an isomorphism $f_\sigma$. Find the least $\alpha \leq \sigma$ such that

- $R_\alpha$ is currently unsatisfied; and
- $\varphi_{\alpha,\sigma}$ respects the addition and multiplication operations in $E_\sigma$ and $F_\sigma$ (at all inputs from $E_\sigma$ for which $\varphi_{\alpha,\sigma}$ converges); and
- for some $y \in F_\sigma$, $\varphi_{\alpha,\sigma}(\sqrt{x + z_{\alpha}(\tau_\alpha)}) \downarrow = y$ and $\varphi_{\alpha,\sigma}(x + z_{\alpha}(\tau_\alpha)) \downarrow = y^2$; and
- $\varphi_{\alpha,\sigma}(\xi) \uparrow$, where $\xi$ is the least element of $\omega_1$ such that $\varphi_{\alpha,\sigma}(\xi) \uparrow$ at the last stage $\sigma' \leq \sigma$ at which $R_\alpha$ either was initialized or received attention.

If there is no such $\alpha \leq \sigma$, then do nothing. Otherwise, $R_\alpha$ receives attention according to the following instructions.

1. If $y = \varphi_{\alpha}(\sqrt{x + z_{\alpha}(\tau_\alpha)})$ is algebraically dependent over $\{y_{\beta,\sigma}: \beta < \alpha\}$, and the set $\{\varphi_{\alpha}(\sqrt{x + z_{\alpha}(\rho)}): \rho \leq \tau_\alpha\}$ is algebraically independent in $F_\sigma$, then increment $\tau_\alpha$ by 1, and do nothing else.

2. If this $y$ is dependent over $\{y_{\beta,\sigma}: \beta < \alpha\}$, and the set $\{\varphi_{\alpha}(\sqrt{x + z_{\alpha}(\rho)}): \rho \leq \tau_\alpha\}$ is algebraically dependent in $F_\sigma$, then declare $R_\alpha$ satisfied, set $y_{\alpha,\sigma} = y$, and do nothing else.

3. If $y$ is algebraically independent over $\{y_{\beta,\sigma}: \beta < \alpha\}$, then check whether $y$ and/or $(-y)$ has a square root in $F_\sigma$. If either one does, then declare $R_\alpha$ satisfied (since $x_{\alpha}(\tau_\alpha)$ has no square root in $E_\sigma$). If neither of $\pm y$ has a square root in $F$, then:

   a. Adjoin to $E_\sigma$ a square root $a$ of $\sqrt{x + z_{\alpha}(\tau_\alpha)}$, and adjoin to $F_\sigma$ a square root $b$ of $f_\sigma(\sqrt{x + z_{\alpha}(\tau_\alpha)})$ (in which case $f_{\sigma + 1} \supseteq f_\sigma$, with $f_{\sigma + 1}(a) = b$), provided that the adjoinment
of this square root in $F$ does not generate a square root of $\psi_\alpha(\sqrt{x + z(\alpha, \tau_\alpha)})$, nor of $\psi_\beta(\sqrt{x + z(\beta, \tau_\beta)})$ for any $\beta < \alpha$; or

(b) adjoin to $E_\sigma$ a square root $a$ of $\sqrt{x + z(\alpha, \tau_\alpha)}$, and adjoin to $F_\sigma$ a square root $b$ of $-f_\sigma(\sqrt{x + z(\alpha, \tau_\alpha)})$ (in which case $f_{\sigma + 1}(a) = b$ and $f_{\sigma + 1} F_\sigma = f_\sigma \circ \psi$, where $\psi$ is the automorphism of $F_{\sigma + 1}$ interchanging $\pm \sqrt{x + z(\alpha, \tau_\alpha)}$ and fixing everything else), provided that the adjoinment of this square root in $F$ does not generate a square root of $\psi_\alpha(\sqrt{x + z(\alpha, \tau_\alpha)})$, nor of $\psi_\beta(\sqrt{x + z(\beta, \tau_\beta)})$ for any $\beta < \alpha$; or

(c) increment $\tau_\alpha$ by 1, if neither (a) nor (b) holds.

If (3)(a) or (3)(b) applied, then $R_\alpha$ is declared satisfied, with $\gamma_{\alpha, \sigma + 1} \equiv \psi_\alpha(\sqrt{x + z(\alpha, \tau_\alpha)})$, and every $R_\beta$ with $\beta > \alpha$ is injured at this stage: $R_\beta$ is initialized, with $\tau_\beta$ being incremented by 1 (instead of being reset to 0).

Only in Steps (3)(a) and (3)(b) is any element adjoined to either $E$ or $F$. When one of these applies at a stage $\sigma + 1$, $f_{\sigma + 1}$ is redefined only on $\pm \sqrt{x + z(\alpha, \tau_\alpha)}$, not on $E_{\sigma + 1}$ nor on any $\sqrt{x + z(\beta)}$ with $\beta \neq \alpha, \tau_\alpha$. $f$ will never subsequently be redefined on $\pm \sqrt{x + z(\alpha, \tau_\alpha)}$ (since either $R_\alpha$ remains satisfied forever, or else it is subsequently injured and $\tau_\alpha$ is incremented). Therefore $f(x) = \lim_{\sigma \to \infty} f_\sigma(x)$ exists for all $x \in E$. Since every $f_\sigma : E_\sigma \to F_\sigma$ was an isomorphism, this limit $f$ is an isomorphism from $E$ onto $F$.

However, we claim by induction on $\alpha$ that every $\mathcal{R}_\alpha$ holds, and that it injures the requirements $\mathcal{R}_\beta$ with $\beta > \alpha$ at only countably many stages. Assume that this holds for all $\alpha' < \alpha$ (so that $\tau_\alpha$ is incremented on account of injury at only countably many stages).

If $\psi_\alpha$ is not total, then eventually the construction finds an $\xi$ on which it diverges, and thereafter it never receives attention again. Likewise, if $\psi_\alpha$ fails to respect the field operations, then at some stage we will discover this and $\mathcal{R}_\alpha$ will never again receive attention. (Of course, in both of these cases, $\psi_\alpha$ cannot be an isomorphism.) So suppose that $\psi_\alpha$ is a field embedding from $E$ into $F$. Once the injuries by higher-priority requirements have ceased (according to our inductive hypothesis), $\tau_\alpha$ can only be incremented by Steps (1) or (3)(c) at stages where $\mathcal{R}_\alpha$ receives attention. But if there are uncountably many such stages, then uncountably many algebraically independent elements $\sqrt{x + z(\alpha, \tau_\alpha)}$ in $E$ are mapped to elements algebraically dependent over the countable subset $\{y_{\beta, \sigma} : \beta < \alpha\}$ of $F$. No field embedding can do this, so by assumption, Step (1) applies to $\mathcal{R}_\alpha$ at only countably many stages, and therefore the construction must eventually reach Step (2) or Step (3).

Of course, if Step (2) ever happens, then $\mathcal{R}_\alpha$ is satisfied right then and never again becomes unsatisfied. (Indeed, in this case $\psi_\alpha$ cannot have been a field embedding, since it maps an algebraically independent set to an algebraically dependent set.) So assume that eventually we reach Step (3) for $\mathcal{R}_\alpha$. If we execute either Step (3)(a) or (3)(b) there, then $\sqrt{x + z(\alpha, \tau_\alpha)}$ has a square root in $E$, but $\psi_\alpha(\sqrt{x + z(\alpha, \tau_\alpha)})$ has no square root in $F$, which will ensure that $\psi_\alpha$ is not an isomorphism. So we must show that Step (3)(c) cannot apply forever. But this is easy. First, the proposed adjoinments of square roots in Steps (3)(a) and (3)(b) cannot both generate square roots of $\psi_\alpha(\sqrt{x + z(\alpha, \tau_\alpha)})$: the two proposed square roots do not generate each other (as neither $E$ nor $F$ contains any $\sqrt{-1}$), so the two proposed square roots generate distinct field extensions, and since each of these proposed extensions has degree 2, they intersect only in the ground field. Therefore, when (3)(c) applies, one or the other proposed square root must be algebraic over $\{\psi_\beta(\sqrt{x + z(\beta, \tau_\beta)}) : \beta < \alpha\}$. If Step (3)(c) applied at uncountably many stages $\sigma + 1$, then for uncountably many distinct values of $\tau_\alpha$, $f(\sqrt{x + z(\alpha, \tau_\alpha)})$ would be dependent over $\{\psi_\beta(\sqrt{x + z(\beta, \tau_\beta)}) : \beta < \alpha\}$ (since all $\tau_\beta$ converge to limits) which would mean that $\psi_\alpha$ would map an uncountable, algebraically independent set to a set of elements all of which are dependent over a countable set. No field embedding can do this, so in this case $\psi_\alpha$ would eventually
show itself not to be an embedding. Therefore, eventually either Step (3)(a) or Step (3)(b) must apply, at which stage $R_\alpha$ is declared satisfied and never again receives attention. Subsequent adjoinments to $E$ and $F$ are done only in Steps (3)(a) or (3)(b) by lower-priority requirements, which are always careful not to adjoin elements which would cause $R_\alpha$ to become unsatisfied. (This is the reason for the existence of Step (3)(c).) Our induction is now complete, and the theorem is proven. ■

At the other extreme from algebraically closed fields, namely fields purely transcendental over $Q$, computable categoricity fails again.

**Proposition 4.7**

If $F = \mathbb{Q}(X_\alpha : \alpha < \omega_1)$ is an $\omega_1$-computable field and is purely transcendental over the rationals with transcendence degree $\omega_1$, then $F$ is not $\omega_1$-computably categorical.

**Proof.** We take $F$ itself to be a presentation with the transcendence basis $\{X_\alpha : \alpha < \omega_1\}$ computable. (Lemma 4.1 only guarantees the existence of some computable transcendence basis, not necessarily of one generating the entire field.) We build a computable field $E \cong F$ with no computable isomorphism from $E$ onto $F$. $X_\alpha$ will be our witness that the computable function $\phi_\alpha$ is not such an isomorphism.

At the start, we build $E_0$ to be $F$ itself, although we only use the elements of $\omega_1^{[0]}$ to do so. (Let $E_0$ be the isomorphic image of $F$ under the map $\lambda + n \mapsto \lambda + 2n$ for all limit ordinals $\lambda$.) We write $y_\alpha \in E_0$ for the image of $x_\alpha$ under this map. Then, for each $\alpha$, we wait for $\phi_\alpha(y_\alpha)$ to converge, say to some $z_\alpha \in F$. When this happens, we find $x_1, \ldots, x_n$ such that $z_\alpha \in \mathbb{Q}(x_{\beta_1}, \ldots, x_{\beta_n})$, and ask whether the polynomial $p(x) = x^2 - z_\alpha$ factors over the subfield $\mathbb{Q}(x_{\beta_1}, \ldots, x_{\beta_n})$. (Kronecker gives a splitting algorithm for this field in [15], since we know the elements $x_{\beta_i}$ to be algebraically independent over $\mathbb{Q}$.) If so, then $z_\alpha$ has a square root in $F$, and so we do not change anything in $E$, but define $y'_\alpha = y_\alpha$. If not, then we adjoin to $E$ a new element $y'_\alpha$ whose square in $E$ is $y_\alpha$, and use the next row of currently unused elements to close $E$ under the field operations. (This must happen at $\omega_1$-many stages, so all rows eventually get used.) Formally, the existing field $E_\sigma$ is extended to $E_{\sigma + 1} = E_\sigma[X]/(X^2 - y_\alpha)$, which is a field because the quadratic polynomial $(X^2 - y_\alpha)$, having no roots in $E_\sigma$, must be irreducible in $E_\sigma[X]$. This completes the construction.

Now $E = \mathbb{Q}(y'_\alpha : \alpha < \omega_1)$ is isomorphic to $F$ via the map $y'_\alpha \mapsto x_\alpha$. However, if $\phi_\alpha(y_\alpha)$ ↓, then $y_\alpha$ has a square root in $E$ iff $\phi_\alpha(y_\alpha)$ has no square root in $F$. Thus no $\phi_\alpha$ can be an isomorphism from $E$ onto $F$. ■

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**References**


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